Chapter 0: Fluency

In grade 6 students bring together the arithmetic that they have learned over the preceding six years of education. Flexibility in the understanding of arithmetic procedures transmits to new contexts (for example, linear functions in grade 8, complex numbers in grade 9, and polynomials and matrix algebra later on). The key to understanding arithmetic operations at a higher level is fluency in performing those procedures. So we interpret *fluency* to mean more than automaticity. Students need to be able to move painlessly from exact computation to estimation of arithmetic results in context (for example: 95×32 is around $100 \times 30 = 3000$). Students need to be able to select the most efficient method of computation when needed (for example: $99 + 99 + 99 = 3(100 - 1) = 300 - 3 = 297$).

These talents can only result from the kind of confidence in knowledge that allows - even encourages - explorations like these: a) input any positive number in their calculator, and then punch the square root key over and over again to see what happens; b) alternate between punching in $1/x$ and the square root key repeatedly, and not be surprised by the result.

Because of its importance and relevance throughout grade 6, we have put the chapter on fluency at the front end of these materials. Because of the nature of fluency - that it is developed by repetition - we do not intend for this chapter to be developed as a unit, but rather that the text and exercises be inserted into the natural flow of the course where appropriate, but frequently enough to be completely covered.

The standards (*6NS.2,3,4*) provide specific fluency objectives. The overall objective is this:

Compute fluently with multidigit numbers and find common factors and multiples,

where we take the word *fluently* as the confidence described above. The specific objectives are:

6.NS.2. Fluently divide multidigit numbers using the standard algorithm.

6.NS.3. Fluently add, subtract, multiply, and divide multidigit decimals using the standard algorithm for each operation.

6.NS.4. Find the greatest common factor of two whole numbers less than or equal to 100 and the least common multiple of two whole numbers less than or equal to 12. Use the distributive property to express a sum of two whole numbers 1 to 100 with a common factor as a multiple of a sum of two whole numbers with no common factor. For example, express $36 + 8$ *as* $4(9 + 2)$.

Chapter 0 is divided into three sections, one for each of the stated objectives.

The 6.NS content standards on fluency significantly involve the standards of practice, for the concept of fluency has to do with the practice of mathematical skills and knowledge. The standards of practice that are specifically addressed in this appendix are:

- **Standard of Practice 5:**
  it Use appropriate tools strategically.
  In particular, the calculator should be used: a) to complete precise, but routine, calculations for which

the practitioner already has a sense of what to expect; b) to explore the consequences of hypotheses; c) to explore questions that arise out of natural inquisitiveness. Calculators should never be used to avoid thinking; that is neither its role in science, nor its capability.

- **Standard of Practice 6:** *Attend to precision.*
  This standard is concerned with precision in communication, as well as precision in computation. In posing a question or describing a solution, it is necessary that ambiguity in concepts be as minimal as possible. For this, students learn to understand concepts with precision, and express that understanding with clarity. This is something learned through practice and thus, explains why this is a standard of practice. Often, a student will relegate a routine or tedious computation to the calculator. That is appropriate - that is the purpose of the tools we have. Students should be able to recognize whether or not the output is in the expected range. For example, a calculation of 432×27 should produce a number between $400 \times 20 = 8000$ and $500 \times 30 = 15000$. In short, it is a good idea to practice mental arithmetic for estimation, and to double check the output of procedure or calculation.

- **Standard of Practice 7:** *Look for and make use of structure.*
  This is probably the most significant of these standards for grade 6. In this grade, we develop understanding of the structure of arithmetic operations so that the student can make effective use of that understanding, not just in sixth grade but throughout their education. In sixth grade we build an understanding of the structure of arithmetic so that they are willing to accept adaptation of that structure to new and broader contexts.

In view of 6.NS.3, we take the word "number" in 6.NS.2 to mean *whole number* and for *division* to mean: given numbers A and B, find numbers Q and R such that $A = QB + R$ and $R < B$. Note: if $A < B$, then $Q = 0$ and $R = A$, so in the sequel we will tacitly assume that we are dividing a given positive integer by a smaller positive integer.

We take 6.NS.3 to mean fluency in conversion between decimals and fractions as well: for example, multiplication by 3/4 is the same as multiplication by 0.75, or division by 4/3.

The first section is concerned with standard 6.NS.2. Attention here is restricted to division, assuming that fluency in the other arithmetic operations has been attended to in prior grades. However, since 6.NS.3 speaks of all the operations, and because of our focus on understanding the operations well enough to be able to mentally estimate a range for the answer, we here review all the operations on whole numbers, providing geometric interpretations that, together with place value, help to explain the algorithms.

In the second section we turn to standard 6.NS.4, as a natural sequel to the discussion of the arithmetic of whole numbers. This standard asks for fluency in the multiplication and division of small whole numbers, for a certain purpose: to find commonalities between two given numbers. Therefore, we concentrate on the factoring of whole numbers, starting with finding all factors, and then all prime factors. The prime factorization of a whole number leads easily to listing all factors, and to the discussion of commonalities between two numbers. An important point in finding the factors of a whole number A is that we need only divide A by primes up to $\sqrt{A}$. However, $\sqrt{A}$ is not in the sixth grade vocabulary. We avoid this problem by concentrating on the search for *factor pairs* (as discussed in fourth grade), and see that we can stop the search when the next factor pair is the same as a prior one, but in reverse order.

Finally, we return to 6.NS.3 which is about the operations on decimals, with attention to place value. Now, place values not only proceed to the left by multiples of 10, but also proceed to the right by multiples of 1/10. If we had negative exponents at our disposal, it would help to explain the algorithms. Since negative numbers will be brought up later in sixth grade mathematics, we shall revisit this at that time. At first, our approach will be to replace an operation with decimal numbers by the same operation with whole numbers, by moving the decimal point (to the left or the right as necessary). In the end, we move the decimal point in the reverse direction to where it belongs.

Section 1. Arithmetic Operations with Whole Numbers

6.NS: Compute fluently with multi-digit numbers . . .

6.NS.2. Fluently divide multi-digit numbers using the standard algorithm.

Lets start with a discussion about the decimal representation of whole numbers, and then return to the implementation of arithmetic operations (addition, subtraction and division) in terms of this representation.

Counting is the first arithmetic skill; it is already observed in infants. One counts a collection of like things (for example, a box filled with red cubes) by (figuratively, if not actually) picking a member and putting it aside, and then moving another member to that aside, and in this way, one by one, recognizing all the members of the collection. As this is done, the members are given a numerical name: the first is 1, the next is 2, then 3, and so forth. We then recognize this movement of pieces as addition: one by one we add the pieces in our collection to the "counted" set. So, 2 = 1 + 1, 3 = 2 + 1 and so on. For large aggregates, we run out of symbols very soon; we cannot expect, in counting three dozen eggs, to use three dozen symbols.

Let's take a look at Roman Numerals to better understand this issue. One is represented by I, and so a collection is represented by iterations of I for each member of the collection. So two is represented by II, three by III, and so forth. But it is difficult to distinguish between two long strings of Is, so a shorthand was devised by the Romans. The first four counting numbers are I, II, III, IIII, but a new symbol was devised for the successor of IIII, namely V. Then six is VI, seven is VII and so forth. But now, after we've added four Is to V, we get to the next number by using the V again: VIIIIs successor is VV. But the Romans saw that eventually there would be the same problem of too many V's, so they nipped that in the bud, by designating VV by X. And so it goes: five Xs are replaced by L, two L's by C, five Cs by a D and two Ds by M. In decimal notation,

I = 1,   V = 5,   X = 10,   L = 50,   C = 100,   D = 500 ,   M = 1000,   and so forth

Here we are concentrating on the creation of new symbols to make the representation of the whole number reasonable, but we should not that there were other simplification strategies. For example, IIII was simplified to IV, XXXX to XL, and so forth. The scheme was this: if a numeral was placed before another numeral representing a larger number, then the lower numeral is to be subtracted; if after, to be added. So DC represents 600, while CD represents 400.

This worked in those days, because there was essentially no reason to count beyond so many thousands. But, what if there were? Do we keep inventing new symbols for larger and larger numbers? There was no algorithm to use to represent any number, no matter how large, for there was no symbols for billion or trillion, and so forth.

During the middle ages, this issue of effective representation of counting numbers was taken up by the Asian civilizations. By effective, we mean a system of enumeration that does not require the continual invention of new symbols as counting numbers grew larger and larger, and one which facilitates addition. The solution is that of place value. Simply put: pick a collection of symbols, starting with 0, with the property that each new symbol represents one more than the preceding, and 1 represents the successor of 0. Then when we've used up all our symbols we move a place over: 10 represents the successor of the last available symbol.

That describes the decimal system. We start with the set of symbols $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, where 0 represents the absence of anything, and each successive symbol represents one more than the predecessor. These are the *digits*. The successor of 9 makes use of the space to the left: 10 meaning 1 of the new kind, and zero digits. The successor increases by one digit, so is 11. When we get to 19, the successor puts a 0 in the first place and a 1+1 in the space to the left: that is 20.

Historical note: The system just described, that of *Arabic numerals*, underwent several significant changes before settling down to our present notation (in the 16th century). Most importantly, at first there was no symbol for "nothing" which was symbolized by an empty place. To illustrate, the numbers 101, 1201, 11, 121 were originally written as

$$1\quad 1 \qquad 12\quad 1 \qquad 11 \qquad 121,$$

which led to confusion in the hands of an unskilled scribe: is it 101 or 11? Finally a symbol was introduced to represent the absence of a numeral, and that is our zero.

To summarize: the decimal system is an algorithm that, in principle, allows us to assign a numeric symbol to any aggregate of objects, no matter how large. The algorithm is: start with the digits $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, where 0 represents the absence of members, and each of the others is just one more than the preceding. A number is a sequence of digits. The successor of that number is found by just taking the successor in the rightmost digit. If that is a 9, then replace the 9 by 0 and increase the digit to the left by 1. If the digit to the left is a 9, do the same in that place.

This is an algorithmic description of our system of representation of numbers. There are also these descriptions:

Geometric: draw a line and pick a point on the line. This point is designated "0." Then pick another point to the right of 0, and call it "1." Now, continue this process: move to the right by the same distance (as 0 is from 1), draw a point (call it "2"). Then repeat and repeat. Each point represents exactly one more than the preceding point.

Algebraic: Each place represents 10 times the place just to the right. So, for example, 3041 represents 1 unit plus 4 tens + no hundreds + 3 thousands." In symbols:

$$3041 = 3 \times 1000 + 0 \times 100 + 4 \times 10 + 1.$$

Students entering grade 6 have an intuitive understanding of place value, how they need to understand this displayed equation, and how it leads into complex algorithms for the algebraic operations.

We start with the students' understanding of addition and multiplication: addition of two whole numbers corresponds to the joining of one group with the other. Multiplication is repeated addition of a group to itself.

As for addition: first of all, the *digits* are the numbers $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 0\}$. Each place in a whole number represents a power of ten: the rightmost digit is the "ones" place, next to the left is the "tens," and to the left of that, the "hundreds" and so forth. Since 10 "ones" is the same as 1 "ten," and so forth, we can replace any multiple of 10 in one place by the same multiple of 1 in the place just to the left.

Example 1. Figure 1 below illustrates addition, using place value, for the sum 15 + 28. The ones place is represented by little ovals, and the tens place by a dark square: 15 is the same as 1 ten and 5 ones; 28 consists of 2 tens and 8 ones. Since addition is performed by putting two groups together, In the second image we have moved the representation so the tens columns are together, as are the ones. Since the sum 5 + 8 is not a digit, we create a group of 10 ovals (as shown in the third image) to get 10+3. Finally, 10 ovals are equivalent to 1 dark square, as shown in the last image as 43.

