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Chapter 0: Fluency

Common Core Standard(s)

- Fluently divide multi-digit numbers using the standard algorithm. (6.NS.2)
- Fluently add, subtract, multiply, and divide multi-digit decimals using the standard algorithm for each operation. (6.NS.3)
- Find the greatest common factor of two whole numbers less than or equal to 100 and the least common multiple of two whole numbers less than or equal to 12. Use the distributive property to express a sum of two whole numbers 1-100 with a common factor as a multiple of a sum of two whole numbers with no common factor. *For example, express* $36 + 8$ *as* $4(9 + 2)$. (6.NS.4)

Vocabulary: sum, addends, difference, divisor, dividend, quotient, product, factor, algorithm, greatest common factor, least common multiple, prime number, prime factorization, distributive property,

Appendix Overview:

It is important to note that the standards addressed in this appendix are often best taught within a specific context or standard where they are used. For example, finding greatest common factors or least common multiples are often an important component in simplifying fractions, finding least common denominators, or rewriting an algebraic expression by factoring. It would be appropriate to address these standards within the unit that focuses on fractions or algebraic expressions. For this purpose we offer these standards not only as a stand-alone unit but encourage teachers to address these standards where appropriate and maintained as a subtext throughout the whole grade.

In 6^{th} grade students consolidate the work done in previous grades on operations with whole numbers and decimals to become fluent with all four operations with these numbers. It is important to understand that in order to be fluent in these operations a student can not only implore a standard algorithm to execute an operation but they can choose methods of computation that help them arrive at their desired outcome most efficiently. This might be estimating or using mental math or they might decide to change decimals to fractions or vice versa. In addition, a student can reflect upon their answer and determine if it makes sense given the context. Throughout this chapter students will begin to see whole numbers, fractions, and decimals all as numbers that belong to the same number system, just represented in different ways. This understanding is crucial for their fluency with arithmetic operations to grow.

The chapter begins by reviewing arithmetic operation with whole numbers. In previous grades students have used models, place value charts, and properties of operations to find sums, differences, products, and quotients. They have then connected these models to the standard algorithms for all operations except division. It is likely that your students have some experience with the standard long division algorithm previous to 6^{th} grade but they have not yet reached fluency. After students have reviewed the standard algorithms for addition, subtraction, and multiplication students will connect place value models with the standard long division algorithm. They will draw on their knowledge of place value as they aim for fluency in dividing multi-digit numbers with the algorithm, addressing standard 6.NS.2

The next section focuses on finding the least common multiple, greatest common factor, and using these skills to rewrite a the sum of two whole numbers using the distributive property as multiple of a sum of two whole numbers with no common factor. This prepares them for writing equivalent algebraic expressions later on in 6^{th} grade and in future grades.

In the last section of this appendix students return to working with arithmetic operations. They apply the knowledge and skills learned in previous grades and from the first section of this appendix to add, subtract, multiply, and divide multi-digit decimals. They practice each of these operations and aim to obtain fluency not only in mastering algorithms but through estimating and using mental math.

Connections to Content:

Prior Knowledge:

In previous grades students have achieved fluency in adding, subtracting, and multiplying multi-digit whole numbers. They have founds whole-number quotients with up to four-digit dividends and two-digit divisors, using strategies based on place value, the properties of operations, modeling, and/or the relationship between multiplication and division. It is not until 6th grade that they connect these strategies to a division algorithm for multi-digit numbers and extend their work to numbers with any given number of digits. In 5th grade students add, subtract, multiply, and divided decimals to hundredths, using concrete models or drawing and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction. They relate these strategies to a written method but have not yet solidified a connection to the standard algorithm. Again, it is in 6th grade that students solidify this connection and extend their knowledge to work with decimals beyond hundredths and ultimately gain fluency.

In 4th grade students study factors and multiples and the dual relationship that they hold. They have also learned how to determine whether a given whole number in the range 1-100 is prime or composite. In 3rd grade students relate multiplication to the area of a rectangle and use area models to represent the distributive property.

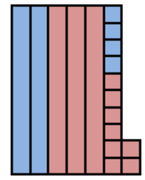
Future Knowledge:

Fluency in the arithmetic operations for multi-digit whole numbers will help students to perform these operations with integers and rational numbers in 7th grade. Operations with decimals are particularly helpful when students learn to convert a rational number to a decimal using long division and in turn determine whether or not a number is rational. It is true that in the future many decimal operations will be performed with a calculator; however fluency will help a student reflect upon whether the answer the calculator gives them makes sense. Fluency also helps them to use strategies of mental math and estimation to obtain answers with ease and efficiency rather than always having to reach for a calculator. Also if students are fluent in changing decimals to fractions and vise versa they add to another great strategy for problem solving to their "tool box"

Using the distributive property to "factor out" a greatest common factor from a sum two whole numbers prepares students to write equivalent expressions for polynomials. This in turn lays a foundation for simplifying rational expressions with polynomials. In addition students will use a greatest common factor to simplify fractions and the least common multiple to find a least common denominator.

