2.1 Log Logic
A Develop Understanding Task

We began thinking about logarithms as inverse functions for exponentials in Tracking the Tortoise. Logarithmic functions are interesting and useful on their own. In the next few tasks, we will be working on understanding logarithmic expressions, logarithmic functions, and logarithmic operations on equations.

We showed the inverse relationship between exponential and logarithmic functions using a diagram like the one below:

\[ f(x) = 2^x \]

We could summarize this relationship by saying:

\[ 2^3 = 8 \text{ so, } \log_2 8 = 3 \]

Logarithms can be defined for any base used for an exponential function. Base 10 is popular. Using base 10, you can write statements like these:

\[ 10^1 = 10 \text{ so, } \log_{10} 10 = 1 \]
\[ 10^2 = 100 \text{ so, } \log_{10} 100 = 2 \]
\[ 10^3 = 1000 \text{ so, } \log_{10} 1000 = 3 \]

The notation is a little strange, but you can see the inverse pattern of switching the inputs and outputs.

The next few problems will give you an opportunity to practice thinking about this pattern and possibly make a few conjectures about other patterns that you may notice with logarithms.
Place the following expressions on the number line. Use the space below the number line to explain how you knew where to place each expression.

1. A. \( \log_3 3 \)  B. \( \log_3 9 \)  C. \( \log_3 \frac{1}{3} \)  D. \( \log_3 1 \)  E. \( \log_3 \frac{1}{9} \)

Explain: ____________________________________________________________________________________________________

2. A. \( \log_3 81 \)  B. \( \log_{10} 100 \)  C. \( \log_8 8 \)  D. \( \log_5 25 \)  E. \( \log_2 32 \)

Explain: ____________________________________________________________________________________________________

3. A. \( \log_7 7 \)  B. \( \log_9 9 \)  C. \( \log_{11} 1 \)  D. \( \log_{10} 1 \)

Explain: ____________________________________________________________________________________________________

4. A. \( \log_2 \left( \frac{1}{4} \right) \)  B. \( \log_{10} \left( \frac{1}{1000} \right) \)  C. \( \log_5 \left( \frac{1}{125} \right) \)  D. \( \log_6 \left( \frac{1}{6} \right) \)

Explain: ____________________________________________________________________________________________________
5. A. \( \log_{4} 16 \)  B. \( \log_{2} 16 \)  C. \( \log_{9} 16 \)  D. \( \log_{16} 16 \)

Explain: ____________________________________________________________________________________________________

6. A. \( \log_{2} 5 \)  B. \( \log_{5} 10 \)  C. \( \log_{6} 1 \)  D. \( \log_{5} 5 \)  E. \( \log_{10} 5 \)

Explain: ____________________________________________________________________________________________________

7. A. \( \log_{10} 50 \)  B. \( \log_{10} 150 \)  C. \( \log_{10} 1000 \)  D. \( \log_{10} 500 \)

Explain: ____________________________________________________________________________________________________

8. A. \( \log_{3} 3^{2} \)  B. \( \log_{5} 5^{-2} \)  C. \( \log_{6} 6^{0} \)  D. \( \log_{4} 4^{-1} \)  E. \( \log_{2} 2^{3} \)

Explain: ____________________________________________________________________________________________________
Based on your work with logarithmic expressions, determine whether each of these statements is always true, sometimes true, or never true. If the statement is sometimes true, describe the conditions that make it true. Explain your answers.

9. The value of $\log_b x$ is positive.

Explain: ______________________________________________________________________________________

10. $\log_b x$ is not a valid expression if $x$ is a negative number.

Explain: ______________________________________________________________________________________

11. $\log_b 1 = 0$ for any base, $b > 1$.

Explain: ______________________________________________________________________________________

12. $\log_b b = 1$ for any $b > 1$.

Explain: ______________________________________________________________________________________

13. $\log_2 x < \log_3 x$ for any value of $x$.

Explain: ______________________________________________________________________________________

14. $\log_b b^n = n$ for any $b > 1$.

Explain: ______________________________________________________________________________________
Log Logic – Teacher Notes
A Develop Understanding Task

**Purpose:** The purpose of this task is to develop students understanding of logarithmic expressions and to make sense of the notation. In addition to evaluating log expressions, student will compare expressions that they cannot evaluate explicitly. They will also use patterns that they have seen in the task and the definition of a logarithm to justify some properties of logarithms.

**Core Standards Focus:**

**F.BF.5** (+) Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.

**F.LE.4** For exponential models, express as a logarithm the solution to \(ab^c = d\) where \(a, c,\) and \(d\) are numbers and the base \(b\) is 2, 10, or \(e\); evaluate the logarithm using technology.

Note for F.LE.4: Consider extending this unit to include the relationship between properties of logarithms and properties of exponents, such as the connection between the properties of exponents and the basic logarithm property that \(\log xy = \log x + \log y\).

**Related Standards:** **F.BF.4**

**Launch (Whole Class):** Begin the task by working through each of the examples on page 1 of the task with students. Tell them that since we know that logarithmic functions and exponential functions are inverses, the definition of a logarithm is:

\[
\text{If } b^x = n \text{ then } \log_b n = x \text{ for } b > 1
\]

Keep this relationship posted where students can refer to it during their work on the task.