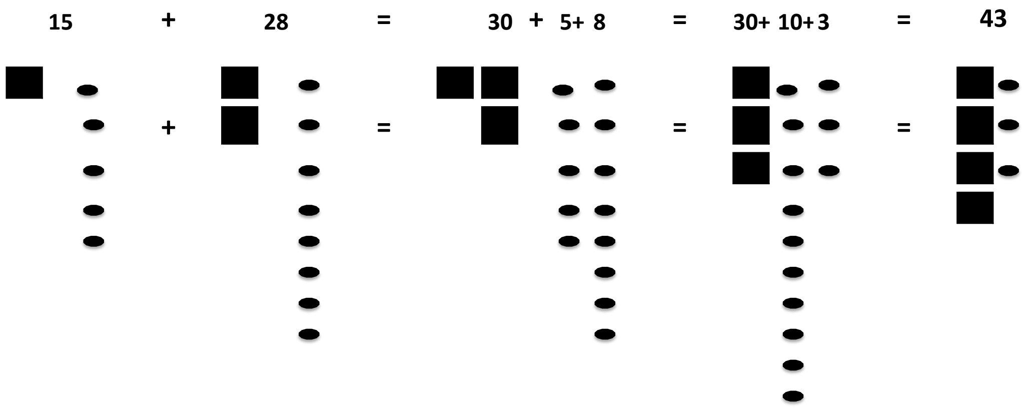


Figure 1

Of course, these images are used to graphically illustrate the procedure for addition; by this time the students should be skilled in the procedure, but might have forgotten the logic behind it. It should be noted that the numbers 15 and 28 are assumed to represent groups of *like* objects: we are not adding 28 pears to 15 trucks.

Students should also be skilled in column addition: to review, we here give an example and explanation.

Example 2. 36 + 47 + 21 + 18 = ?

Solution. Figure 2 below illustrates the logic behind the procedure; fluency requires that students are automatic in doing this in one column, without explicitly separating the tens and ones as we do in Figure 2. In the first image we have written the four numbers in columns representing the tens and ones. Now we add down the columns, giving the sums of 10 tens and 22 ones. Now we use the knowledge that 22 ones is the same as 2 tens and 2 ones: this is displayed in the third image, where the column sums give us 12 tens and 2 ones. This is the same as 1 hundred and 2 tens and 2 ones, and that is the final answer.

10s	1s
3	6
4	7
2	1
1	8

=

10s	1s
3	6
4	7
2	1
1	8
10	22

=

10s	1s
3	
4	
2	
1	
2	2
12	2

=

100s	10s	1s
	3	
	4	
	2	
	1	
	2	2
1	2	2

Figure 2

Multiplication as Repeated Addition

When we contemplate, for whole numbers A and B, A repetitions of the set B, then we say that the number of the entire aggregate is $A \times B$, literally, B repeated A times. This has the same value as Arepeated B times, known as *the commutative property of multiplication.*

For example, if we have 15 copies of a set of 18 items, how many items do we have in all? Figure 3 shows this. The problem has been broken down into four pieces to show how we actually compute the product. Since $15 = 10 + 5$, we can transform 15×18 into $(10 \times 18) + (5 \times 18)$. Then we can use the fact that $18 = 10 + 8$, to get

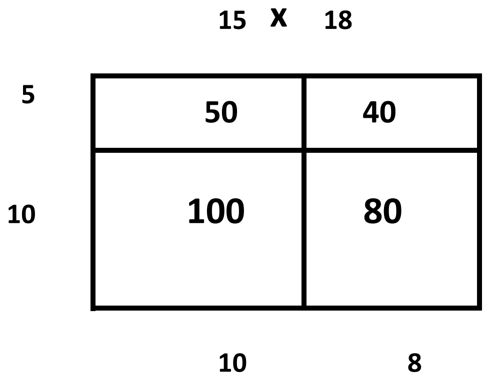


Figure 3: $15 \times 18 = 100 + 80 + 50 + 40 = 270$

$$15 \times 18 = (10 \times 10) + (10 \times 8) + (5 \times 10) + (5 \times 8)$$

where each summand is represented by a box in Figure 3. That is, the product can be visualized as the sum of four components, for each of which we need only multiply digits and keep track of powers of ten.

This example illustrates - in the context of place value - a fundamental arithmetic fact: the *distributive property* that relates multiplication and addition:

$$(A + B)C = AC + BC$$

and the more seemingly complicated version

$$(A + B)(C + D) = A(C + D) + B(C + D) = AC + AD + BC + BD,$$

which just amounts to two successive applications of the distributive property. With $A = 10, B = 5, C = 10, D = 8$ this becomes:

$$(10 + 5)(10 + 8) = 10 \times 10 + 10 \times 8 + 5 \times 10 + 5 \times 8 = 720$$

as shown in Figure 3.

When we turn to the standard algorithms for addition and multiplication, we will see that that the driving force that makes those algorithms work is the distributive property, which, when stated in generality, says:

Distributive Property: If two expressions are to be multiplied, and each expression is a sum of terms, the result is the sum of all products of a term from the first expression and a term from the second expression.

More on Place Value

Now let us go into the place value representation of whole numbers a little more deeply, seeking to expose the structure of addition.

Example 3. 97 + 523 =?

SOLUTION. The standard algorithm asks us to think of the addends in place value:

$$523 = 5 \cdot 100 + 2 \cdot 10 + 3 \cdot 1$$

$$97 = 0 \cdot 100 + 9 \cdot 10 + 7 \cdot 1$$

and then add like quantities to get

$$523 + 97 = 5 \cdot 100 + 11 \cdot 10 + 10 \cdot 1.$$

Now, we rearrange the right hand side, so that we have only single digits. $11 \cdot 10 = 1 \cdot 100 + 1 \cdot 10$, and $10 \cdot 1 = 1 \cdot 10$. These changes lead us to

$$523 + 97 = 5 \cdot 100 + 1 \cdot 100 + 1 \cdot 10 + 1 \cdot 10.$$

Now, collecting like terms (hundreds with hundreds, tens with tens, etc.),we have

$$523 + 97 = 6 \cdot 100 + 2 \cdot 10 + 0 \cdot 1 = 620.$$

Hidden behind the above discussion are properties of addition that now should be exposed. These amount to the assertion that we can add a list of numbers in any order that works for us. More formally, these are the properties of *commutativity* and *associativity*. Precisely, we have:

Commutative Property of Addition: For any two numbers A and B: $A + B = B + A$.

Associative Property of Addition: For any three numbers A, B and C: $(A + B) + C = A + (B + C)$.

To illustrate with Example 3: this can be done mentally. To calculate 97 + 523, first take the 3 from 523 and add it to 97, giving us the sum 100 + 520 to calculate. But clearly all we now need to do is add the 100 to the 500 (of 520) to get the answer: 620.

The fundamental principal is that a whole number is written as a sequence of digits (numbers between 0 and 9), where the place indicates a power of ten. For this purpose we will use exponential notation for the powers of 10. Reading from the right, the first digit is the number of units (multiples of 10^0), the place to the left is the number of tens (multiples of 10^1), the place to the left of that is the number of hundreds (multiples of 10^2), and going forward, each new digit represents the number of the next power of 10. We finish the algorithm by collecting the summands together in a sequence of digit multiples of powers of ten. The fundamental rule for this is

$$10 \cdot 1 = 10, \quad 10 \cdot 10 = 1 \cdot 10^2, \quad \ldots, \quad 10 \cdot 10^n = 10^{n+1}$$

So, for example if taking the sum produces the number 27 in the hundreds place, we consider 27 = 20+7, and so can trade 20 hundreds (10^2) for 2 thousands (10^3), thus placing the 2 in the thousands place, while the 7 remains in the hundreds place: $27 \times 10^2 = 20 \times 10^2 + 7 \times 10^2 = 2 \times 10 \times 10^2 + 7 \times 10^2 = 2 \times 10^3 + 7 \times 10^2 = 2700$.

The student learns to do this without being explicit, using a well developed shorthand (or just memory) at each step. The point in writing this down in explicit detail is to illustrate how the student is to understand these operations, rather than doing them by rote. It is that understanding that is the key to the fluency that allows appropriate flexibility in estimating the result. Let us illustrate by example.

Example 4. At 9:00 on election night, the ballot count is: candidate A: 47560; candidate B: 44127. But now the returns from District 10 are just in: 2316 votes in District 10 cast for candidate A, and 7387 for candidate B. Who is now in the lead?

Solution. The question is "*who is now in the lead?*", not "*by how many votes*?" Since we are looking for an estimate, we say that A's new tally is about 47,500 + 2,300 = 49,800 votes, while that for B is about 44,000 + 7,000 = 51,000 votes, telling us that candidate B has taken the lead! If the actual sums are needed, we could use the algorithm above or a calculator to find them to be 49,876 and 51,514.

Example 5. Gianni, Sylvia and Lester are at the amusement park, and they want to ride the Loop-de-loop. Tickets are \$5.50 each. Gianni has \$4.50, Sylvia has \$7.50 and Lester has \$4.50. Can they pool their resources and all ride the Loop-de-loop?

Solution. In order for all to ride, they need 3 tickets at \$5.50 apiece, or $3 \times 5.50 = 16.50$ dollars. Since we are discussing "fluency," we'd like to point out that this multiplication is easily computed by using the distributive property:

$$3 \times 5.50 = (3 \times 5) + 3 \times 0.50 = 15 + 1.50 = 16.50.$$

All together they have, in dollars:

$$4.50 + 7.50 + 4.50 = 4 + 7 + 4 + 3 \times 0.50 = 15 + 1.50 = 16.50,$$

exactly what they need to buy 3 tickets.

Subtraction

As with addition, we focus on the place value representation of number, so the actual subtraction we do is that between digits. Now, in any place, the digit for the number to be subtracted may be greater than the digit in that place for the number from which we are subtracting. In this case, we regroup *from* the place value just to the left. Note that, just as subtraction is the inverse of addition, the regrouping in subtraction is the inverse of that for addition. Let us illustrate:

Example 6. a) Subtract 21 from 53; b) Subtract 28 from 53; c) Subtract 46 from 201

SOLUTION. a) To subtract 21 from 53, we write the numbers in place value, rearrange by place value and subtract in place value:

$$53 - 21 = (50 + 3) - (20 + 1) = (50 - 20) + (3 - 1) = 30 + 2 = 32.$$

This is graphically illustrated in Figure 5. This is the same as the figures for addition with these changes: numbers to be subtracted are in outline *and* the pairing of a black figure with an outline figure of the same type requires the removal of both (reflecting that 1 − 1 = 0, 10 − 10 = 0 and so forth).

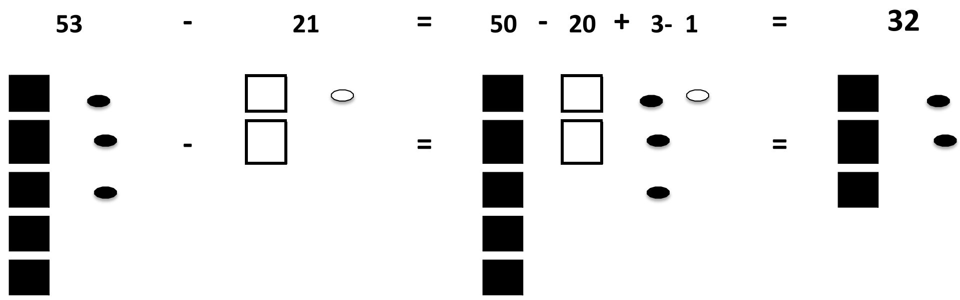


Figure 4

b) Now let's subtract 28 from 53, using the same ideas:

$$53 - 28 = (50 + 3) - (20 + 8) = (50 - 20) + (3 - 8) = 30 + (3 - 8).$$

To handle the term, 3 − 8, we regroup one ten from the 50 as ten ones and add those to the 3 to get (40 − 20) + (13 − 8) = 20 + 5 = 25, as illustrated in Figure 5.