MATHEMATICAL PRACTICE STANDARDS

	Make sense of problems and persevere in solving them.	Roxy has created a new cherry chocolate treat to sell in her store. She packages the cherry chocolates into tubes, boxes, and cases. Each tube contains 10 cherry chocolates, each box contains 10 tubes, and each case contains 10 boxes. Roxy has made 1851 cherry chocolates and has received requests from 12 schools for the treats. She would like to give each school the same number of cherry chocolates. Determine how many cherry chocolates each school will receive and the number of cases, boxes, tubes, and pieces of cherry chocolates each school will receive. *As students work on this problem they must draw upon their experience with place value. They must infer what arithmetic operation will give them the desired outcome and also how they can express the desired outcome in terms of the number of boxes, tubes, and loose pieces of chocolate. They must be able to explain the correspondence between any modeling they have done and the division algorithm. Finally they must reflect upon their answer given the context of the problem being sure to make sense of the remainder left over.*
	Reason abstractly and quantitatively.	Explain in your own words how you know that a number is divisible by 4 if the last two digits form a number that is divisible by 4. *As students explore why different divisibility rules work they break down numbers into powers of 10 or 100. They must think abstractly as they analyze the powers of 100 (as is the case for divisibility of 4) and infer that all powers or multiples of 100 are divisible by 4. They decontextualize the given number and reason that this is true for all numbers that are divisible by 4. They can then conclude that since this is true they need only consider tet divisibility of the last two digits.*
	Construct viable arguments and critique the reasoning of others.	Roxy's cashier has made some calculations for some of the purchases at the candy store and has made some mistakes, his work is shown below. For problems 5, 6, and 7 go through each transaction and determine the mistake, explain how to perform the calculation correctly and fix the mistake. *Throughout this appendix you will find several problems that take on the form of "Find, Fix, and Justify" For these problems students analyze another student's work and must identify mistakes in the work. They make arguments as to why something is wrong by pointing out explicit errors observed. Once they fix the mistake they must justify why their reasoning is correct.*
	Look for and express regularity in repeated reasoning.	Describe each pattern given below, then find the next two terms. a. $1, 0.3, 0.09, 0.027$ ... b. $17, 0.17, 0.0017$... c. $10, 15, 22.5, 33.75$ ... *When multiplying decimals students must recognize the regularity in how the placement of decimal in the final product behaves. In this example not only do students see a pattern in the sequence of given numbers but they must know how multiplying each term by a decimal factor will produce the next term.*

<table>
<tr>
<td></td>
<td>Model with mathematics.</td>
<td>Marta has created the model below. She claims that this model can be used to represent the sum of 24 and 38.

1. If Marta’s claim is true, what is the value of the small square?

2. What is the value of a rod (long rectangle)?

3. Find the sum of 24 and 38 using the addition algorithm and discuss how this relates to the model above.

Modeling is an important component in helping students understand the fluency standards. In previous grades students used base-ten blocks to model addition for whole numbers and numbers with decimals up to hundredths. The blocks and model help students to understand why we “carry” as we bundle into groups of ten when using the addition algorithm. As students analyze these models they extend this understanding to adding decimals greater than hundredths. Modeling is used to help students connect all of the arithmetic operations to their respective algorithm.</td>
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<tr>
<td></td>
<td>Attend to precision.</td>
<td>After delivering all of the candy to the schools Roxy has earned $73 in tips for making the deliveries. She decides to split the tip money between she and her 3 employees. How much tip money will each person receive?

In order to communicate the answer to this problem with precision students must reflect upon the context of the problem For this context it makes sense to report the quotient as a decimal and not as a remainder because we want to know exactly how much money everyone will receive.

After making the calculation above Roxy realizes that she forgot to include the two delivery truck drivers when making her tip calculation. How much money will each person receive if she tips herself, her three employees, and the two delivery truck drivers? Check your answer with a calculator.

Once again students must not only reason about the correct way (quotient with a decimal, fraction, or remainder) to report this answer but also understand that they need only find the quotient to the nearest hundredth. Since this context deals with money we only care about the amount to the nearest cent since you cannot break up a penny. This is an example of how they must accurately and effectively express a numerical answer with a degree of precision appropriate for the problem context.</td>
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<tr>
<td></td>
<td>Look for and make use of structure.</td>
<td>Find the GCF of the two numbers in each given sum. Use the distributive property to write an equivalent expression to the sum that contains the GCF as one of its factors. How do you know that you found the correct equivalent expression?
<table>
<tr><td>a. $42 + 14$</td><td>b. $36 + 27$</td></tr>
<tr><td>c. $55 + 44$</td><td>d. $16 + 72$</td></tr>
</table>
In the example above students must rewrite a given sum as an equivalent expression using the distributive property. To do so they must understand how you can decompose a number into a product of its factors. This understanding demonstrates knowledge about the structure of these sums and products. In part a above should be able to see the expression 42+14 as a single sum of 56, as adding to positive whole numbers, or as a multiple of a sum of two whole numbers with no common factor.</td>
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<tr>
<td></td>
<td>Use appropriate tools strategically.</td>
<td>After making the calculation above Roxy realizes that she forgot to include the two delivery truck drivers when making her tip calculation. How much money will each person receive if she tips herself, her three employees, and the two delivery truck drivers? Check your answer with a calculator.
The quotient in this problem will contain a repeating decimal of 6. When students check their answer with a calculator they will see that the calculator often reports a digit of 7 at the end of their calculator screen. This gives them the opportunity to discuss how the calculator has rounded their answer and why it would do this. It is also important to discuss how different key strokes on a calculator can indicate division. For example inputting numbers as fractions.</td>
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Section 0.1: Arithmetic Operations with Whole Numbers

Section Overview:
This section specifically addresses a student's ability to fluently add, subtract, multiply, and divide multi-digit decimals using the standard algorithm for each operation. In the first lesson students review how models connect to the standard algorithms for addition, subtraction, and multiplication. In previous grades students have done extensive work in relating these models to the standard algorithm for these operations so this lesson gives students the opportunity for them to now solidify their knowledge and practice fluency. In the next lesson students develop the standard algorithm for division using place value models. While they have found quotients of multi-digit numbers in previous grades their work with a standard algorithm for division has not been as extensive has with the other operations. This second lesson provides students with the opportunity to practice the algorithm and work on fluency with division.

