7.2 Circle Dilations
A Develop Understanding Task

The statement "all circles are similar" may seem intuitively obvious, since all circles have the same shape even though they may be different sizes. However, we can learn a lot about the properties of circles by working on the proof of this statement.

Remember that the definition of similarity requires us to find a sequence of dilations and rigid motion transformations that superimposes one figure onto the other.

Zac is describing to Sione how he would prove that circle \( A \) is similar to circle \( B \).

Zac: “Translate circle \( A \) until its center coincides with the center of circle \( B \). Then enlarge circle \( A \) by dilation until the points on circle \( A \) coincide with the points on circle \( B \). Or, you could shrink circle \( B \) by dilation until the points on circle \( B \) coincide with the points on circle \( A \).”

Sione has some questions: “After the translation, what is the scale factor for the enlargement that carries circle \( A \) onto circle \( B \)? And, what is the scale factor for the reduction that carries circle \( B \) onto circle \( A \)?

1. How would you answer Sione’s questions?
Based on Zac and Sione’s discussion, we are probably convinced that circle $A$ and circle $B$ are similar. Another way we might convince ourselves that the two circles are similar would be to find the center of dilation that maps pre-image points from circle $A$ onto corresponding image points on circle $B$.

2. Locate the center of dilation that carries circle $A$ onto circle $B$. Explain how you know the point you found is the center of dilation. (Note that both circles have been drawn tangent to $RS$.)

3. Draw some chords, triangles or other polygons in each circle that would be similar to each other. Explain how you know these corresponding figures are similar.

4. Based on the figures you drew in question 3, write some proportionality statements that you know are true.

5. Here is a proportionality statement you may not have considered. What convinces you that it is true?

\[
\frac{\text{circumference of circle } A}{\text{diameter of circle } A} = \frac{\text{circumference of circle } B}{\text{diameter of circle } B}
\]

Since this ratio of circumference to diameter is the same scale factor for all circles, this ratio has been given the name $\pi$ (pi).

6. How much larger is the circumference of circle $B$ than the circumference of circle $A$?

7. Do you think the following proportion is true or false? Why?

\[
\frac{\text{area of circle } B}{\text{area of circle } A} = \frac{\text{circumference of circle } B}{\text{circumference of circle } A}
\]
7.2 Circle Dilations – Teacher Notes

*A Develop Understanding Task*

**Note to teachers:** You may want to provide students with several copies of the diagram in the task to facilitate multiple drawings in response to question 3.

**Purpose:** In this task students consider the similarity of circles by examining two different transformation strategies that map one circle onto another. In the first strategy one circle is translated so that the center of the circles coincide. The inner circle can then be enlarged to carry it onto the outer circle, or the outer circle can be shrunk to carry it onto the inner circle. Students are asked to determine the scale factors for both the enlargement and the reduction. In the second strategy students observe that any circle can be mapped onto any other circle by dilation. Students are also asked to find the scale factor of this dilation, which is the same scale factor as the enlargement (or reduction) factor used in the first strategy. Students are also given the opportunity to draw similar figures inscribed within the two circles, which has the potential of surfacing some observations about central and inscribed angles, and the relationship between tangent lines and radii.

**Core Standards Focus:**

**G.C.1** Prove that all circles are similar.

**Related Standards:** **G.C.2**

**Launch (Whole Class):**
Students should have access to this task based on the work with dilations in module 6. After posing the main question to be explored in this task, "Are all circles similar?", set students to work on the task.

**Explore (Small Group):**
Students may be confused about finding the scale factors asked for in question 1, since no numbers are given. Ask what they would do if the specific radii of the two circles were given. Help students recognize the scale factor for the enlargement would be \( \frac{r_2}{r_1} \) and for the reduction it would be \( \frac{r_1}{r_2} \).

For question 2 students need to recognize that the line \( RS \) will pass through the center of dilation, as will the line \( AB \) which contains the centers of the circles, since both lines map a point on circle \( A \) to its image on circle \( B \). The point of intersection of these two lines will be the center of dilation.