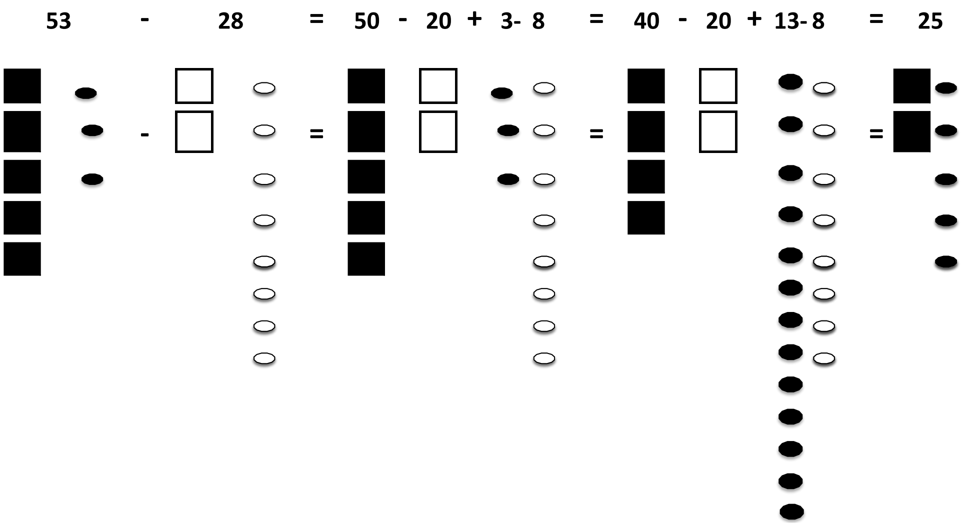


Figure 5

Of course, using negative numbers, we can replace 3 – 8 by −5 to get 30 + (3 – 8) = 30 – 5 = 25. However, this approach is not of value in the standard algorithm, and can get us in trouble with subtraction of multi-digit numbers, as is illustrated by the next problem.

c) 201 – 46 is a little harder, since there are no "tens" digits to regroup. So, we go one step further to the left and regroup the "hundreds" digit:

$201 - 46 = (200 + 1) - (40 + 6) = (200 - 40) + (1 - 6) = 160 + (1 - 6) = 150 + (10 + 1 - 6) = 150 + (11 - 6) = 155.$

An alternative to the standard algorithm) is illustrated in the image to the left in Figure 6: always regroup from the left. Start the computation with 1 – 6, which we don't know how to handle. So we add a 10 in the "ones" place for 201, and a 1 in the "tens" place for 46. This is legitimate since the net change in the expression is 10 – 10 = 0. In effect, we have replaced the subtraction 200 + 1 – (40 + 6) = (200 – 40) + (1 – 6) by the subtraction (200 – 50) + 11 – 6 = 150 + 5 = 155. Of course the 200 – 50 term is easy to calculate. However, applying the standard algorithm consistently (which is the intent of the word "algorithm") we would write, 200 – 50 = 100 + 100 – 50 = 100 + 50 = 150.

The image on the right in Figure 6 below presents another way of thinking of the standard algorithm. We solve the dilemma of the subtraction 1 – 6, by trading one group of ten (from the 20 tens) for ten ones and displaying and adding it to the 1, giving us an eleven in the "ones" place. Since we took 1 away from 20 we are left with 19. This is shown by crossing through the 20 and putting a 19 above it. Now do the subtraction: 201 – 46 = (19 – 4) × 10 + (11 – 6) = 155.

$$\begin{array}{r} 2^{1}0^{1}1 \\ -\ {}_{1}4_{1}6 \\ \hline 155 \end{array} \qquad \begin{array}{r} 19\phantom{^{1}1} \\ \not{2}\not{0}^{1}1 \\ -\ 4\ 6 \\ \hline 155 \end{array}$$

Figure 6

What is interesting about this example is the illustration of the nature of an algorithm as something that is reliably accurate in all contexts, even though in some particular contexts it isn't the easiest way to proceed. So, if we are asked $100 - 5 = ?$, we'd simply respond: 95. But the standard algorithm tells us to calculate this way: $100 - 5 = 90 + 10 - 5 = 90 + (10 - 5) = 90 + 5 = 95$. The message here is: in calculations, don't hesitate to use mathematical properties to make calculations easier, but you can always count on the standard algorithm.

To illustrate, let's return to part c) of Example 6: $201 - 46 = ?$ If we put this on a number line (see Figure 7) we see that 46 is 4 less than 50 and 201 is 1 more than 200. Now, $200 - 50 = 150$ is easy, and what we are looking for has an additional 1 unit on one side and 4 on the other. Thus $200 - 46 = 155$. We elaborate on this in the following paragraph.

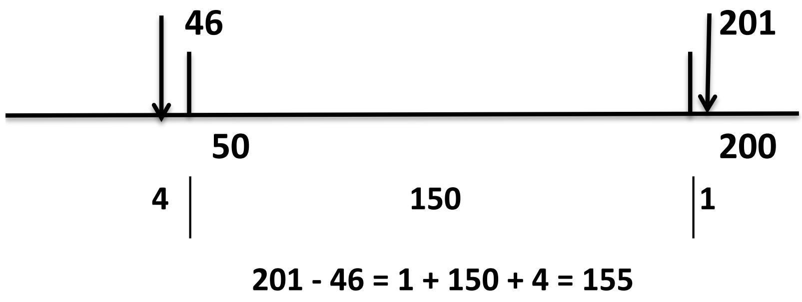


Figure 7

Subtraction as Addition

If we represent a subtraction of one positive number from a larger one on the number line, we recognize the result as the calculation of the length of the line segment from one number to the other. This gives us a new, and often easier, way of getting the result: by breaking that line segment up into easily calculate pieces (as exhibited in Figure 7). Here are two more illustrations:

Example 7. Subtract: $1243 - 587$.

SOLUTION. First, imagine the operation to be performed along a horizontal line, as in Figure 8 (top image). Our object is to find the length of the grey section. We reduce this to simple additions as shown in the bottom image. Draw the lines at 1200 and 600. The piece of the grey section between these two lines is 600. The grey part to the left of this piece has length 13, and that to the right, 43. Thus

$$1243 - 587 = 13 + 600 + 43 = 656.$$

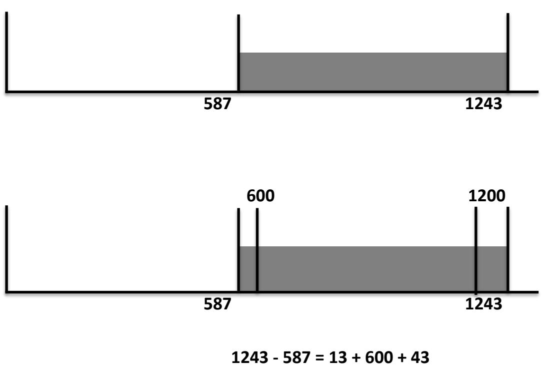


Figure 8

Example 8. Subtract: 4007 − 1991.

Solution. Place the numbers on a number line and partition the line interval from 1991 to 4007 conveniently, as in Figure 9.

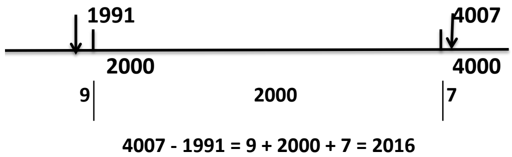


Figure 9

A significant reason for introducing this image is to deepen the understanding of subtraction as the calculation of the length of the line segment between the two points on the number line. This will help students to understand arithmetic computations on the set of integers in grade 7.

Multiplication

Once we understand the role the distributive property plays in the multiplication algorithm for place-value numbers, we can see the basics of the algorithm. To illustrate:

Example 9. Calculate 532×69.

Solution. First, estimate the result, as a check on our calculation (whether by hand or calculator). Rounding to the nearest one-digit numbers gives us 500×70, which is 35000. This provides a reality check on our use of a calculator.

Now let's do the calculation using the distributive property to simplify the problem. Expand the numbers explicitly in place value and then take the sum of all the products of one term from the first factor by a term from the second factor:

$$532 \times 69 = (500 + 30 + 2)(60 + 9) = (500 \cdot 60) + (500 \cdot 9) + (30 \cdot 60) + (30 \cdot 9) + (2 \cdot 60) + (2 \cdot 9) =$$

$$30000 + 4500 + 1800 + 270 = 120 + 18 = 36,708.$$

Figure 10 is a geometric illustration of the use of the distributive property in this problem.

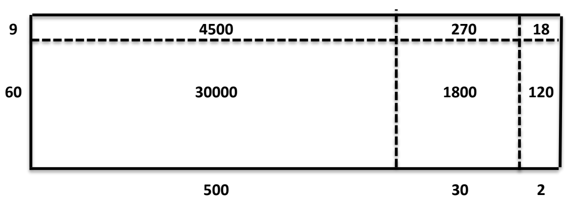


Figure 10

Of course, the point of the multiplication algorithm is to use place value to simplify this formula. Figure 11 provides the detail. In the top two rows we have inserted the two numbers to be multiplied, where the place value is indicated by the boxes. In row (1) we have copied the numbers in the top rectangles of Figure 10, and then put in the numbers from the bottom three rectangles in row (2). Row (3) is the sum of rows (1) and (2). In the remaining rows we illustrate, step by step, starting with the ones column, the process of shifting a 10 in one column to a 1 in the column just to the left. Ordinarily this level of detail is not made explicit, and the computation looks more like the table on the right.

			5	3	2
				6	9
(1)			45	27	18
(2)	30	18	12		
(3)	30	63	39	18	
(4)	30	63	40	8	
(5)	30	67	0	8	
(6)	36	7	0	8	

		5	3	2
			6	9
(1)	4	7	8	8
(2)	31	9	2	
(3)	36	7	0	8

Figure 11

Once again, the numbers appear, with the proper digits in their proper places. On row (1) of right image, we see the result of multiplying the first factor by the ones digit of the second factor. Reading from the right, $9 \times 2 = 18$, which appears as an 8 in the ones place. Next, $9 \times 3 = 27$, to which we have added the 1 group of ten traded from ten ones from the first calculation, and then moved the 2 of 28 over to the third column. Continuing in this way, we get row (2) on the right, and finally, row (3) is the sum of rows (1) and (2). Notice that we have not made explicit the digits carried over (as shown for addition in Figure 6). When students began these computations in grade 4, they at first needed to show the carried figure, but in grade 6, they should have achieved a level of automaticity for which that is not needed.

As mentioned earlier, we initially estimate the answer to check that our calculation is reasonable. This estimation is particularly desirable when using a calculator, so as to provide a ball-park figure for the desired result. To do that, we round to the nearest single digit number, as described at the beginning of this problem, getting $500 \times 70 = 35,000$. Since the calculator shows 36,708, we can have confidence that that calculation is correct. Notice that the leading terms multiply whereas the number of following zeroes add. So, to estimate 532×69, we multiply $5 \times 7 = 35$, and follow it by 3 zeroes (since the 5 is followed by two places, and the 6 by one).

Example 10. Evaluate: a) 39×7 b) 140×50 c) 63×48.

Solution. a). Start with an estimate of the leading figures: 39 is close to 40, which is 4 in the tens space. $4 \times 7 = 28$, so $40 \times 7 = 280$. But (thinking of multiplication as repeated addition), we've counted one "7" too many, so we subtract 7 to get $39 \times 7 = 280 - 7 = 273$. At this point it is important that the students see that they have not just done some cute tricks, but have employed the distributive property:

$$39 \times 7 = (40 - 1) \times 7 = 40 \times 7 - 1 \times 7 = 280 - 9 = 273.$$

b) Multiplying by 50 is the same as multiplying by 100 and dividing by 2, and multiplying by 100 is the same as appending two 0s on the right. So:

$$140 \times 50 = (140 \div 2) \times 100 = 70 \times 100 = 7000.$$

c) This problem is not as easy, for we have to round and then compensate in both factors. But still, with a piece of paper handy, we can draw Figure 12 to the right. We want the value of the area of the gray figure. First, we calculate the area of the whole figure: $60 \times 50 + 3 \times 50 = 3150$. Now we calculate the area of the figure to the right of the dashed line, and then subtract it from the preceding calculation: $63 \times 2 = 126$, and so the area of the gray figure is $3150 - 126 = 3024$. The last subtraction is also easy mentally: $150 - 125 = 25$, but since we had to subtract one more, we have $150 - 126 = 24$, giving the result.

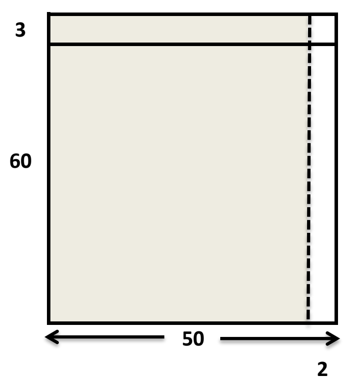


Figure 12

Example 11. Mentally estimate the following products:

a) 847×632 b) $6,017 \times 47$ c) $12 \times 51,000$ d) $143,000 \times 8$.

The word *estimate* should have some qualification as to how good it is to be. The qualifier could be "within two significant figures" or "up to the nearest hundreds (or thousands or thousandths)." Here we have no qualifiers, so we have to guess what is good enough. What we have done in the solution is to accept the best estimate that can be arrived at through mental arithmetic. At the minimum, we use the leading digit as the "significant figure" and then count the places to find the estimate.