Concepts and Skills to Master in this Section:

By the end of this section, students should be able to:

1. Fluently add multi-digit numbers using the standard algorithm.
2. Fluently subtract multi-digit numbers using the standard algorithm.
3. Fluently multiply multi-digit numbers using the standard algorithm.
4. Fluently divide multi-digit numbers using the standard algorithm.

0.1a Class Activity: Arithmetics Operations with Whole Numbers

Prior to 6th grade a big part of multi-digit arithmetic has been done with modeling. This section reviews how models connect to the basic algorithms for addition, subtraction and multiplication. Throughout this section encourage your student to estimate their answers before calculating as this will give them a sense of whether or not their answer makes sense.

Roxy is ordering candy to re-stock some items at her candy shop. Answer each of her questions below without using a calculator. Show your work and be ready to discuss how you answered each question.

This task gives students the opportunity to review what they know about operations with multi-digit numbers. In 4th and 5th grade students performed operations with multi-digit whole numbers and with decimals to hundredths. They used strategies based on place value, the properties of operations, and/or the relationship between inverse operations to find sums, differences, products, and quotients. As students work through these problems review with them how you can relate these strategies to the standard algorithms. These models will be used in later sections to extend their understanding to operations with multi-digit decimals that extend beyond hundredths.

1. A case of Mega Mania Jaw Breakers cost \$12 each, how much will 26 cases Mega Mania Jaw Breakers cost?

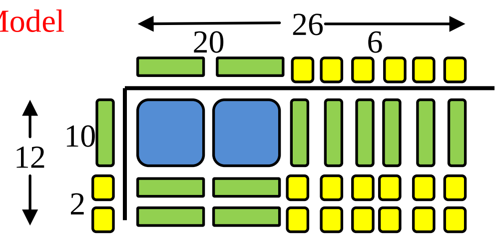
Algorithm

$$\begin{array}{r} {}^{1}12 \\ \times\ \ 26 \\ \hline {}^{1}72 \\ +240 \\ \hline 312 \end{array}$$

Partial Product

$$\begin{aligned} 12 \times 26 &= (10 + 2) \times (20 + 6) \\ &= 200 + 60 + 40 + 12 \\ &= 312 \end{aligned}$$

Area Model

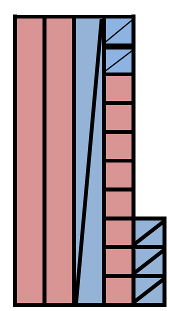


$= 200 + 60 + 40 + 12 = 312$

Encourage students to estimate the product first. We are multiplying 26 and 12. You can round 26 to 25, so how much is 12 groups of 25? We know that 4 groups of 25 is 100 (think quarters and dollars) and there are 3 of these 4 group bundles in 12. $3 \times 100 = 300$. The answer should be around 300. As you are reviewing the multiplication algorithm discuss how place value plays a role in the placement of each number. If needed, review how to multiply multi-digit numbers using area models or partial products and how these strategies relate to the algorithm.

2. An entire case of Old Fashion Root Beer is \$43. If Roxy gets a discount of \$15 off, how much will the case of root beer cost?

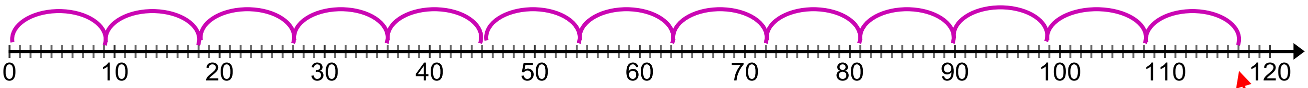
$$\begin{array}{r} {}^{3}\cancel{4}{}^{1}3 \\ -\,15 \\ \hline 28 \end{array}$$



Students might estimate this difference by rounding 43 to 45. The difference of 45 and 15 is 30 so the answer should be a little bit less than 30 because we rounded up. This is a model with base ten blocks that represents $43 - 15 = 28$. The entire model represents the minuend and the subtrahend is shaded blue. In the model you can observe unbundling and borrowing of a ten as used in the algorithm.

3. I spent \$117 on boxes of Tangy Licorice Ropes, each box costs \$9. How many boxes of licorice ropes did I buy?

117 can be split into 13 groups of 9.

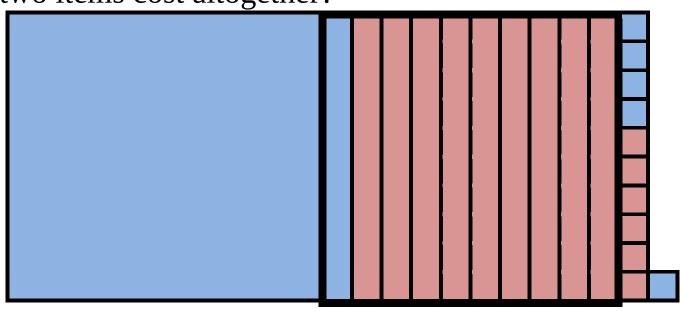


$117 \div 9 = 13$

You can round 117 to 120, how many times will 9 go into 120? We know that 9 goes into 100 about 11 times and it goes into 20 about 2 times. $11 + 2 = 13$, so our answer should be around 13. Students have been exposed to, but have not yet solidified an algorithm for division of multi-digit numbers in previous grades. This is an example of using number line model to find the quotient. The division algorithm will be solidified in the next lesson.