**Explore (Small Group):** The task begins with expressions that will generate integer values. In the beginning, encourage students to use the pattern expressed in the definition to help find the values. If they don’t know the powers of the base numbers, they may need to use calculators to identify them. For instance, if they are asked to evaluate \(\log_2 32\), they may need to use the calculator to find that \(2^5 = 32\). (Author’s note: I hope this wouldn’t be the case, but the emphasis in this task is on reasoning, not on arithmetic skill.) Thinking about these values will help to review integer exponents.

Starting at #5, there are expressions that can only be estimated and placed on the number line in a reasonable location. Don’t give students a way to use the calculator to evaluate these expressions directly; again the emphasis is on reasoning and comparing.
As you monitor students as they work, keep track of students that have interesting justifications for their answers on problems #9 – 15 so that they can be included in the class discussion.

**Discuss (Whole Class):** Begin the discussion with #2. For each log expression, write the equivalent exponential equation like so:

\[ \log_3 81 = 4 \quad 3^4 = 81 \]

This will give students practice in seeing the relationship between exponential functions and logarithmic functions. Place each of the values on the number line.

Move the discussion to #4 and proceed in the same way, giving students a brush-up on negative exponents.

Next, work with question #5. Since students can’t calculate these expressions directly, they will have to use logic to figure this out. One strategy is to first put the expressions in order from smallest to biggest based on the idea that the bigger the base, the smaller the exponent will need to be to get 16. (Be sure this idea is generalized by the end of the discussion of #5.) Once the numbers are in order, then the approximate values can be considered based upon known values for a particular base.

Work on #7 next. In this problem, the bases are the same, but the arguments are different. The expressions can be ordered based on the idea that for a given base, \( b > 1 \), the greater the argument, the greater the exponent will need to be.

Finally, work through each of problems 9 – 15. This is an opportunity to develop a number of the properties of logarithms from the definitions. After students have justified each of the properties that are always true (#10, 11, 12, and 14), these should be posted in the classroom as agreed-upon properties that can be used in future work.

**Aligned Ready, Set, Go: Logarithmic Functions 2.1**
Ready, Set, Go!

Ready
Topic: Graphing exponential equations

Graph each function over the domain \(-4 \leq x \leq 4\).

1. \(y = 2^x\)
2. \(y = 2 \cdot 2^x\)
3. \(y = \left(\frac{1}{2}\right)^x\)
4. \(y = 2 \left(\frac{1}{2}\right)^x\)

5. Compare graph #1 to graph #2. Multiplying by 2 should generate a dilation of the graph, but the graph looks like it has been translated vertically. How do you explain that?

6. Compare graph #3 to graph #4. Is your explanation in #5 still valid for these two graphs? Explain.
Set

Topic: Evaluating logarithmic functions

Arrange the following expressions in numerical order from smallest to largest. Do not use a calculator. Be prepared to explain your logic.

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>\log_2{32}</td>
<td>\log_7{343}</td>
<td>\log_3{5}</td>
<td>\log_{15}{225}</td>
<td>\log_{11}{11}</td>
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<tr>
<td>8</td>
<td>\log_3{81}</td>
<td>\log_5{125}</td>
<td>\log_8{8}</td>
<td>\log_4{1}</td>
<td>\log_{100}{1}</td>
</tr>
<tr>
<td>9</td>
<td>\log_7{45}</td>
<td>\log_3{12}</td>
<td>\log_4{12}</td>
<td>\log_3{30}</td>
<td>\log_x{x}</td>
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<tr>
<td>10</td>
<td>\log_x{\frac{1}{x^2}}</td>
<td>\log_5{\frac{1}{5}}</td>
<td>\log_2{\frac{1}{8}}</td>
<td>\log_{\frac{1}{10,000}}{1}</td>
<td>\log_x{1}</td>
</tr>
<tr>
<td>11</td>
<td>\log{200}</td>
<td>\log{0.02}</td>
<td>\log_2{10}</td>
<td>\log_2{\frac{1}{10}}</td>
<td>\log_2{200}</td>
</tr>
</tbody>
</table>

Answer the following questions. If yes, give an example or the answer. If no, explain why not.

12. Is it possible for a logarithm to equal a negative number?
13. Is it possible for a logarithm to equal zero?
14. Does \( \log_x{0} \) have an answer?
15. Does \( \log_x{1} \) have an answer?
16. Does \( \log_x{x^5} \) have an answer?
Go

Topic: Properties of Exponents

Write each expression as an integer or a simple fraction.

17. \(27^0\)  
18. \(11(-6)^0\)  
19. \(-3^{-2}\)

20. \(4^{-3}\)  
21. \(\frac{9}{2^{-1}}\)  
22. \(\frac{4^3}{8^0}\)

23. \(\frac{4^0}{2^{-5}}\)  
24. \(3\left(\frac{293}{115}\right)^0\)  
25. \(42 \cdot 6^{-4}\)

26. \(\frac{3}{6^{-1}}\)  
27. \(\frac{7^{-2}}{4^{-1}}\)  
28. \(\frac{32^{-1}}{4^{-1}}\)