Solution. a) The leading digits are 8 and 6, and they are each followed by 2 places. So a product estimate is 48 followed by 4 zeroes: 480,000. In detail, $4800 \times 600 = 8 \times 100 \times 6 \times 100 = 8 \times 6 \times 100 \times 100 = 480000$.We also know that this estimate is low, since we rounded down in both factors. The calculator gives 535,304, so we probably did the calculation correctly.

b) The best estimates for the leading digits are 6 and 5, since 47 is closer to 50 than 40. The 6 is followed by three places, and the 5 by one. So the product estimate is 30 followed by 4 zeroes: 300,000. The calculator gives 282,799. Note that the actual value is lower than the estimate. That is because we estimated 47 by the higher number 50. In general, when rounding one number up and the other number down, the estimate could be lower or higher than the actual value; what is important is that it is close.

c) To estimate this, we don't have to work at all: 10 is close to 12, and since "multiplication by 10" amounts to putting another zero at the end, we immediately get the estimate 510,000. The actual multiplication also isn't very hard:

$$12 \times 51,000 = [(10 + 2) \times 51] \times 1000 = [10 \times 51 + 2 \times 51] \times 1000 = [510 + 102] \times 1000 = 612,000.$$

d) Again, the estimate is easy: estimate 8 by 10, and conclude that the product is about, and less than 1,430,000. The actual product is 1,144,000.

We want to point out that, in multiplying two numbers on the calculator, it is easy to lose track of the number of places the result should have. So, if we multiply 113,000,000 by 704, when we punch in the first number, we may not punch in the right number of zeros. Estimating helps to check whether or not our input was correct. For this multiplication, the answer has to be close to 7 followed by 10 zero place values. So, just counting place values can help us decide whether or not we punched in the right number of zeroes. Alternatively, we can just multiply 113 by 704, and remember that the result has to be followed by 6 zeros. Here we get $113 \times 704 = 79,552$, so the exact answer is 79,552,000,000 (which agrees with our estimate of 7 followed by 10 place values).

Division as Repeated Subtraction

When we ask to find $A \div B$ for natural numbers A and B with $A \geq B$, we are asking: how many groups of size B can be formed from a group of size A? So, we subtract a group of size B, and if what is left over is not less than B, we subtract another group of size B. We continue in this manner, counting the groups of size B formed, until the final piece that is left over is less than B. The number of times we have subtracted B is called the *quotient*, denoted Q, and the piece that is left is the *remainder*, usually denoted by R. This process is embodied in the division theorem, which says:

Division Theorem. Given two whole numbers A and B, with $A \geq B$, there are whole numbers Q and R with $R < B$ such that $A = QB + R$. Q and R are uniquely determined by these conditions.

Figure 13 illustrates the theorem with $A = 128, B = 12$. The quotient Q is the largest number of pieces of size 12 that can be formed from A, and the remainder R is what is left over. So, in this case, $Q = 10$ and $R = 8$ because $128 = 10 \times 12 + 8$.

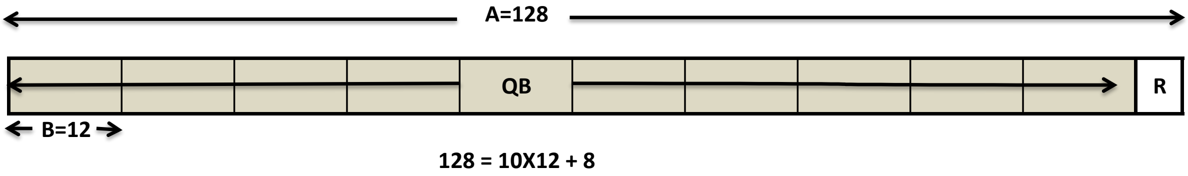


Figure 13

Example 12. Demonstrate the Division Theorem with $A = 35$ and

a) $B = 2$ b) $B = 5$ c) $B = 9$ d) $B = 11$ e) $B = 14$.

Solution.

- a) 2 divides 35 17 times with a remainder of 1. So $Q = 17$ and $R = 1$, and the statement of the division theorem is $35 = 17 \cdot 2 + 1$.
- b) 5 divides 35 7 times with no remainder. So, here we have $Q = 7, R = 0$, and $35 = 7 \cdot 5 + 0$.
- c) Here we may have to count the multiples of 9: $9, 18, 27, 36$. Since 35 is between the third number and the fourth, we conclude that $Q = 3, R = 8$, and $35 = 3 \cdot 9 + 8$.
- d) Same here: Three elevens (3×11) gives 33 which is 2 less than 35, so $35 = 3 \cdot 11 + 2$, and we have a quotient of 3 and a remainder of 2.
- e) The multiples of 14 are $14, 28, 42, \ldots$, so now $Q = 2$, and since $35 - 2 \cdot 14 = 7$, we have a quotient of 2 and a remainder of 7.

The operation called *the division algorithm* is a systematic technique to solve the problem: given A and B with $A \geq B$, find Q and R such that $A = QB + R$ with $R < B$. The purpose of Example 13 was to illustrate in simple contexts the way the division algorithm works. Basically it is this: make a *good* guess q for Q and let $R' = A - qB$. If $R' < B$, you're done. If not, apply the same technique to R', and continue the process until we end up with an R that is less than B.

Of course, the crux of this description is the phrase *good guess*; when do we know we've made a good guess? The answer is that that we guess one place value at a time. Starting with the top place values (a for A and b for the top one or two place values for B), we solve $a = qb + r$ and then calculate qB, subtract that from A to get $R = A - qB$. We've now found the top place value of the quotient, to proceed we now go through the same procedure with R replacing B. And so forth. This description is still too vague, but the best way to clarify it is to go through examples.

Example 13: $3547 \div 75 = ?$

Solution. Let's follow Figure 14 (next page). In a) we have set the problem up for long division. We concentrate on the highest place value of both divisor and dividend. Since 7 is not less than 3, we look at $35 \div 7 = 5$, so we first consider 5. We've written down that step in b), but now we can see that 5 is too big, because 70×5 is 350, which is just below 354, so 75×5 has to be too big.

Now, we try 4, and in c) we have multiplied 75 by 4 to get 300. Subtracting that from 354, we get a remainder of 54 (keep in mind that this is 54 in the tens place, so actually 540). We haven't accounted for the 7 ones so we bring them down, giving a (temporary) remainder of 547. Since that is (much) bigger than 75 we continue in the same way. $7 \times 7 = 49$, so that is a good guess, since 527 is bigger than 490 by about 30 - less than 75. So, in step d) we put 7 in the tens slot, calculate $75 \times 7 = 525$, which works, since the remainder, 22, is now less than 75. In the form of the division theorem:

$$3547 = 47 \times 75 + 22 .$$

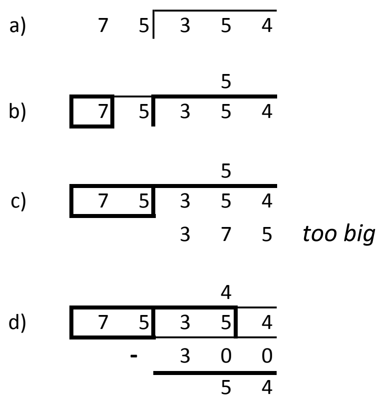


Figure 14

Example 14. Hercule's class has 23 students. His class is going to make a field trip to the state capitol which will cost \$32 per student. The class had a fundraiser that produced \$654. Is that enough? If not, how much more has to be raised?

Solution. For a quick calculation, multiply 23 by 30 using the distributive property: $23 \times 30 = (20 \times 3 + 3 \times 3) \times 10 = 690$, which is more than the amount Hercule's class has, and less than 23×32, the amount needed. To find the amount needed, since we know that $23 \times 30 = 690$, we need to just add $23 \times 2 = 46$ to 690, giving us 736.

So the trip will cost \$736. The class needs an additional $736 - 654 = 36 + 46 = 82$ dollars for the excursion for the whole class.

Since we are talking about long division, let's solve this as a division problem. We can phrase the problem in these two ways:

a) How much do we have available per student?

b) For how many students can we pay the full price with the current funds?

Then we calculate the shortfall.

a) There is \$654 available, and we have 23 students, so we have calculate $654 \div 23$. Let's complete the division, following along in Figure 15 on the next page.

. In line a) we've written the problem in computational form. In line b) we've found that 3 as the first place value of the quotient is too big. So we try 2, and the first three lines show that $2 \times 23 = 46$, which works fine. This appears on line 2 of c); the remainder ($654 - 2 \times 46 \times 10 = 194$) we get is in the next line. Now repeat this reasoning for $194 \div 23$, to find a quotient of 8 and a remainder of 10. That's less than 23, so we are done, the quotient is $2 \times 10 + 8 = 28$. We have $28 per student, with $10 left over. That means that we need to raise another 4×23 less 10, which is $82.

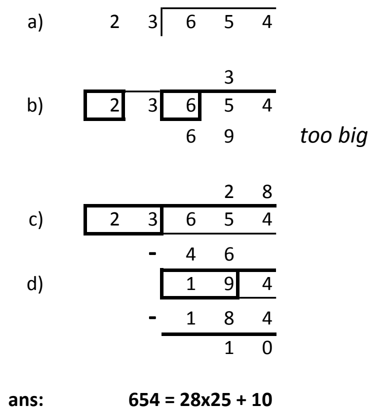


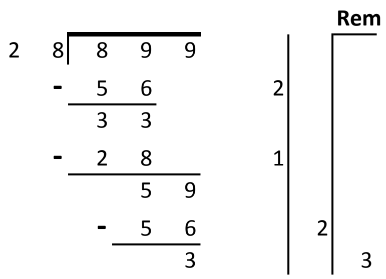
Figure 15 : $654 = 28 \times 23 + 10$

b) To find out how many students $654 will pay for, where the price per student is $32, we calculate $654 \div 32$ using the very same algorithm. We get $654 = 20 \times 32 + 14$. So with the available funds, we can send 20 students, and have $14 left over. Thus, to send all 23 students we need an additional $3 \times 32 - 14 = 82$ dollars.

To summarize: the idea behind the division algorithm is to find the best quotient for the leading place values of the divisor and dividend, then take the product of this best quotient with the divisor, subtract from the dividend and repeat. But why work so hard to find the *best guess*? Why not try any guess, so long as it gives us a product (with the divisor) that is less than the dividend. In this case, we have to keep track of place value. To illustrate:

Example 15. What is $899 \div 28$?

SOLUTION. On the right we first indicate the intended division, and to the right of that we've drawn two vertical lines, separating the tens place, the one place and remainder (a quick estimate, like $900 \div 30$ tells us to anticipate a two place quotient). Now look at $89 \div 28$, and guess a quotient of 2. Put 2 in the tens place to the right; then multiply $28 \times 2 = 56$ and put 56 under the 89 (because the 9 is in the tens place value). Subtract to get 33. That is still larger than 23 (but not by much), so assume a quotient of 1 and repeat the process. Notice, we are still in the tens place, so that is where we put the 1, just beneath the 2. Subtract 28 from 33 to get 5 (in the tens place), and then bring down the 9 ones, so that now we have the number (59) yet to be divided by 28. From what we know, that gives a quotient of 2 and a remainder of 3. That is recorded in lines 6 and 7. The right diagram tells us that we are done: $899 \div 28$ produces a quotient of 32 with a remainder of 3 We already know 28×2, so we use that guess. But now we are in the ones column, so we put the 2 to the right of the first line. Subtract 56 from 59 to get the remainder 3, which is placed to the right of the second vertical bar. Now add along the answer columns to get 32 with a remainder of 3 (lines 5-7). That is $899 \div 28 = 32$ with a remainder of 3.



In this example we keep track of the development of the quotient oon the side because we are collecting addends as we move along. The bars are used to separate place values; it is more common to use zeros to indicate place value, simply beccause it makes the process more transparent.

Example 16. Let's try something harder: divide 72381 by 114.