4. I would like to buy a case of Chocolate Nut Clusters for \$115 and a case of Whopper Hoppers for \$86. How much will these two items cost altogether?

$$\begin{array}{r} {\scriptstyle 1\,1\,\,} \\ 115 \\ +86 \\ \hline 201 \end{array}$$

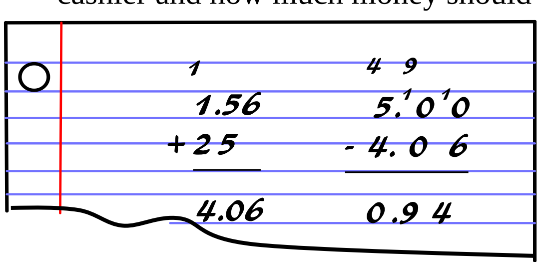


Round 86 to 90 and 115 to 110. $90 + 110 = 200$. Our answer should be around 200. This base ten model shows that the sum of 115 and 86 is 201. In the model each addend is shaded a different color and the sum is represented by the entire model. You can see how the ones form a group of 10 that is carried to the tens place in the algorithm. You can also see how there are ten rods of ten that form a hundred cube. This relates to carrying the one into the hundreds place in the algorithm.

Find, Fix, and Justify

Roxy's cashier has made some calculations for some of the purchases at the candy store and has made some mistakes, his work is shown below. For problems 5, 6, and 7 go through each transaction and determine the mistake, explain how to perform the calculation correctly and fix the mistake.

5. Corey buys two different candies at Roxy's store. One is a box of Fruity Frogs for \$1.56 and the other is a piece of Tangy Taffy for \$0.25. Corey has a 5-dollar bill. How much money does Corey owe the cashier and how much money should he get back?

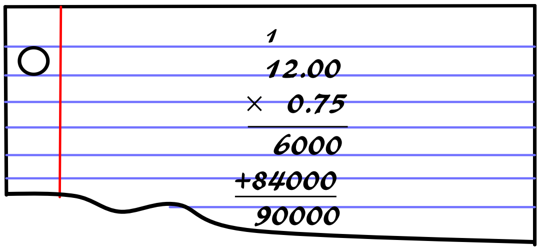


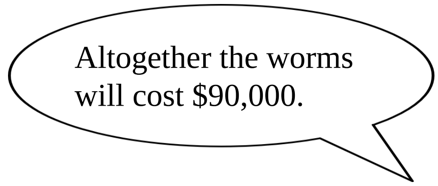
Corey should get \$0.94 back in change.

The cashier did not line up the decimals when adding. You have to make sure you line up the decimals when adding and subtracting to ensure that you are combining numbers within the same place value. Corey owes the cashier \$1.81 and should get back \$3.19.

Find, Fix, and Justify Continued

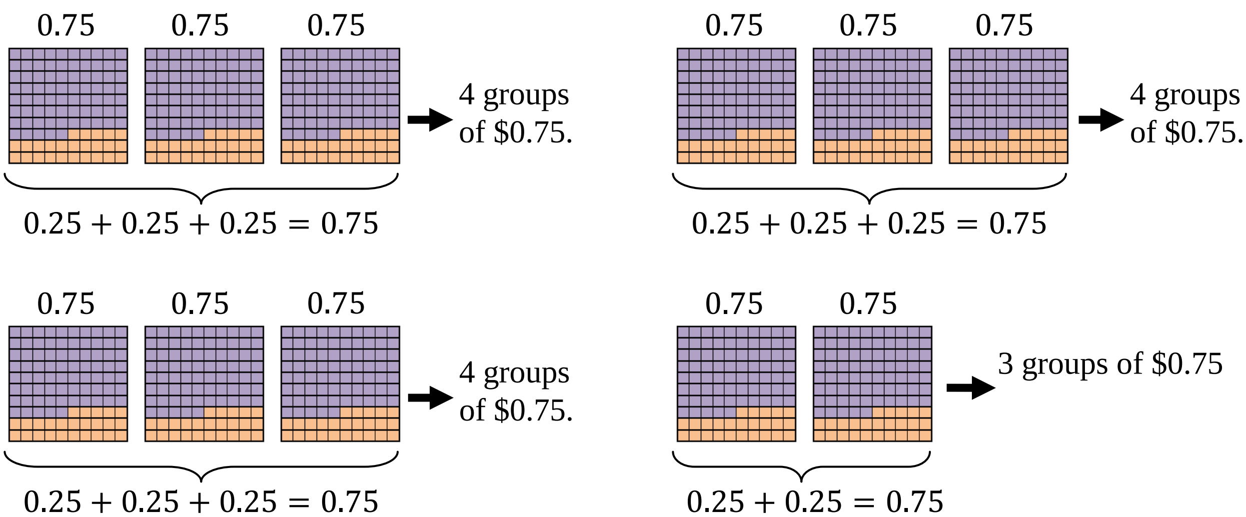
6. Lola is buying Gooey Glow Worms for each of her 12 cousins. The worms cost \$0.75 each. How much will all the worms cost together?



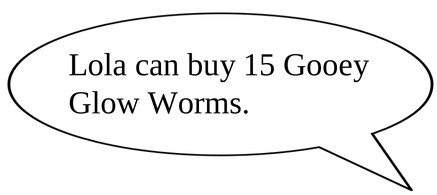


The cashier made the mistake of thinking they needed to line up the decimals when multiplying. He also did not take into account where the decimal needs to go in the product. Obviously \$90,000 is way too much if the worms only cost \$.75 each. The cost of the worms is \$9.00, you must consider where to put the decimal point in the product either by deducing that when multiplying by a decimal your product will be smaller than your beginning factor and or by reasoning about the position the decimal point is in the factors and how that effects the placement of the decimal point in the product.

7. Lola decides to just spend all the money she has on the \$0.75 Gooey Glow Worms. She has \$11, how many worms can she get?



