7.4 Off on a Tangent

A Develop and Solidify Understanding Task

Recall that the right triangle definition of the tangent ratio is:

\[ \tan(A) = \frac{\text{length of side opposite angle } A}{\text{length of side adjacent to angle } A} \]

1. Revise this definition to find the tangent of any angle of rotation, given in either radians or degrees. Explain why your definition is reasonable.

2. Revise this definition to find the tangent of any angle of rotation drawn in standard position on the unit circle. Explain why your definition is reasonable.
We have observed that on the unit circle the value of sine and cosine can be represented with the length of a line segment.

3. Indicate on the following diagram which segment’s length represents the value of $\sin(\theta)$ and which represents the value of $\cos(\theta)$ for the given angle $\theta$.

There is also a line segment that can be defined on the unit circle so that its length represents the value of $\tan(\theta)$. Consider the length of $\overline{DE}$ in the unit circle diagram below. Note that $\triangle ADE$ and $\triangle ABC$ are right triangles. Write a convincing argument explaining why the length of segment $DE$ is equivalent to the value of $\tan(\theta)$ for the given angle $\theta$. 
4. On the coordinate axes below sketch the graph of \( y = \tan(\theta) \) by considering the length of segment \( DE \) as \( \theta \) rotates through angles from 0 radians to \( 2\pi \) radians. Explain any interesting features you notice in your graph.

Extend your graph of \( y = \tan(\theta) \) by considering the length of segment \( DE \) as \( \theta \) rotates through negative angles from 0 radians to \(-2\pi \) radians.

5. Using your unit circle diagrams from the task Water Wheels and the Unit Circle, give exact values for the following trigonometric expressions:

   a. \( \tan\left(\frac{\pi}{6}\right) = \) 
   b. \( \tan\left(\frac{5\pi}{6}\right) = \) 
   c. \( \tan\left(\frac{7\pi}{6}\right) = \) 

   d. \( \tan\left(\frac{\pi}{4}\right) = \) 
   e. \( \tan\left(\frac{3\pi}{4}\right) = \) 
   f. \( \tan\left(\frac{11\pi}{6}\right) = \) 

   g. \( \tan\left(\frac{\pi}{2}\right) = \) 
   h. \( \tan(\pi) = \) 
   i. \( \tan\left(\frac{7\pi}{3}\right) = \)
Functions are often classified based on the following definitions:

- A function $f(x)$ is classified as an **odd function** if $f(-\theta) = -f(\theta)$
- A function $f(x)$ is classified as an **even function** if $f(-\theta) = f(\theta)$

6. Based on these definitions and your work in this module, determine how to classify each of the following trigonometric functions.

- The function $y = \sin(x)$ would be classified as an [odd function, even function, neither an odd or even function]. Give evidence for your response.

- The function $y = \cos(x)$ would be classified as an [odd function, even function, neither an odd or even function]. Give evidence for your response.

- The function $y = \tan(x)$ would be classified as an [odd function, even function, neither an odd or even function]. Give evidence for your response.
Purpose: The purpose of this task is to extend the definition of the tangent from the right triangle trigonometric ratio definition, \( \tan(\theta) = \frac{\text{length of side opposite angle } \theta}{\text{length of side adjacent to angle } \theta} \), to an angle of rotation definition: \( \tan(\theta) = \frac{y}{x} \). The graph of the tangent function is obtained by representing the tangent of an angle of rotation by the length of a line segment related to the unit circle, and tracking the length of the line segment as the angle of rotation increases around the unit circle. The trigonometric identity \( \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} \) is also explored in terms of the unit circle.

Core Standards Focus:

F.TF.2 Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.

F.TF.3 (+) Use special triangles to determine geometrically the values of sine, cosine, tangent for \( \pi/3, \pi/4 \) and \( \pi/6 \), and use the unit circle to express the values of sine, cosine, and tangent for \( \pi-x \), \( \pi+x \), and \( 2\pi-x \) in terms of their values for \( x \), where \( x \) is any real number.

F.TF.4 (+) Use the unit circle to explain symmetry (odd and even) and periodicity of trigonometric functions.

F.IF.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes.

Related Standards: F.IF.4, F.IF.7, F.IF.9
Standards for Mathematical Practice:
SMP 7 – Use appropriate tools strategically

Vocabulary: Students will define the tangent function for an angle of rotation as \( \tan \theta = \frac{y}{x} \) where \( x \) and \( y \) are the coordinates of a point on a circle where the terminal ray of the angle of rotation intersects the circle when the angle is drawn in standard position (i.e., the vertex of the angle is at the origin and the initial ray lies along the positive \( x \)-axis.)

The Teaching Cycle:
Launch (Whole Class):
Remind students that we have redefined sine and cosine for angles of rotation drawn in standard position by using the values of \( x, y \) and \( r \). Ask how they might redefine tangent using these same values. Students should note that the definition \( \tan(\theta) = \frac{y}{x} \) is independent of the value of \( r \).

Examine the two unit circle drawings in question 3 together as a class. In the drawings label segment \( AC \) as \( x = \cos(\theta) \), segment \( BC \) as \( y = \sin(\theta) \) and segment \( AB \) as \( r = 1 \). In the second drawing note that \( \triangle ABC \) is similar to \( \triangle ADE \) and that the measure of segment \( AE \) is 1. Using this information ask students to consider what this implies about the measure of segment \( DE \). Give students a couple of minutes to suggest that since the triangles are similar they can write the proportion \( \frac{DE}{AE} = \frac{BC}{AC} \) or \( \frac{DE}{1} = \frac{y}{x} \). They should recognize that the length of segment \( DE \) is defined in the same way that we have defined \( \tan(\theta) \). That is, the length of segment \( DE \) represents the value of \( \tan(\theta) \) in the same way that the length of segment \( AC \) represents the value of \( \cos(\theta) \) and the length of segment \( BC \) represents the value of \( \sin(\theta) \). You may also want to point out that the trigonometric identity \( \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} \) is present in this diagram.