Question 3 may challenge students until they realize that any line through the center of dilation will map points on circle \( A \) to their images on circle \( B \). Such secant lines will identify vertices of similar figures that can be inscribed in circle \( A \) and circle \( B \). Therefore, encourage students to focus on
inscribed triangles and inscribed polygons—figures that have their vertices on the circles—so they can map corresponding pre-image points to image points in the other circle. Once students have created one or more inscribed polygons that are similar to each other, they should be able to write some proportionality statements for question 4. Watch for two types of proportionality statements: ones where the ratios consist of two segments from the same figure, and ones where the ratios consist of corresponding segments from similar figures.

In question 6 help students recognize that while the ratio of circumference to diameter of each circle is \( \pi \), the ratio of the circumference of the larger circle to the circumference of the smaller circle is \( \frac{r_2}{r_1} \).

To help resolve question 7 you may refer back to task 6.1, *Photocopy Faux Pas*, where it was observed that the scale factor for area is the square of the scale factor for a line segment. Since the circumference is a linear measurement, the ratio of the areas of the circles will be the square of the ratio of the circumferences.

**Discuss (Whole Class):**

Much of the whole class discussion should focus on the similar figures students created for question 3 in order to talk about inscribed angles and intercepted arcs, and possibly surface the relationship between inscribed angles and central angles. Pose a possible pair of similar figures such as the following (or use a student-generated pair of similar figures that will get at these same issues).

Ask students to identify some similar triangles in this diagram. Use these triangles to write some proportionality statements.

Ask students to identify some isosceles triangles in this diagram and explain how they know they are isosceles.

Ask students to identify any inscribed angles.

Ask students to identify any central angles and point out that a central angle and the intercepted arc have the same degree measure.

Ask students if they think there are any right triangles in this figure, and why they think it contains a right angle.
Students may suggest that $\triangle HGF$ is a right triangle by the way it looks. Point out that $\angle HGF$ is an inscribed angle that intercepts the semicircular arc $FH$ and that the triangle is inscribed in a semicircle. If $\angle HGF$ is a right angle, then it measures half of the $180^\circ$ arc it intercepts.

Either end the discussion with this last observation, which will be revisited in the next task, or you may choose to continue this discussion about the relationship between inscribed angles and their intercepted arcs. If you choose to continue the discussion, label the angles of the triangles in circle $B$ as shown in the following diagram. Ask students the following questions to generate a conjecture and proof about the relationship between the degree measure of an inscribed angle and its intercepted arc:

How do we know $x = y$? (Because $\triangle BFG$ is isosceles.)
How do we know $w = x + y$? (An external angle of a triangle measures the sum of the two remote interior angles.)
What does this imply about the relationship between $x$ and $w$? (By substitution, $w = 2x$ or $x = \frac{1}{2}w$.)
What does this imply about the relationship between an inscribed angle and its intercepted arc?
Ready, Set, Go!

Ready
Topic: Finding missing angles, rotational symmetry, regular polygons

Find the missing angle in each of the figures below.

1. \[ \angle x = \]

2. \[ \angle x = \]

3. \[ \angle x = \]

4. \[ \angle x = \]

Find the angles of rotational symmetry for the regular polygons. Rotational symmetry means that the polygon rotates the indicated number of degrees to land on itself and all points in the image coincide with the pre-image.

5. \[ \angle q = \]

6. \[ \angle s = \]
Set

Topic: Dilations, proportionality between similar figures.

For each set of similar figures complete the proportionality statements.

7. \( \triangle ABC \sim \triangle CDE \)
   a. \( \frac{AB}{BC} = \frac{?}{?} \)
   b. \( \frac{AC}{AB} = \frac{?}{CD} \)
   c. \( \frac{BC}{AC} = \frac{DE}{?} \)

8a. \( \frac{AB}{BC} = \frac{?}{B'C'} \)
   b. \( \frac{BC}{B'C'} = \frac{?}{?} \)

9. Quadrilateral \( ABCD \sim \) Quadrilateral \( EFGH \)
   a. \( \frac{EF}{?} = \frac{GH}{CD} \)
   b. \( \frac{\text{Circumference Large Circle}}{\text{Circumference Small Circle}} = \frac{?}{?} \)
Go

Topic: Finding lines of reflection, finding the center of a circle.

Find the line of reflection between the image and the pre-image.

Find the center of each circle. (Hint: rotations happen on circles and so finding the center of a circle is like finding the center of rotation between pairs of point on the circle.)

12. Use the given chords to assist you.

13. Draw two chords to assist you.