4 + 4 + 4 + 3 = 15 Gooey Glow Worms



On Lola's last two dollars he states that there are 3 groups of \$0.75 this in not correct. He is reasoning that she can buy half a glow worm but it is actually 2/3 of a glow worm. You cannot buy partial glow worms. There are two groups of 0.75 with \$0.50 leftover. That means that she can buy 14 worms and will have \$0.50 leftover.

0.1a Homework: Arithmetic Operations with Whole Numbers

Directions: Estimate each sum, difference, product or quotient. Then use an algorithm or model to find the exact answer.

1. $435 + 269$	2. 269×4	3. $435 - 269$ 166
4. $450 \div 15$	5. $43500 - 26$	6. 435×269 117,015

7. At 9:00 on election night, the ballot count is shown in the table below.

Candidate A	Candidate B
47560	44127

But now the returns from District 10 are just in: 2316 votes in District 10 cast for candidate A, and 7387 for candidate B: Who is now in the lead and by how many votes?

8. To date, my orchard has produced 23,420 bushels of apples and 16,870 bushels of pears. A bushel of apples brings in $7.00 and a bushel of pears brings in $10. To sustain my orchard, I need to make in $300,000. Assuming the ratio of apples to pears remains the same; about how many more bushels of apples and pears do I need to collect to sustain the orchard? Justify your answer.
You do not need to bring in any more bushels of fruit. Your profit is already $332,640, this is $32,640 over the required amount.

9. Gianni, Sylvia and Lester are at Lagoon, and they want to ride the Loop-de-loop and tickets are $5.50 each. Gianni has $4.50, Sylvia has $7.50 and Lester has $4.50. Can they pool their resources and all ride the Loop-de-loop? Justify your answer.

Find, Fix, and Justify

A cashier has made some calculations for some of the purchases at the hardware store and has made some mistakes, her work is shown below. For each problem go through every transaction and determine the mistake, explain how to perform the calculation correctly, and fix the mistake.

10. Clara buys a part for the faucet on her kitchen sink that costs $1.81. She gives the cashier a 10-dollar bill. How much money should the cashier give her back?

If it costs $1.81 and she gave me $10.00, how much is between 1.81 and 10.00? The difference between 10 and 1 is 9 and the difference between 100 and 81 is 19. If I add those two differences together I get $9.19. She should get $9.19 back.

The cashier made the mistake of not accounting for borrowing a dollar in order to subtract the $0.81. If you take $0.81 away from $1.00 that reduces the $10 to $9. So you should consider the difference of 9 and 1 which is 8. Thus the cashier should give her $8.19. This can be shown with the subtraction algorithm.

11. Jackson buys 3 large pieces of plywood for $8.05 each. He has a 20-dollar bill and a 5-dollar bill which he gives the cashier. He then insists on digging around in his pocket to find an extra $0.15 to give the cashier as well. Jackson says it will make the calculation easier. How much money does he get back?

$$8.05 \times 3 = 24.15$$

$$25.00 - 24.15 = 0.85$$

8.05 × 3 is $24.15. Jackson has $25 to give me. If I subtract $24.15 from $25 I get $0.85. Why is he insisting on giving me an extra $0.15 too?

12. Nancy finds some really great light fixtures on sale for her new home? The sign says that you can 4 lights fixtures for $404.00. She asks the cashier how much each light fixture costs.

$$\frac{404}{4} = \frac{40+4}{4} = \frac{40}{4} + \frac{4}{4} = 10 + 1 = 11$$

I can use partial quotients by splitting my dividend up. 404 = 40 + 4. Then I can divide each piece by 4 and add my two quotients together to get the cost of each fixture. Each fixtures csots $11.

0.1b Class Activity: Dividing Multi-Digit Numbers

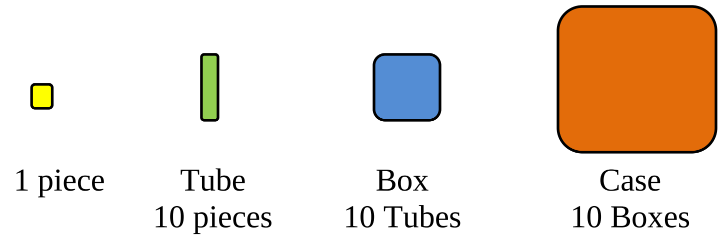
Part 1: Roxy has received orders from several different schools requesting candy from her store for their school fundraiser. She has recorded the amount of each type of candy she has in stock and the number of schools that want each type of candy in the table below. She would like to give each school that requests a certain type of candy the exact same amount of candy to be fair. Study the table and without calculating, estimate if the amount of candy that each school receives is correct. Justify your answer.

Type of Candy	Truffle Troll Treats	Rainbow Drops	Lemon Swirly Pops
Amount in Stock	6255	154	4950
Number of schools requesting this candy	15	9	20
Amount of candy each school receives	417	14	247.5

The amount of candy each school receives is correct for the Truffle Troll Treats and the Lemon Swirly Pops. You can determine this by using estimation. Ask students to share their estimation strategies with the class. Possible arguments might be that for the Truffle Troll Treats we know that 15 will go into 60 four times, this means that 15 will go into 6255 a little more than four hundred times. Thus 417 is a reasonable answer. For the Lemon Swirly Pops students might argue that 4950 is pretty close to 5000 and 20 goes into 5000 two hundred and fifty times. Since 247.5 is a little less than 250 this amount makes sense. We know that the amount received per school for the Rainbow drops it not correct because 9 will go into 100 almost exactly 11 times. 9 goes into 54 exactly 6 times. Thus the amount of candy that each school receives should be very close to $11 + 6 = 17$.

Discuss with students how estimation plays an important role in division and also how good problem solving includes reflecting upon whether your answer makes sense given the context.