SOLUTION. In line a) we've written the division to be performed. Now, pick a multiple of 114 that is less than 732: let's try 3 . $3 \times 114 = 342$, so in the next line, b), we write 34200 since $300 \times 114 = 34200$ and record the 300 to the right, for that is, so far, our quotient.

a)	114	7 3 2 8 1	
b)		- 3 4 2 0 0	3 0 0
c)		3 9 0 8 1	
d)		- 3 4 2 0 0	3 0 0
e)		4 8 8 1	
f)		- 3 4 2 0	3 0
g)		1 4 6 1	
h)		- 1 4 0	1 0
i)		3 2 1	
j)		- 2 2 8	2
		0 9 3	6 4 2

73281 = 642x114 + 93

c) Now subtract line b) from line a) to get the first remainder of 39081.
d) From what we've done so far, it is clear that we want to put in another 34200, which is 300×114. We see we have another summand of 300 to our developing quotient.
e) Subtracting line d) from line c) we get a new remainder of 4881.
f) Repeating the process, from what we've done we again want to use $3 \times 114 = 342$, but in a place to the left (that is, $30 \times 114 = 3420$). Put 3420 under 4881, and put a 30 to the right of that (the last 0 is put in because now there is only one place needing a value).
g) This is the new remainder, the difference between the two lines.
h) 114 is the biggest multiple of 114 that is less than 146, so we write in 1140, with a 10 to its right, signifying the new addend we have for the quotient.
i) This is the difference between the preceding two lines, and j) is 2×114, the multiple of 114 less than 321. To the right we put a 2: the last addend of the quotient, for the new remainder (last line, 93) is less than 114. To the right of this last remainder we put the sum of all the quotient addends: 642, and conclude that $73281 = 642 \times 114 + 93$.

Divisibility

There are some observations to be made that make finding digit factors of whole numbers relatively easy. For example, we know that any number, no matter how many places there are, is even if the digit in the one's place (10^0 position) is even; that is, one of $0, 2, 4, 6, 8$. To see why this is, we return to place value.

Let's start with an example: $2,458$. This is an even number because 8 is even. But why? Because

$$2,458 = 245 \times 10 + 8.$$

Now since 10 is even, the first term is divisible by 2, and since 8 is also even, the sum, $2,458$ is even. In fact:

$$2,458 \div 2 = 245 \times (10 \div 2) + 8 \div 2 = 245 \times 5 + 4 = 1225 + 4 = 1229.$$

This argument carries forth for any number no matter how many digits:

$$3,578,647 = 357,864 \times 10 + 7$$

is odd because the first term is even, but the second, 7, is odd - so the sum is odd.

This same trick works for divisilbilty by 5. Writing $2,458 = 245 \times 10 + 8$, since 10 is divisible by 5, the first summand on the right hand side is divisible by 5, but the second, 8, is not, so the sum is not divisible by 5. In fact, since the only digits divisible by 5 are 0 and 5, we can conclude that a whole number is divisible by 5 precisely when it ends in a 0 or a 5.

Similarly, a whole number is divisible by 10 precisely when it ends in a zero (in fact this follows directly from the concept of place value).

Since 1, 2, 5 and 10 are the only factors of 10, this argument will not work for any other digit. However, a slight generalization works for 4: Since 100 is divisible by 4, we need to look only at the last two digits of a number to see if it is divisible by 4:

$$2,358 = 23 \times 100 + 58.$$

Since the first term is divisible by 4, we need only ask if 58 is divisible by 4. It isn't, so 4 is not a factor of 2,358 (but is of 2,348, since $48 \div 4 = 12$).

A similar argument works for divisibility by 8: Since 8 divides 1000, we have only to look at the the last three digits. As an example, let's take 46,348. Since the last digit is 8, this is divisible by 2. Now we look at $46348 \div 2 = 23174$. Since the last digit is even, we know that 23174 is divisible by 2, so $46348 = 23174 \times 2$. But now, since 74 is not divisible by 4, the evenness ends here, and 46248 is not divisible by 8. Or we could have continued the division by 2: $46348 = 23174 \times 2 = 11587 \times 4$, and 11587 is *not* divisible by 2. Another way to look at it is this: $8 = 4 \times 2$, so if a whole number fails the 4-test, it isn't divisible by 8. If it passes the 4-test and the quotient ends up in a 0.2,4,6,8, then it is divisible by 8; otherwise not.

For 3 and 9 there is a neat little trick. Note that

$$10 = 9 + 1; \ 100 = 99 + 11; \ 1,000 = 999 + 11; \ 10,000 = 9,999 + 1 \text{ , etc.}$$

Every positive power of 10 is the sum of a number whose only digit is 9: a 1 followed by k 0's is equal to k 9's plus 1. A number whose only digit is 9 is divisible by both 3 and 9. It follows that, for any number, if the sum of its digits is divisible by 3 or 9, the number is divisible by 3 or 9, respectively. Let's illustrate:

$$3,567 = 3\times10^3+5\times10^2+6\times10+7 = 3\times999+3+5\times99+5+6\times9+6+7 = \text{(something divisible by 3) plus } (3+5+6+7) .$$

Since $3 + 5 + 6 + 7 = 21$, we can conclude that 3,567 is divisible by 3. But 21 is not divisible by 9, so 3,567 is not divisible by 9.

The remaining divisibility test is for the digit 7. We could examine place value very carefully and come up with a protocol for determining divisibility by 7, but the protocol is far more difficult to execute than actual division. So, we do not state a rule.

In summary,

Tests for Divisibility of Whole Numbers

a) Divisibility by 2: The number ends in one of 0, 2, 4, 6, 8.

b) Divisibility by 3: The sum of the digits of the number is divisible by 3.

c) Divisibility by 4: The number given by last two digits is divisible by 4.

d) Divisibility by 5: The last digit of the number is 0 or 5.

e) Divisibility by 6: The number is even and the sum of its digits is divisible by 3.

f) Divisibility by 7: No trick beats the division algorithm.

g) Divisibility by 8: The number given by last three digits is divisible by 8.

h) Divisibility by 9: The sum of the digits of the number is divisible by 9.

i) Divisibility by any power of two: Divide number by 2 consecutivelly until the last digit is odd. If you make k successive divisions the number is divisible by 2^k.

Example 17. Test each of the following numbers for divisibility by a positive digit.

a) 3,467,002 b) 77,760 c) 33,000,333 d) 40,446.

Solution. a) The last digit is 2, so the number is divisible by 2 but not by 5 or 10. Divide again by 2: the last digit is odd. Therefore, this number is divisible by 2, but not by 4 or 8. The sum of digits is 22, so the number is not divisible by 3 or 9. Being not divisible by 3, it is not divisible by 6. Finally, divide by 7 and get a whole number. Thus the digits that divide 3,467,002 are 2 and 7.

b) Since last digit is a zero, 77760 is divisible by 10, and therefore, also divisible by 2 and 5. The last two digits are 60, which are divisible by 4, leaving a quotient of 19420 (which is even so 77760 is also divisible by 8). The sum of the digits is 27 so the number 77760 is divisible by 3 and 9. Because it is divisible by both 2 and 3, 77760 is divisible by 6. Now the first three digits are divisible by 7 (that is 77700 is divisible by 7), but 60 is not, so our number does not have 7 as a factor. In summary, the basic divisors of 77760 are 2, 3, 4, 5, 6, 8, 9, and 10.

c) This number ends in 3, so is not divisible by 2, 5 or 10, and by extension, by none of 4, 6 or 8. The sum of the digits is 15, so it is divisible by 3 (but *not* 9), giving a quotient of 11,000,111 which (after trying to divide by 7) has no digit factors other than 3.

d) This number ends in a 6, so is divisible by 2, but not by 5. The quotient by 2 ends in a three, so 40,446 is not divisible by 4 or 8. The sum of the digits is 18, so our number is also divisible by 3 and 9. Being divisible by 2 and 3, it is also divisible by 6. By calculation, the number is not divisible by 7. In summary, the digit divisors are 2, 3, 6 and 9.

Example 18. You are the senior member of a team of 9 realtors in an agricultural realty agency. When a sale is executed, all partners receive equal shares, and if there is a remainder, the remainder goes to you as senior partner. In the month of October last year, the following sales were made:

a) 14,017 acres b) 77,760 acres c)11,010 acres d) 40,100 acres.

in the month of October, how many more acres did you accrue than did your partners?

Solution. a) The sum the digits is 13. This is 9+4, so when the equal allotments are done, there are 4 acres left over: these are yours. b) The sum of the digits is 27, divisible by 9, so you receive no bonus. c) The sum of the digits is 3, so the number is not divisible by 9. The highest number less that 11,010 that is divisible by 9 is 11, 007. Thus you receive 3 acres more than do your partners. d) By the same reasoning, 40,095 acres can be evenly divided among the partners, so the senior member receives an additional 5 acres. In total, the senior partner has received 4 + 0 + 3 + 5 = 12 acres more than the other partners.

Section 2. Factors and Multiples of Whole Numbers

Students first started learning about factoring whole numbers in third grade. Then, in fourth grade students understand that factors come in pairs and can find all factor pairs of numbers up to 100. The use of factor pairs shows students that to find all factors of a number less than 100, they need only actually divide by single digits. In sixth grade, we develop this idea and strive for fluency in finding factors for all numbers up to 144, and multiples of all number up to 12.

Equally important is that students in grade 6 begin to understand the difference between finding addends or finding factors of a whole number. For example, students know that $12 = 10 + 2$ and $12 = 6 \times 2$, and that this is useful information for mental calculation. For example, to calculate 12×15, here are some options:

$$12 \times 15 = (10 + 2) \times 15 = 10 \times 15 + 2 \times 15 = 150 + 30 = 180, \text{ or}$$

$$12 \times 15 = (6 \times 2) \times 15 = 6 \times (2 \times 15) = 6 \times 30 = 180.$$

However, to calculate $2/12$, we are looking for common factors, so use of the expression $12 = 6 \times 2$ leads to the simplification of $2/12$ as $1/6$.

Developing this intuition at this time leads ultimately to greater success in grade 10 in factoring quadratics like: $x^2 - 8x + 12$. Finding two numbers whose product is 12 and whose sum is 8 can be better understood when the difference between addends and factors is clear.

6.NS.4. Find the greatest common factor of two whole numbers less than or equal to 100 and the least common multiple of two whole numbers less than or equal to 12. Use the distributive property to express a sum of two whole numbers 1 to 100 with a common factor as a multiple of a sum of two whole numbers with no common factor. For example, express $36 + 8$ *as* $4(9 + 2)$.

This standard shows us that in grade 6 students see that *factor* and *multiple* enjoy a dual relationship (as in *3 is a factor of 18* and *18 is a multiple of 3*).

When we see a multiplication of whole numbers, such as $a \cdot b = c$ we say that a and b are a *factor pair* for c. We say that a and b are *factors* of c. Note that since $1 \cdot c = c$, 1 is a factor of every whole number, and any whole number is a factor of itself. In fact, 1 and c are always a factor pair for c. To test if a is a factor of c, we divide c by a. If the remainder is zero, a is a factor of c, as is the quotient $b = c/a$, and a and b are a factor pair.

Example 19: How many rectangles are there with whole number side lengths and area 24 square feet.

Solution. The question is: in how many ways can we write $24 = A \times B$, with A and B whole numbers. To solve this, start with $A = 1$ (and thus $B = 24$), and then divide 24 by successive whole numbers until we get them all. The answer is

$$1 \times 24, \qquad 2 \times 12, \qquad 3 \times 8, \qquad 4 \times 6.$$

The next whole number is 5, which we know does not divide 24 because it ends in a 4. Next is 6, which is a factor of 24 but, since $6 \times 4 = 4 \times 6$, it represents the same rectangle as 4×6. The commutativity property tells us that, as we go through testing numbers increasingly, one we've repeated a factorization, all further attempts simply reproduce preceding factor pairs.

A *prime number* is a whole number greater than 1 that has only two distinct factors, 1 and itself. Otherwise said, a number N is prime if it has only the factor pair, $1, N$. A number which is not prime is said to be *composite*.

Example 20. Of the following numbers, identify the primes:

a) 12 b) 17 c) 19 d) 21 e) 23 f) 57 g) 93 h) 97.

Solution. All numbers in this list are less than 100. So, if any of them are composite, they have to have a factor strictly between 1 and 10. This means that we can use our tests for divisibility by digits.

a) 12 is composite, because it is even: divisible by 2.

b) and c) both 17 and 19 fail all our divisibility tests, so are prime.

d) the sum of the digits of 21 is 3, so 21 is composite.

e) 23 fails all our divisibility tests, so is prime.

f) and g) For both 57 and 93, the sum of the digits is divisible by 3, so they are composite (divisible by 3).

h) 97 fails all our digit divisibility tests, so is prime.

Example 21. Find all prime numbers less than 100.

