6.7 Pythagoras by Proportions
A Practice Understanding Task

There are many different proofs of the Pythagorean Theorem. Here is one based on similar triangles.

Step 1: Cut a $4 \times 6$ index card along one of its diagonals to form two congruent right triangles.

Step 2: In each right triangle, draw an altitude from the right angle vertex to the hypotenuse.

Step 3: Label each triangle as shown in the following diagram. Flip each triangle over and label the matching sides and angles with the same names on the back as on the front.

Step 4: Cut one of the right triangles along the altitude to form two smaller right triangles.

Step 5: Arrange the three triangles in a way that convinces you that all three right triangles are similar. You may need to reflect and/or rotate one or more triangles to form this arrangement.

Step 6: Write proportionality statements to represent relationships between the labeled sides of the triangles.

Step 7: Solve one of your proportions for $x$ and the other proportion for $y$. (If you have not written proportions that involve $x$ and $y$, study your set of triangles until you can do so.)

Step 8: Work with the equations you wrote in step 7 until you can show algebraically that $a^2 + b^2 = c^2$. (Remember, $x + y = c$.)
Use your set of triangles to help you prove the following two theorems algebraically. For this work, you will want to label the length of the altitude of the original right triangle $h$. The appropriate legs of the smaller right triangles should also be labeled $h$.

**Right Triangle Altitude Theorem 1:** If an altitude is drawn to the hypotenuse of a right triangle, the length of the altitude is the geometric mean between the lengths of the two segments formed on the hypotenuse.

**Right Triangle Altitude Theorem 2:** If an altitude is drawn to the hypotenuse of a right triangle, the length of each leg of the right triangle is the geometric mean between the length of the hypotenuse and the length of the segment on the hypotenuse adjacent to the leg.

Use your set of triangles to help you find the values of $x$ and $y$ in the following diagram.
6.7 Pythagoras by Proportions – Teacher Notes
A Practice Understanding Task

Purpose: The purpose of this task is to give students additional practice with writing proportionality statements about similar triangles. Students will generate a new proof of the Pythagorean theorem that is based on similar triangles, rather than area. They will also explore a geometric way of representing the geometric mean between two numbers. Students may have previously worked with the geometric mean algebraically in the task Geometric Meanies found in the Mathematics Vision Project, Secondary I curriculum.

Core Standards Focus:

G.SRT.4 Prove theorems about triangles. Theorems include: the Pythagorean theorem proved using triangle similarity.

G.SRT.5 Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.

Related Standards:

Launch (Whole Class):
Give each student a 4 × 6 index card and provide rulers and scissors for the construction of the set of three triangles described in steps 1-4 of the task. Make sure that students label the sides and angles of the triangles correctly, according to the diagram. (Note: Students are to label the two triangles formed from cutting the index card along one of its diagonals before they cut one of the triangles along its altitude to form two smaller triangles.) Once students have correctly created and labeled their set of three triangles, have them work on steps 5-8. Students who finish this work quickly should work on the Right Triangle Altitude Theorems, but not all students need to do so.

Explore (Small Group):
If students arrange all three right triangles on top of each other with the right angles superimposed on top of each other, it will be apparent that the triangles are similar. This can be verified using the AA Similarity Theorem for Triangles. With the triangles arranged in this way, students should be able to write the following proportions by comparing the two smaller triangles to the largest triangle: \( \frac{c}{b} = \frac{b}{y} \) and \( \frac{c}{a} = \frac{a}{x} \). The remainder of the proof may take a lot of prompting and guidance. Don’t be discouraged by this, but try to use as much student thinking as possible as you help students work through the theorem. Decide when it might be appropriate to bring the class together for a whole class discussion.
Discuss (Whole Class):

You might begin the discussion by having students list all of the proportionality statements they found. If no one has written these two proportions, \( \frac{c}{b} = \frac{b}{y} \) and \( \frac{c}{a} = \frac{a}{x} \), ask students to re-examine their triangles to see if they can find how these proportions show up in the triangles they have constructed. Ask students how they might rearrange these proportions to solve them for \( x \) and \( y \). This should lead to the equations \( b^2 = cy \) and \( a^2 = cx \). Adding these equations together yields \( a^2 + b^2 = cx + cy \) or \( a^2 + b^2 = c(x + y) \). Since \( x + y \) is another name for side \( c \) of the largest triangle, we can rewrite this equation as \( a^2 + b^2 = c^2 \) by substitution. We have arrived at a relationship between the lengths of the sides of a right triangle—the Pythagorean theorem—without referring to the area of the sides.

Right Triangle Altitude Theorem 1 may already have been included on the list of proportionality statements students have written: \( \frac{x}{h} = \frac{y}{h} \). If not, ask students if they can find how this proportion shows up in the triangles they have constructed. (They will need to compare sides of the two smaller triangles.)

The proportions used to prove the Pythagorean theorem, although it may take students awhile to recognize this, represent Right Triangle Altitude Theorem 2.

**Aligned Ready, Set, Go: Similarity and Right Triangle Trig 6.7**
Ready, Set, Go!

**Ready**

Topic: Determining similarity and congruence in triangles.

1. Determine which of the triangles below are similar and which are congruent. Justify your conclusions. Give your reasoning for the triangles you pick to be similar and congruent.
Use the given right triangles with altitudes drawn to the hypotenuse to correctly complete the proportions.

2. \( \frac{a}{c} = \frac{f}{?} \)

3. \( \frac{a}{f} = \frac{c}{?} \)

4. \( \frac{a}{b} = \frac{f}{?} \)

5. \( \frac{a}{d} = \frac{c}{?} \)

6. \( \frac{f}{d} = \frac{e}{?} \)

7. \( \frac{b}{c} = \frac{e}{?} \)

Find the missing value for each right triangle with altitude.

8. 

9.
Go

Topic: Using similarity and parallel lines to solve problems
Finding Geometric and Arithmetic Means

In each problem determine the desired values using the similar triangles parallel lines and proportional relationships. Write a proportion and solve.

10.

11.

Analyze each table below closely and determine the missing values based on the given information and values in the table.

12. An Arithmetic Sequence

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13. A Geometric Sequence

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14. An Arithmetic Sequence

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15. A Geometric Sequence

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