Part 2:
Roxy has created a new cherry chocolate treat to sell in her store. She packages the cherry chocolates into tubes, boxes, and cases. Each tube contains 10 cherry chocolates, each box contains 10 tubes, and each case contains 10 boxes.

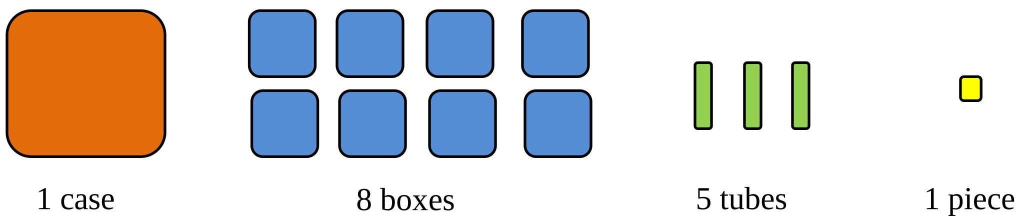


**Figures not drawn to scale*

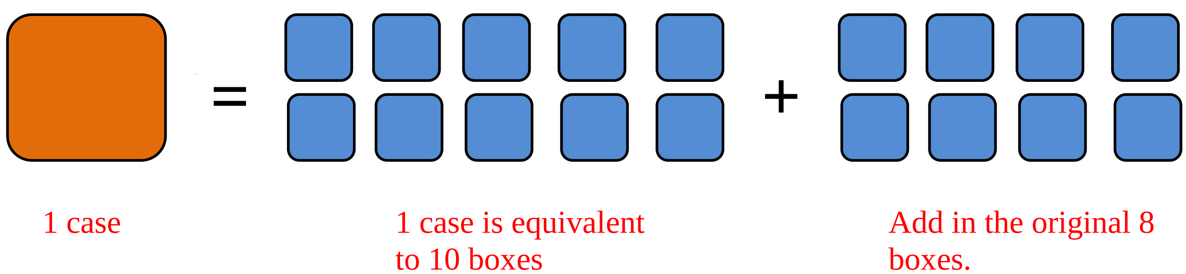
Roxy has made 1851 cherry chocolates and has received requests from 12 schools for the treats. She would like to give each school the same number of cherry chocolates. Determine how many cherry chocolates each school will receive and the number of cases, boxes, tubes, and pieces of cherry chocolates each school will receive.

Ask your student to estimate the solution first and share their estimation reasoning. The task is worked out in detail below showing how you can use place value models to solve the problem. In previous grades students used models to solve long division problems and have begun to relate these models to the long division algorithm. It is completely acceptable for students to solve with an area model or with partial quotients instead of with a place value model. As you work through this task with your student relate the models to the standard algorithm.

According to Roxy's packaging guidelines 1831 cherry chocolates would be packaged in 1 case, 8 boxes, 5 tubes, and 1 individual candy.



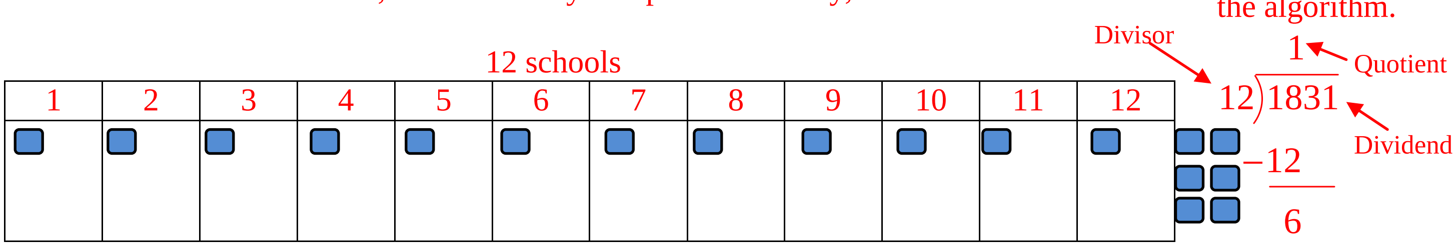
This is essentially breaking the dividend into its place value parts.
1 case cannot be divided amongst the 12 schools evenly. We must "unpack" the case. There are 10 boxes in a case.



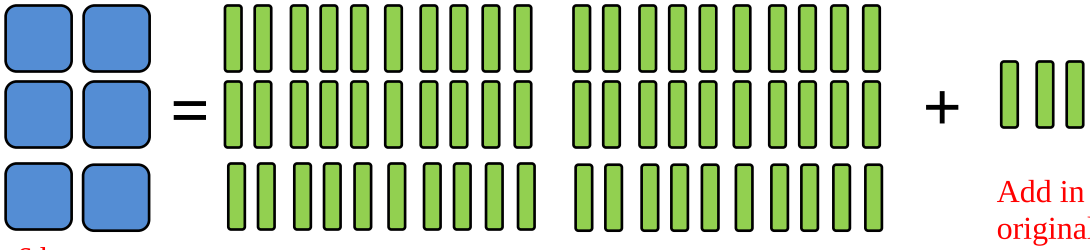
This makes 18 boxes that can be distributed among the 12 schools.

If each school receives 1 box, which is really 100 pieces of candy, then 6 boxes are leftover.

Relate this to the algorithm.



We must now divide the 6 boxes leftover amongst the 12 schools. Since the 6 boxes cannot be evenly distributed amongst the 12 schools than we must “unpack” the boxes into tubes. There are 10 tubes in a box.