Solution. Using our divisibility tests, the number cannot end in a 2,4,5,6,8 or 0, and the sum of the digits must not be divisible by 3. This leaves 28 numbers. Applying the "divisibility by 7" test, eliminates 32 of those numbers (49 and 77). The remaining numbers must be prime. The are:

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 91, 97.$$

Example 22: Find all the factors of 84.

Solution. To simplify the problem, we search for factor pairs. First, 1 and 84 form a factor pair. Since 84 is even, we get the factor pair 2 and 42. Now 84 satisfies the divisibility by 3 test, giving us the factor pair 3 and 28. We now divide 84 by the digits up to 7; each time we get a remainder of 0, we get a factor pair. Here is the list of factor pairs:

1	2	3	4	6	7	12	...
84	42	28	21	14	12	7	...

We have ended at the pair 12,7 because that is the same as the preceding pair, but in reverse order. Furthermore, continuing would only be the same as reading the list of pairs backwards and in reverse order. Thus, 84 has these 12 factors:

1   2   3   4   6   7   12   14   21   28   42   84

Example 23: Find all the factors of 132.

Solution. We proceed in the same way, stopping once we've reached the turning point:

1	2	3	4	6	11
132	66	44	33	22	12

and stop there, because the next factor pair (12,11) just repeats the last pair in this list. So, the factors of 132 are

1   2   3   4   6   11   12   22   33   44   66   132

It is a fact that any number can be written as a product of primes (allowing repetitions). It is easy to see why this is so: given a number N, either it is prime, or it can be written as a product of two numbers, $N = A \times B$, neither of which is 1. If A and B are prime, we are done. If not, we can factor one or both of them, and then ask if any of these factors is not prime. Eventually the procedure ends.

Looking back at the examples above: the prime factors of 84 are $2, 3, 7$, and indeed $84 = 2 \cdot 2 \cdot 3 \cdot 7$. The prime factors of 132 are 2, 3, 11 and $132 = 2 \cdot 2 \cdot 3 \cdot 11$.

This leads to another way of finding all factors of a given number: find the prime factorization. Then any factor is a product of some of those primes. Let's repeat Example 22, using this method.

Example 24. Once again, find the prime factorization of 84.

Solution. 84 is even, so $84 = 2{\cdot}42$. Next, 42 is even, so $84 = 2{\cdot}2{\cdot}21$. 21 has the factor pair 3,7, so $84 = 2{\cdot}2{\cdot}3{\cdot}7$, and we are done.

Common Factors and Multiples

In this subsection we use both common symbols for multiplication: "3 times 7" can be written as "3×7" or "$3 \cdot 7$." We do this as preparation for algebraic notation, where the preferred symbol is "$\cdot$," simply because, the symbol $\times$ can be mistaken as the variable x.

Example 25. Find all numbers that are factors of *both* 84 and 132.

SOLUTION. We can just look at the lists in Examples 23 and 24 and see that the list of factors of both 84 and 132 is $\{1, 2, 3, 4, 6, 12\}$. Note that the greatest common factor of 84 and 132 is 12.

Example 26. Find the common factors of 75 and 100.

SOLUTION. Both numbers end in a 5 or 0, so 5 is a factor of both: $75 = 5 \cdot 15, 100 = 5 \cdot 20$. Now let's find the common factors of 15 and 20: again 5 divides both, so $15 = 5 \cdot 3, 20 = 5 \cdot 4$. Thus $5 \times 5 = 25$ is also a common factor of 75 and 100. There are no further common factors, since 3 and 4 have no common factors. Thus the factors common to 75 and 100 are 1, 5 and 25, and 25 is their greatest common factor. Finally, we've also found the prime factorization of both numbers: $75 = 3 \cdot 5 \cdot 5, 100 = 2 \cdot 2 \cdot 5 \cdot 5$.

The *greatest common factor, GCF*, of two numbers (that is, that common factor that is the largest) plays an important role in much of arithmetic. For example, if we want to divide two numbers A and B and happen to know the GCF, then we can divide out the GCF and look at the quotient of the other factors.

Example 27. a) $100 \div 75 = ?$ b) $132 \div 84 = ?$

SOLUTION. a) This is easy: the GCF of 100 and 75 is 25, so we can write $100 = 75+25$, so $100 \div 75$ has a quotient of 1 and a remainder of 25.

b) Similarly, from Examples 23 and 24, we know that the GCF of 132 and 84 is 12, we know that $132 \div 84 = 11 \div 7$. Now, $11 = 1 \cdot 7 + 4$; multiplying by 12, we get $132 = 1 \cdot 84 + 48$.

At the other side of finding common factors of two numbers is that of finding common multiples of two numbers. Of course, there are infinitely many: if C is a common multiple of A and B, then so are $2 \cdot C, 3 \cdot C, 4 \cdot C, \dots$. But there will be a *least common multiple* and that is designated the LCM.

Example 28. What is the LCM of 75 and 100?

SOLUTION. One way is to take one of the numbers and find the first multiple of that number that is divisible by the other. So, look at 75, 150, 225, 300, Since 300 is the first multiple of 75 that is a multiple of 100, it is the LCM.

Alternatively, we can first find the prime factorization of the two numbers:

$$75 = 5 \times 5 \times 3, \qquad 100 = 2 \times 2 \times 5 \times 5.$$

Now, since 75 is the smaller number, the LCM has to have $5 \times 5 \times 3$ in it. To be divisible by 100 we also need the factor 2×2, so the LCM is $2 \times 2 \times 5 \times 5 \times 3 = 300$.

Example 29. What are the GCF and LCM of a) 84 and 132; 168 and 132?

SOLUTION. a) We have the prime factorizations

$$84 = 2 \cdot 2 \cdot 3 \cdot 7\ , \quad 132 = 2 \cdot 2 \cdot 3 \cdot 11.$$

Once we have the prime factorization of two numbers, the GCF is the product of all the prime factors common to both numbers. The common factors are $2 \cdot 2 \cdot 3$, sothe GCF is 12. The LCM is the smallest number that contains all the prime factors of both 84 and 132 and that is: $2 \cdot 2 \cdot 3 \cdot 7 \cdot 11 = 924$.

It should be noted that some care should be taken when we have a prime as a multiple factor of both numbers. In this case, the GCF takes the smaller of the multiples, and the LCM takes the larger of the multiples. In the above case 2 is a double factor of both, 3 is a single factor of both, and 7 and 11 are neither factors of both. So the GCD gets as factors, two 2s and one 3, and the LCM gets two 2s, a 3, a 7 and an 11.

b) The prime factorizations for168 abd 132 are:

$$168 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 7\ , \quad 132 = 2 \cdot 2 \cdot 3 \cdot 11.$$

In this case, the prime factors of the GCD are: two 2s and a 3, so the GCD is 12. The prime factors of the LCM consist of: three 2s, one 3, one 7 and one 11, giving us 1848.

It is a fact that the product of two whole numbers is equal to the product of the GCD and the LCM.

> For two whole numbers A and B, if the GCD is D and the LCM is M, this equality holds:
>
> $$A \cdot B = D \cdot M.$$

Let's look at this in the last computation:

$$168 \cdot 132 = (2 \cdot 2 \cdot 2 \cdot 3 \cdot 7) \cdot (2 \cdot 2 \cdot 3 \cdot 11),$$

and the product of the GCD and LCM is

$$12 \cdot 1848 = (2 \cdot 2 \cdot 3) \cdot (2 \cdot 2 \cdot 2 \cdot 3 \cdot 7 \cdot 11).$$

Both expressions have as factors: five 2s, one 3, one 7, one 11, so they compute to the same number. In the following problem it could be worthwhile to confirm this in each case.

Example 30. Find the GCF and LCM of the pair of numbers:

a) 80, 100   b) 16, 132   c) 24, 72   d) 42, 72

Solution. We will try to find the simplest way to the answers, using the prime factorization when it is easiest.

a) Find the GCF: look for the largest factor of 80 that is a factor of 100. 80 and 40 fail, but 20 works. So GCD(80,100) = 20.

Find the LCM: The first multiple of 100 that is divisible by 80 is 400, so LCM(80,100) = 400.

b) Look at prime factors: $16 = 2 \cdot 2 \cdot 2 \cdot 2$, $132 = 2 \cdot 2 \cdot 3 \cdot 11$. The GCF is the product of common factors, and the only common prime factor is 2. 16 has four of them, and 132 has two of them, so the GCD is 4. The LCM has to be a multiple of 16, so has to have four 2 factors, and also a multiple of 132, so has to have factors 3 and 11. The LCM this is $2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 11 = 528$.

c) $24 = 3 \cdot 8$, $72 = 3 \cdot 3 \cdot 8$, so 24 divides 72 and this is the GCD. It also tells us that 72 is the LCM.

d) It is easy to see that 6 divides both numbers: $42 = 6 \cdot 7$, $72 = 6 \cdot 12$. Since 7 and 12 have no common divisors, the GCD is 6. Thus the LCM is $6 \cdot 7 \cdot 12 = 504$.

The subject of LCM leads to easy ways to add fractions. Let's review what is involved in adding fractions. When we add two whole numbers, we do so, implicitly understanding that we are adding like terms. 3 + 4 means that

3 dogs plus 4 dogs gives 7 dogs, and for "dogs" we can put in *any* noun, so long as we do so for both addends. We can't add 3 dogs to 4 laws of physics. So, when it comes to fractions, the expression $1/c$, where c is a whole number means the cth part of the unit. We can add fractions of the form a/c and b/c by adding the numerators:

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}.$$

This is because we read this as a repetitions of the cth part of the unit plus b repetitions of the cth part of the unit, giving us $a + b$ repetitions of the cth part of the unit.

But what about the addition of fractions with different denominators? Each fraction represents a length on the number line, and the sum of two lengths is the length of the interval obtained by appending the two lengths. If these lengths are given arithmetically by fractions, how do we provide an arithmetic answer? The answer to this is that we can replace the fractions with fractions with the same denominator. Once this replacement is made, we can just add the numerators.

To make this clear, we return to the representation of a fraction p/q (with p and q whole numbers with $q \neq 0$) as a point on the line: First we subdivide the unit length into q segments of the same length, and associate the right endpoint of the first segment with the fraction $1/q$. Then p/q is the sum of p copies of this length. Now, we note that if we replace both p and q with np and nq, for any whole number n, then we arrive at the same point (as 4/6 brings us to the same point as 2/3). So, fractions can be different and still represent the same point of the number line. In this case, we say that the fractions are equivalent.

Two fractions $\frac{a}{b}$ and $\frac{c}{d}$ are *equivalent* if they represent the same point on the number line.

The fact is that we can always replace any two fractions by equivalent fractions with the same denominator. We have just observed that for whole numbers a, b, n with b and n nonzero:

$$\frac{a}{b} \quad \text{is equivalent to} \quad \frac{na}{nb}.$$

We will develop a criterion for equivalence, but first let's look at an illustrative example.

Example 31. Of these pairs of fractions determine whether or not they are equivalent

$$\text{a) } \frac{2}{8}, \text{ and } \frac{6}{24}, \qquad \text{b) } \frac{2}{8} \text{ and } \frac{3}{12}, \qquad \text{c) } \frac{2}{8} \text{ and } \frac{7}{12}.$$

SOLUTION. a) Since $6 = 2 \cdot 3$,, we check $8 \cdot 3$. Since that is 24, we have shown equivalence:

$$\frac{2}{8} = \frac{2 \cdot 3}{8 \cdot 3} = \frac{6}{24}.$$

We could also have factored a 2 out of the numerator and denominator of the first fraction, and a 6 out of each part of the second fraction:

$$\frac{2}{8} = \frac{1 \cdot 2}{4 \cdot 2} = \frac{1}{4} \quad \text{and} \quad \frac{6}{24} = \frac{1 \cdot 6}{4 \cdot 6} = \frac{1}{4},$$

so the two fractions are equivalent, for the are both equivalent to $\frac{1}{4}$.

b) Similarly, factoring a 2 out of numerator and denominator of the first fraction, and 3 out of the second, once again we get that both are equivalent to $\frac{1}{4}$ We also could have multiplied the first by $3/3$ and the second by $2/2$ to find that both are equivalent to $\frac{6}{24}$.

c) Following these ideas, we can change both fractions to equivalent fractions with the same denominator:

$$\frac{2}{8} = \frac{2 \cdot 15}{8 \cdot 15} = \frac{30}{120}, \quad \frac{7}{30} = \frac{7 \cdot 4}{30 \cdot 4} = \frac{28}{120},$$

so the fractions are not equivalent.