Now that we have a way of visually representing the magnitude of the value of \( \tan(\theta) \), assign students to work on determining what this implies about the shape and features of the graph of \( y = \tan(\theta) \). Also have them work on the rest of the task by using their unit circle diagrams.
Explore (Small Group):
If students are having a hard time sketching the graph, focus their attention on small intervals of \( \theta \). For example, what happens to the length of segment DE as \( \theta \) increases from 0 radians to \( \frac{\pi}{2} \) radians? What happens when \( \theta = \frac{\pi}{2} \)? What happens when \( \theta \) increases from \( \frac{\pi}{2} \) to \( \pi \)? How would you draw \( \Delta AD \) on this interval? What about negative angles of rotation?

Watch as students compute values of \( \tan(\theta) \) using information recorded on their unit circle diagrams. Students may need help simplifying the ratios formed by \( \frac{y}{x} \). Allow students to leave these ratios unsimplified until the whole class discussion when you can discuss some of the arithmetic involved, hopefully by using work from students who are successful at simplifying these ratios. Look for such students.

Listen for how students apply the definitions of odd and even functions to the sine, cosine and tangent functions. What representations do they draw upon to make these decisions: the symmetry of points around the unit circle, a graph of the function, or some other ways of reasoning?

Discuss (Whole Class):
Focus the whole class discussion on the following three items:

- The graph of the tangent function, including the period of \( \pi \) and the behavior of the graph near and at \( \pm \frac{\pi}{2} \) and \( \pm \frac{3\pi}{2} \) (the vertical asymptotes).
- The values of the tangent function at angles that are multiples of \( \frac{\pi}{6} \) and \( \frac{\pi}{4} \), including the arithmetic of simplifying these ratios.
- The classification of sine, cosine and tangent as even or odd functions and the evidence used to support these classifications (e.g., the graph of the function or the symmetry of the unit circle).

Aligned Ready, Set, Go: Trigonometric Functions, Equations and Identities 7.4


**READY**

Topic: Making rigid and non-rigid transformations on functions

The equation of a parent function is given. Write a new equation with the given transformations. Then sketch the new function on the same graph as the parent function. (If the function has asymptotes, sketch them in.)

1. \( y = x^2 \)

   Vertical shift: up 8
   
   horizontal shift: left 3
   
   dilation: \( \frac{1}{4} \)
   
   Equation:
   
   Domain:
   
   Range:

2. \( y = \frac{1}{x} \)

   Vertical shift: up 4
   
   horizontal shift: right 3
   
   dilation: \(-1\)
   
   Equation:
   
   Domain:
   
   Range:

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3. \( y = \sqrt{x} \)

Vertical shift: none.  
Horizontal shift: left 5  
Dilation: 3

Equation:  
Domain:  
Range:

4. \( y = \sin x \)

Vertical shift: 1  
Horizontal shift: left \( \frac{\pi}{2} \)  
Dilation (amplitude): 3

Equation:  
Domain:  
Range:
SET

Topic: Connecting values in the special triangles with radian measures

5. Triangle ABC is a right triangle. AB = 1.

Use the information in the figure to label the length of the sides and measure of the angles.

6. Triangle RST is an equilateral triangle. RS = 1

S\\overrightarrow{A} is an altitude

Use the information in the figure to label the length of the sides, the length of S\\overrightarrow{A}, and the exact length of S\\overrightarrow{A}.

Label the measure of angles RSA and SRA.

7. Use what you know about the unit circle and the information from the figures in problems 5 and 6 to fill in the table. Some values will be undefined.

<table>
<thead>
<tr>
<th>function</th>
<th>$\theta = \frac{\pi}{6}$</th>
<th>$\theta = \frac{\pi}{4}$</th>
<th>$\theta = \frac{\pi}{3}$</th>
<th>$\theta = \frac{\pi}{2}$</th>
<th>$\theta = \pi$</th>
<th>$\theta = \frac{3\pi}{2}$</th>
<th>$\theta = 2\pi$</th>
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</thead>
<tbody>
<tr>
<td>$\sin \theta$</td>
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<td>$\tan \theta$</td>
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</tbody>
</table>

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8. Label all of the points and angles of rotation in the given unit circle.

9. **Graph** \( f(x) = \tan \theta \). Use your table of values above for \( f(x) = \tan \theta \). Sketch your asymptotes with dotted lines.

10. Where do asymptotes always occur?
GO
Topic: Recalling trig facts

Answer the questions below. Be sure you can justify your thinking.

11. Given triangle ABC with angle C being the right angle, what is the sum of m∠A + m∠B?

12. Identify the quadrants in which \( \sin \theta \) is positive.

13. Identify the quadrants in which \( \cos \theta \) is negative.

14. Identify the quadrants in which \( \tan \theta \) is positive.

15. Explain why it is impossible for \( \sin \theta > 1 \).

16. Name the angles of rotation (in radians) for when \( \sin \theta = \cos \theta \).

17. For which trig functions do a positive rotation and a negative rotation always give the same value?

18. Explain why in the unit circle \( \tan \theta = \frac{y}{x} \).

19. Which function connects with the slope of the hypotenuse in a right triangle?

20. Explain why \( \sin \theta = \cos(90° - \theta) \).