6 boxes

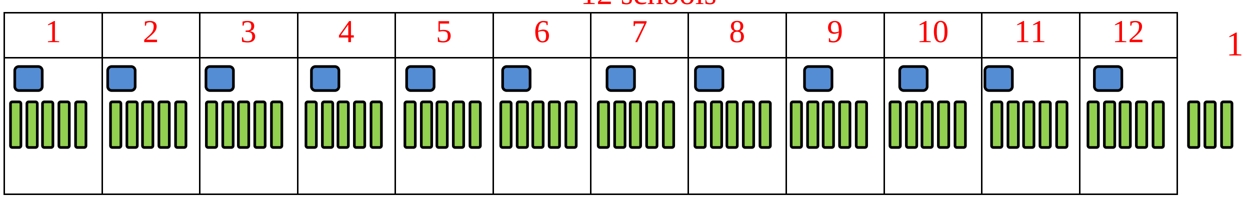
6 boxes is equivalent to 60 tubes

Add in the 3 original tubes

This makes 63 tubes that can be distributed among the 12 schools.

If each school receives 5 tubes, which is really 50 pieces of candy, then 3 tubes are leftover.

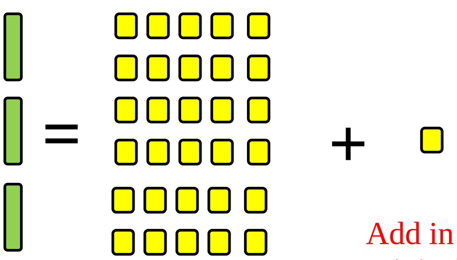
12 schools



Relate this to the algorithm

$$\begin{array}{r} 15 \\ 12\overline{)1831} \\ -12 \\ \hline 63 \\ -60 \\ \hline 3 \end{array}$$

We must now divide the 3 tubes leftover amongst the 12 schools. Since the 3 tubes cannot be “evenly” distributed amongst the 12 schools than we must “unpack” the tubes into pieces. There are 10 pieces in a tube.



Add in the 1 original piece

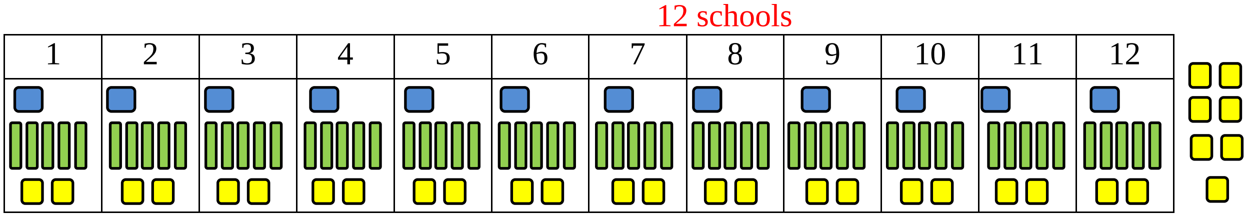
This makes 31 pieces can be distributed among the 12 schools.

3 tubes

3 tubes are equivalent to 30 pieces

If each school receives 2 pieces than 7 pieces are leftover.

12 schools



Relate this to the algorithm

$$\begin{array}{r} 152 \\ 12\overline{)1831} \\ -12 \\ \hline 63 \\ -60 \\ \hline 31 \\ -24 \\ \hline 7 \end{array}$$

This models shows that each school will receive 1 box, 5 tubes, and 2 pieces of chocolate cherries, this is 152 pieces of candy. Be sure to relate this to the algorithm, talking about how the 1 in the quotient is really 100 and the 5 is really 50. Roxy will have 7 pieces of candy leftover. When a division problem results in a quotient that is not a whole number students must be able to decide whether the context dictates them to report the quotient as a whole number with a remainder, as a decimal, or as a fraction. For this context it is not likely that Roxy would break the loose candies into small pieces to divide amongst the 12 schools. So reporting the quotient with a remainder is appropriate. However, students will practice writing quotients with decimals or fractions rather than remainders on the next page.

Part 3

After delivering all of the candy to the schools Roxy has earned $73 in tips for making the deliveries. She decides to split the tip money between she and her 3 employees. How much tip money will each person receive?

$$\begin{array}{r}
18.25 \\
4\overline{)73.00} \\
-4 \\
\hline
33 \\
-32 \\
\hline
1\,0 \\
-8 \\
\hline
20 \\
-20 \\
\hline
0
\end{array}$$

For this context it makes sense to report the quotient as a decimal and not as a remainder because we want to know exactly how much money everyone will receive. If needed, draw a model to illustrate unbundling the remaining 1 dollar into 10 dimes, relate it to bringing down a zero in the standard algorithm. You can then illustrate unbundling the remaining 2 dimes into 20 pennies; this is shown in the algorithm by subtracting and bringing down the second zero. As students practice using long division paying attention to the repetition of "subtract" and "bring down" will aide in achieving fluency and ease when working with the algorithm.

You can also talk about writing the remainder as a fraction and how this fraction is related to the decimal. In this problem you have a remainder of 1. This remainder must be split amongst 4 people; you can represent this with the quotient $1 \div 4 = \frac{1}{4} = 0.25$

Each person will receive a tip of $18.25.

After making the calculation above Roxy realizes that she forgot to include the two delivery truck drivers when making her tip calculation. How much money will each person receive if she tips herself, her three employees, and the two delivery truck drivers?

$$\begin{array}{r}
12.1666... \\
6\overline{)73.0000} \\
-6 \\
\hline
13 \\
-12 \\
\hline
10 \\
-6 \\
\hline
40 \\
-36 \\
\hline
40 \\
-36 \\
\hline
40
\end{array}$$

For this context is makes sense to report the quotient as a decimal and not as a remainder because we want to know exactly how much money everyone will receive. There are two important topics of discussion in this problem. One is that is appears that the decimals will continue on forever. We know this because they begin to repeat themselves. Repeated decimals will be address thoroughly in 7th grade. Secondly, in relation to the context of money we round our answer up to the nearest hundredth because we cannot break up a penny.