In general, we can always replace the denominators by a common multiple, multiply both numerator and denominator of both fractions by the appropriate multiplier, and then, if the numerators are the same, the fractions are equivalent, and if the numerators are not the same, they are inequivalent. We thus have these criteria for equivalence:

> Either of the following two statements is a criterion for the equivalence of two fractions $\frac{a}{b}$ and $\frac{c}{d}$.
>
> a) There are whole numbers m and n such that $bm = dn$.Then if $am = cn$, the fractions are equivalent, otherwise, they are not.
>
> b) If $ad = bc$ the fractions are equivalent, otherwise they are not.

This discussion suggests to us that, to add a/b and c/d we should find a common multiple M of b and d, and represent both fractions as whole number copies of $1/M$. Then we just addd the numeratos. Let us illustrate:

Example 32. $\frac{5}{6} + \frac{2}{9} = ?$

Solution.

Figure 16 is a graphic of the solution. Each of the first four columns is one unit in length. In column **a**, we have divided the unit into sixths in order to exhibit the number 5/6 in gray. In column **b**, we have divided the unit into ninths in order to exhibit the number 2/9 in gray. We cannot add the gray boxes for they are of different sizes. However, we can subdivide these images further so that the gray zones are split up into boxes of the same size. If we divide each box in column a into three boxes of the same size, and each box in column b into two boxes of the same size, then the we can represent the summands as multiples of one particular box. This is shown in columns **c** and **d**, where the first comprises 15 boxes and the second comprises 4 boxes, and the size of each box is 1/18 of the unit. Now we can add the gray boxes and get the answer e: nineteen eighteenths (that is 19/18). This is the algebraic formulation of Figure 16:

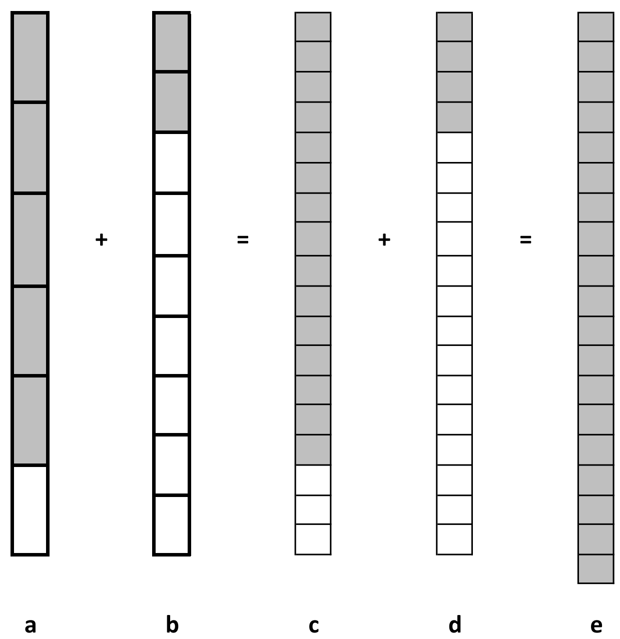


Figure 16

$$\frac{5}{6} + \frac{2}{9} = \frac{15}{18} + \frac{4}{18} = \frac{19}{18}.$$

These procedures are important for student understanding, and it is a good idea to bring up the addition of fractions whenever possible throughout the year. A fraction a/b is talking about a copies of the bth part of the unit, and we can only add two fractions by making the denominators the same (same sized pieces) and then adding the corresponding numerators.

Example 33. Add the fractions:

a) $\frac{1}{3}, \frac{1}{9}$ b) $\frac{3}{4}, \frac{7}{10}$ c) $\frac{7}{6}, \frac{8}{9}$ d) $\frac{11}{42}, \frac{31}{72}$

Solution. In each of these problems, we want to rewrite the fraction so as to have the same denominator. So, for a), since 1/3 and 3/9 represent the same number, and 3/9 and 1/9 have the same denominator we add as follows:

a)
$$\frac{1}{3} + \frac{1}{9} = \frac{3}{9} + \frac{1}{9} = \frac{4}{9}$$

b) Look for a common denominator: that is the same as looking for a common multiple of 4 and 10. Clearly the product, 40, will do. So
$$\frac{3}{4} + \frac{7}{10} = \frac{30}{40} + \frac{28}{40} = \frac{58}{40} .$$

Even though this is not in lowest term, the expression tells us a lot; for example, that $\frac{3}{4} + \frac{7}{10}$ is almost $1\frac{1}{2}$. If we were a little more observant to begin with, we would have noted that the denominators have 20 as a common multiple, and thus could calculate this way:

$$\frac{3}{4} + \frac{7}{10} = \frac{15}{20} + \frac{14}{20} = \frac{29}{20},$$

which gives the sum in lowest terms.

c) Addition of fractions suggest finding the LCM of the denominators; but sometimes it is easier to just take the product. In this case, the product of the denominators is 54, but the least common multiple is 18. Let's compare the two computations:
$$\frac{7}{6} + \frac{8}{9} = \frac{63}{54} + \frac{48}{54} = \frac{63+48}{54} = \frac{111}{54} = \frac{37}{18} ;$$
$$\frac{7}{6} + \frac{8}{9} = \frac{21}{18} + \frac{16}{18} = \frac{21+16}{54} = \frac{37}{18} ;$$

d) We will want to appeal to a calculator for problems like this one. First, let's do an easy mental calculation to get a reasonable approximation We can approximate these fractions by the quotients of the digits in the tens place: $1/4, 3/7$, and then note that $3/7$ is just a little less than $1/2$. So, using the approximations $1/4, 1/2$, we see that our result should be close to $1/4 + 12 = 3/4$.

Now we go to the actual computation. First we find the LCM of the denominators. We have $42 = 6 \cdot 7$, and $72 = 6 \cdot 12$, so these fractions can be represented as fractions with a denominator of $7 \cdot 6 \cdot 12 = 504$. We get

$$\frac{11}{42} = \frac{132}{504}, \quad \frac{31}{72} = \frac{217}{504},$$

so we get the sum by adding the numerators: $132 + 217 = 349$, and the answer is $349/504$. Using a calculator, we can represent that by the decimal $0.692\ldots$, which is close to our estimate $3/4 = 0.75$.

Note: When using the calculator, one could also convert to decimals. However, almost always such a conversion involves an approximation (only fractions with a denominator ending a 5 or a 0 are terminating), and the error in such approximations grows with each calculation. For example, if we want accuracy to three decimal places, we can calculate $11/42 = 0.262, 31/72 = 0.431$ and add: $0.262 + 0.431 = 0.693$ which works fine. But when we need a greater degree of accuracy, or there are many computations to be made, it might not work out. On the other hand, the result when calculating with the fractions and using the calculator only on whole number operations (as we have done above) *always* provides a precisely accurate result.

When performing arithmetic operations, it is almost always the case that the knowledge and use of the arithmetic rules are involved, and often lead to simplifications in procedure when their use is recognized. Let us illustrate:

When performing certain arithmetic computations, it often makes problems easier to be cognizant of factoring and arithmetic rules, particularly when working together. We will demonstrate how this works in a series of examples.

Example 34. a) $84 \div 36 = ?$ b) $24 \times 25 =?$ c) $\frac{28}{5} \times \frac{25}{7} =?$ d) $\frac{21}{4} \div \frac{24}{5} =?$.

Solution. a) It should by now be standard to divide out common factors, but we include this here to illustrate the interaction of factoring and arithmetic rules.

$$84 \div 36 = \frac{84}{36} = \frac{12 \times 7}{12 \times 3} = \frac{7}{3},$$

because the common factor 12 divides both numbers.

b) There are many possible solutions. In this particular method, we use the commutativity of multiplication:

$$24 \times 25 = (2 \times 12) \times (5 \times 5) = (2 \times 5) \times (12 \times 5) = 10 \times 60 = 600.$$

c)

$$\frac{28}{5} \times \frac{25}{7} = \frac{28 \times 25}{7 \times 5} = \frac{28}{7} \times \frac{25}{5} = 4 \times 5 = 20.$$

d)

$$\frac{21}{4} \div \frac{24}{5} = \frac{21}{4} \times \frac{5}{24} = \frac{21 \times 5}{4 \times 24} = \frac{21}{28} \times \frac{5}{4} 4 \times 5 = \frac{7}{8} \times \frac{5}{4} = \frac{35}{32}.$$

In summary: operations involving multiplication and division are simplified by division of common factors. In problems involving addition and subtraction, identification of common factors combined with the distributive law can effect simplifications.

Example 35. a) $48 + 36 =?$ b) $28 + 52 =?$

Solution. a) Since 12 is a common factor of the summands, we can factor it out; what remains is a simpler computation:

$$48 + 36 = 12 \times 4 + 12 \times 3 = 12 \times (4 + 3) = 12 \times 7 = 84.$$

b) Similarly

$$28 + 52 = 4 \times 7 + 4 \times 13 = 4 \times 20 = 80.$$

Example 36. When Troy and Chrissie got married the put together their farm holdings. Troy had six 48-acre plots and Chrissie owned four 18 acre plots. How much acreage do they have together?

Solution. The combined acreage is $6 \times 48 + 4 \times 18$. Now, think factors:

$$6 \times 48 + 4 \times 18 = 6 \times 48 + 4 \times 6 \times 3 = 6 \times 48 + 6 \times 12 = 6 \times (48 + 12) = 6 \times 60 = 360.$$

Just as common divisors are useful in division of whole numbers, common multiples are necessary in addition of fractions. Recall the number line interpretation of fractions: for a positive integer a, $1/a$ refers to one of a equal parts of the unit length, and, for a whole number b, b/a consists of b repetitions of $1/a$. That leads to an understanding of the addition of two fractions with the same denominator: b copies of $1/a$ plus c copies of $1/a$ gives $b + c$ copies of $1/a$. Algebraically this is represented by the distributive property:

$$\frac{b}{a} + \frac{c}{a} = b \cdot \frac{1}{a} + c \cdot \frac{1}{a} = (b + c) \cdot \frac{1}{a} = \frac{b + c}{a}.$$

Now, let's recall how we add disparate fractions, through examples. Keep in mind that in adding fractions we need to find a *common denominator*, and need not always seek for for the least common denominator.

Example 37. a) $\frac{3}{5}+\frac{5}{10}=?$ b) $\frac{1}{6}+\frac{3}{4}=?$ c) $\frac{3}{16}+\frac{3}{24}=?$ d) $\frac{1}{3}+\frac{1}{4}=?$ e) $\frac{1}{6}+\frac{3}{10}=?$

a) the denominators 5 and 10 are factors of 10. So:

$$\frac{3}{5}+\frac{5}{10}=\frac{6}{10}+\frac{5}{10}=\frac{6+5}{10}=\frac{11}{10}.$$

b) the denominators 6 and 4 are factors of 12. So:

$$\frac{1}{6}+\frac{3}{4}=\frac{2}{12}+\frac{9}{12}=\frac{2+9}{12}=\frac{11}{12}.$$

c) the denominators 16 and 24 are factors of 48. So:

$$\frac{3}{16}+\frac{3}{24}=\frac{9}{48}+\frac{6}{48}=\frac{9+6}{48}=\frac{15}{48}.$$

d) the denominators 3 and 4 are factors of 12. So:

$$\frac{1}{3}+\frac{1}{4}=\frac{4}{12}+\frac{3}{12}=\frac{4+3}{12}=\frac{7}{12}.$$

e) the denominators 6 and 10 are factors of 60. So:

$$\frac{1}{6}+\frac{3}{10}=\frac{10}{60}+\frac{18}{60}=\frac{10+18}{60}=\frac{28}{60}.$$

The answers for c) and e) are not in lowest terms, but absolutely correct. If we wanted the answers in lowest terms, we'd have: c) 5/16 and e) 7/15. Note that in e) this reduction was necessary because we did not select the *lowest* common multiple, but the reduction in c) is coincidental.

Section 3. Arithmetic Operations with Decimals

6.NS.3. Fluently add, subtract, multiply, and divide multi-digit decimals using the standard algorithm for each operation.

Addition and Subtraction

The algorithms for the arithmetic operations are the same for multi-digit decimals as the algorithm for whole numbers, with one additional ingredient: the placement of the decimal point in the answer. For addition and subtraction this is a simple task: it goes in the same position as that for the given numbers. If the numbers have different size decimal parts, we make them the same by filling in the empty spaces with zeroes. Here we illustrate the possibilities:

Example 38. Add or subtract as indicated:

a) 3.54 + 7.28 b) 1.89 + 3.6 c) 7.28 – 2.54 d) 3.6 – 1.89

SOLUTION.

a) $\begin{array}{r} 3.54 \\ +\,7.28 \\ \hline 10.82 \end{array}$ a) $\begin{array}{r} 1.89 \\ +\,3.60 \\ \hline 5.49 \end{array}$ a) $\begin{array}{r} 7.28 \\ -\,3.54 \\ \hline 3.74 \end{array}$ a) $\begin{array}{r} 3.60 \\ -\,1.89 \\ \hline 1.71 \end{array}$

We have not made explicit the regrouping that has taken place, for by this time, students should be quite able to see where it occurs.

Example 39. Jerry is making three berry pies for the class picnic. The recipe calls for a cup each of blueberries, raspberries and currants. He goes to the store with \$14.35. He finds that the price of a cup of blueberries is \$1.57, for a cup of raspberries, \$2.09 and for currants \$0.89.

- a) How much will he spend on the berries?
- b) He remembers that his friend Melinda hates currants, so he decides change the recipe to: two cups of blueberries, 1 cup of raspberries and no currants. Now how much will he spend?
- c) In case a) how much change does he have after the purchase?
- d) In case b) how much change does he have after the purchase?

Solution. a) Jerry will get 3 cups of blueberries at \$1.57/cup, 3 cups of raspberries at \$2.09/cup and 3 cups of currants at \$.89/cup. So he will spend:

$$3 \times (1.57 + 2.09 + 0.89) = 3 \times 4.55 = 13.65.$$

b) Each pie now costs $2 \times 1.57 + 2.09 = 5.23$, so the berry cost for 3 pies is $3 \times 5.23 = 15.69$.

c) He came with \$14.35 and spent \$13.65, so he has $35 + 35 = 70$ cents change.

d) He came with \$14.35 and spends \$15.69, so has to borrow \$1.34 from the grocer.

e) OK, there is no part e), but I don't want to leave Jerry between a rock and a hard place. So, I suggest to him that he make one pie *special* for Melinda, and the other two according to the original recipe. A Melinda pie costs \$5.23 (in berries), and a regular pie, \$4.55. So, in this way he spends $5.23 + 2 \times 4.55 = 5.23 + 9.10 = 14.23$, and comes home with 12 cents, a happy friend and a contented class!

Multiplication and Division.

Here the rule is almost as simple, but more difficult to track. The point to keep in mind is that we move decimal points so as to make the computation that of whole numbers, and when the computation is finished, to correct those movements by opposite movements. Remember that a movement of a decimal place to the right corresponds to a multiplication by a whole number power of ten, and movement to the left is division by a whole number power of ten. To multiply two decimals, we move both decimal places to the right so that both numbers are whole numbers. Now multiply those whole numbers. Finally, move the decimal point to the left a number of spaces equal to the sum of the moves to the right of the decimal places in the original two numbers. For division, the algorithm is the same, but with the word *sum* replaced by *difference*. But we'll get to that; let's look at multiplication.

Example 40. Multiply: 36.42×0.75.

Solution. Move the decimal point in 36.42 over two digits to get 3642. This corresponds to multiplying by 10^2. Similarly, moving the decimal point of 0.75 over by two digits gives 75. This corresponds to multiplying by another 10^2. This reduces the problem to multiplication of whole numbers and then compensating for the moves of decimal points. That is,

$$3642 \times 75 = 273150. \tag{a}$$

Now move the decimal point 4 digits to the left on both sides corresponding to dividing by 10^2 twice to undo the initial multiplication. This gives us

$$36.42 \times 0.75 = 27.3150. \tag{b}$$

On the left hand side we moved left 4 digits by moving two digits left in each of the factors, in order to reproduce the original problem. It is woth noting that when we have an equation of the form $A \times B = C$, we can move decimal points for A, B, C in any way we please so long as the sum of the moves of the decimal point in A and B is equal to the move of the decimal point in C. So, equation (a) above imples equation (b), as well as all these:

$$3.642 \times 7.5 = 27.3150, \quad 3642 \times .0075 = 27.3150 \quad 364.2 \times .075 = 27.3150 \quad 364200 \times 750 = 273150000.$$

Example 41. a) Multiply: 4.715×0.8. b) Multiply: 36.35×80.

Solution. a) Multiplying the first number by $10^3$and the second by 10^1, we essentially move the decimal point four places to the right, getting $4715 \times 8 = 37720$. To undo the initial multiplication, we need to divide by 10^4 by moving the decimal place 4 places to the left, resulting in 3.7720. We should always check that we've made the right calculation: eight-tenths of 4.715 has to be about 1/5 smaller than 4.7; so 3.7 something looks correct.

b) $3635 \times 8 = 29080$. We moved the decimal point two places to the right in the first number, and one place to the left for the second number. That gives us an overall net move of 1 place to the right; to correct that we move 1 place to the left in the answer to get 2,908.

Example 42. Lorenzo buys three and a quarter pounds of bananas at \$0.69 per pound. What is Lorenzo's cost?

Solution. First, let's make an estimate: three pounds at \$0.70 per pound comes to \$2.10; we take that as an estimate. Now, performing the multiplication:

$$3.25 \times 0.69 = (325 \cdot 69) \times 10^{-2} \cdot 10^{-2} = 22,425 \times 10^{-4} = 2.2425 ,$$

which the store will round up to \$2.25. This agrees nicely with our estimate of \$2.10.

Division. Let us start at the division theorem for whole numbers: For two whole numbers A and B there exist (unique) whole numbers Q and R such that $A = QB + R$ and $R < B$. We say that "A divided by B is equal to Q with remainder R. Another way to state the division theorem is with fractions:

$$\frac{A}{B} = Q + \frac{R}{B} \quad \text{with} \quad \frac{R}{B} < 1 .$$

Now, if we want to express R/B as a decimal we just continue the long division by introducing decimals. For example, if we want to express $647 \div 5$ as a decimal, we replace 647 by 647.000, with as many zeros as we need for the desired degree of accuracy.

Example 43. Perform the indicated division, correct to 2 decimal places:

a) $647 \div 5$ b) $115.7 \div 5$ c) $115.7 \div 6$ d) $115.7 \div 15$.

Solution.

a)

	1	2	9.	4	
5	6	4	7.	0	0
	5	0	0.		
	1	4	7.		
	1	0	0.		
		4	7.		
		4	5.		
			2.	0	0
			2.	0	0

On the left we display what the division looks like: just as in dividing whole numbers, but now we have included two place values to the right of the decimal point, indicating tenths place and hundredths place. The division proceeds as with whole numbers, while keeping track of the decimal point. Note that it remains in the same place throughout the division. Finally once we've calculated the digit for the tenths place, we have a remainder of zero. Thus we've finished and the answer is $647 \div 5 = 129.4$.

```
b)        2  3.  1  4
    5| 1  1  5.  7  0
       1  0  0.
          1  5.
          1  5.
             0.  7
             0.  5
                 2  0
                 2  0
```

The only difference between b) and a) is that the dividend has a decimal part. We just put that in, add a 0 in the hundredths place, and proceed as above. Again we end up with a remainder of 0, so the result is precise: $115.7 \div 5 = 23.14$.

```
c)          1  9.  2  8  3
       6| 1  1  5.  7  0  0
             6  0.
             5  5.
             5  4.
                1.  7
                1.  2
                    5  0
                    4  8
                       2  0
                       1  8
                          2  0
```

The procedure for c) is once again the same, however, when we get to the second decimal place, we do not have a remainder of zero. Thus the result: $115.7 \div 6 = 19.28$ is correct to 2 decimal places. Should we continue the long division further, we will continue to obtain 3's for as long as we have patience to go.

```
                 7.  7  1  3
d) 1 5 | 1  1  5.  7  0  0
         1  0  5.
            1  0.  7
            1  0.  5
                .  2  0
                   1  5
                      5  0
                      4  5
                         5
```

For d) the procedure remains the same, although the divisor now has two digits. We still proceed with the division algorithm for whole numbers, while attending to the position of the decimal point. We find that, up to two decimal places, $115.7 \div 15 = 7.71$. We should notice that if we continued the long division, we'd get 3's all the way down.

The most general situation along these lines is that of dividing a decimal number by another decimal number as in, say $115.7 \div 5.35$. If we multiply both numbers by the same number, the result of the division is the same. So, we can multiply both numbers by 10 as many times as we need to to make the divisor a whole number. Then we proceed as in Example 41. Restating this procedure algebraically:

$$115.7 \div 5.35 = \frac{115.7}{5.35} = \left(\frac{115.7}{5.35}\right)\left(\frac{10 \cdot 10}{10 \cdot 10}\right) = \frac{11570}{535} = 11,570 \div 535 .$$

Example 44. Perform the indicated division, correct to 2 decimal places:

a) $115.7 \div 5.35$ b) $463.26 \div 1.3$ c) $8500 \div 4.25$ d) $115.7 \div 0.07$.

Solution. The issue here is the precise location of the decimal point. The actual division can be done on a calculator, but the operator must have an estimate of the result as a check on the calculation. Here results are computed to two decimal places.

- a) $115.7 \div 5.35$ is about $100 \div 5 = 20$. Now let's do the actual division and see if the answer is near or about 20:

$$115.7 \div 5.35 = \frac{115.7}{5.35} = \frac{11570}{535} = 21.63.$$

- b) $463.26 \div 1.3$ is going to be about (in fact *more than*) a third less than 450, so about 300. Calculating:

$$463.26 \div 1.3 = \frac{463.26}{1.3} = \frac{4632.6}{13} = 356.35.$$

- c) $8500 \div 4.25$ is going to be around $8000/4 = 2000$. In fact

$$8500 \div 4.25 = \frac{8500}{4.25} = \frac{850000}{425} = 2000.$$

- d) $115.7 \div 0.07$ is going to be *of the order of* $100 \div \frac{1}{10} = 100 \times 10 = 1000$. Calculating:

$$115.7 \div 0.07 = \frac{115.7}{0.07} = \frac{11,570}{7} = 1652.86.$$

Notes

- i) In performing arithmetic operations with decimals, the context will help us to determine the degree of precision we should go to get a useful result. For example, if we are shipping tons of coal, we need not measure weights to the nearest hundredth of a pound, but if we are shipping diamonds, we better measure to the nearest thousandth of an ounce.
- ii) When we convert fractions to decimals, we have to specify the degree of accuracy we want. So, if we convert 1/7 to a decimal on a calculator we get a response like 0.14285714 or 0.1428571428571428, depending upon the accuracy of our calculator. Neither decimal is precise; in fact if we asked for a thousand, or a million decimal places, we still would not have a precise decimal representation. But for most purposes, two to four decimal places is sufficient for calculation, depending upon the context.
- iii) When working with dollars and cents, the decimals are precise: $14.35 means fourteen dollars and thirty-five cents. But now suppose that this is a lunch bill and we want to add a 12% gratuity. We multiply $14.35 by 1.12 to get $16.072 which is precise, but not dollars and cents. So, we either *round up* to $16.08 or *round down* to $16.07. We note that a bank, in any transaction, will round up for income and round down for outgo. Otherwise they will lose (rather than gain) fractions of a cent. For one transaction the difference seems absurd, but the banks typically do tens of thousands of transactions a day.
- iv) Thinking of decimals as approximations of precise numbers, we have to be cognizant of the fact that operations among rounded numbers magnify the error in the approximation. This is illustrated in the following problem.

Example 45. Express $\frac{2}{9} \times \frac{3}{11}$ as a decimal correct to three places.

Solution. There are two ways to go.

- A. Multiply 2/9 by 3/11 to get 2/33, and calculate the long division $2 \div 33$to three places to get 0.061.
- B. Express both 2/9 and 3/11 as numbers correct to three decimal places, and multiply those numbers. We get the multiplication $0.222 \times 0.272 = .060384$, which rounds to 0.060.

Which answer is correct (to three decimal places)? In case A we performed the multiplication with the precise fractions, and then converted to the three decimal approximation. In case B, we first did the decimal conversion, then multiplied and then converted to three decimals. Applying an operation after estimating tends to magnify the estimate's error. So, we were better off with case A, where we did not reduce to three decimals until the end.

The point here is that fractions are precise, whereas the decimal representation is, in most cases, an approximation. So, a good rule to follow is to use precise expressions for numbers until the end of the computation, and *then* convert to decimal approximation. So A is the answer correct to three decimal places.