Ask students to check this problem on their calculator so you can talk about why the answer on the calculator has a 7 at the end.

Each person will receive a tip of $12.66

Directions: Estimate each quotient. Then use the standard algorithm to find the exact quotient.
Encourage students to make a list of the divisor's multiples to reference for each problem.

1.	2.	3.	4.
23 $14\overline{)322}$ − 28 42 − 42 0 Possible estimation reasoning might be that 14 is close to 15. We know that 15 goes into 300 twenty times so our quotient will be a little more than 20.	467 $12\overline{)5484}$ − 48 68 − 60 84 − 84 0 Discuss possible estimation reasoning.	75 $115\overline{)8625}$ − 805 575 − 575 0 Possible estimation reasoning might be to round the divisor to 100 and the dividend to 8000. 100 goes into 800 eighty times. The quotient should be close to 80.	103 $205\overline{)21115}$ − 205 61 − 0 615 − 615 0 In this problem discuss the role that the 0 in the quotient plays. This is obtained because 205 will not go into 61, this means than we must unbundle the 61 tens into 610 ones and then add the remaining 5 ones to get the 615. That is why there is a 0 in the tens place in the quotient.

<table>
<tr>
<td>5.

18.05
20)361.00
– 20
161
– 160
10
– 0
100
– 100
0

Discuss possible estimation reasoning and the zero in the tenths place of the quotient.</td>
<td>6.

121.2195121951
41)4970.0000000000
– 41
87
– 82
50
– 41
90
– 82
80
– 41
390
– 369
210
– 205
50
– 41
90
– 82
80
– 41
390
– 369
210
– 205
50

Discuss the repeating decimals, in this problem the decimal does not begin to repeat right away but eventually a pattern emerges.</td>
<td>7.

19.57386363
352)6890.00000000
– 352
3370
– 3168
2020
– 1760
2600
– 2464
1360
– 1056
3040
– 2816
2240
– 2112
1280
– 1056
2240
– 2112
1280

Possible estimation reasoning might be to round the dividend to 7000 and the divisor to 350. 350 goes into 7000 twenty times. The quotient should be close to 20.</td>
<td>8.

18.125
6)108.750
– 6
48
– 48
07
– 6
15
– 12
30
– 30
0

This problem is a natural extension to the previous problems. If you feel your students are not ready for it yet it will be revisited in section 0.01e.</td>
</tr>
</table>

As you work through the problems above relate the algorithm to place value, this will help the students not only become more fluent in the algorithm but help them to understand why the algorithm works. You can also encourage students to check their answer by multiplying the quotient by the divisor. If this product equals the dividend they have done the problem correctly.

Find, Fix, and Justify

9. Owen has completed the following division problem and has made a mistake. Find the mistake and explain what Owen has done wrong. Then solve the division problem correctly.

$561.\overline{22}$
45)25245.00
– 225
275
– 270
55
– 45
100
– 90
100
– 90
100

Owen should have brought down the 4 and not the 5.

561
45)25245
– 225
274
– 270
45
– 45
0

This problem illustrates a common mistake when doing long division. It is important to keep your numbers lined up. If students struggle give them a piece of graph paper to do their calculations on so they write each number in the appropriate place value column.

0.1b Homework: Dividing Multi-Digit Numbers

Directions: Without calculating determine which of the following quotients are correct. Justify your answer.

1. $152 \div 14 = 2128$
2. $4508 \div 92 = 49$
3. $14880 \div 124 = 1200$
   This quotient is not correct. If you round the dividend to 15,000 and the divisor to 100 then you can estimate the quotient to be around 150. The quotient of 1200 is too big.

4. Marty manufactures special bolts for motorcycles in his garage. He has packaged 1 case, 5 boxes, 1 tube, and 2 individual bolts to be sent to 14 motorcycle shops. Marty packages 10 individual bolts in 1 tube, 10 tubes in one box, and 10 boxes per case. Draw a picture or model that represents how the bolts are packaged, and then determine how many cases, boxes, tubes, and individual bolts will be sent to each of the 14 motorcycle shops Explain why your answer makes sense.

Directions: Estimate each quotient. Then use the standard algorithm to find the exact quotient. Express remainders as decimals.

5. $78 \div 6$	6. $352 \div 16$ 22	7. $540 \div 18$	8. $49{,}815 \div 405$	9. $578 \div 32$ 18.0625

Directions: Estimate each quotient. Then use the standard algorithm to find the exact quotient. Express remainders as decimals.

10. $4{,}635 \div 45$	11. $6{,}996 \div 212$	12. $1{,}018 \div 72$	13. $326 \div 8$	14. $40613601 \div 4263$
103		14.13888…		

Find, Fix, and Justify

15. An airplane travels 18,032 miles in 45 hours. Pablo has completed the following division problem to determine the plane's speed in miles per hour and has made a mistake. Find the mistake and explain what Pablo has done wrong. Then solve the division problem correctly.

$47.\overline{1}$
45)18032.00
−180
0320
− 315
50
− 45
50

Pablo brought down the 2 and 0 too early since 45 does not go into 0. He should have placed a 0 in the tens spot and brought down the 3, then placed another zero in the ones spot and brought down the 2.

$400.7\overline{1}$
45)18032.00
−180
03
− 0
320
− 315
50
− 45
50

The plane travels at a rate of about 400.71 mph

16. Monique has correctly found the quotient below doing long division. Use her work shown to find each product.

172
24)4128
− 24
172
− 168
48
− 48
0

a. 172×24

b. 70×24

c. 2×24

Section 0.2: Factors and Multiples

Section Overview:
The first lesson in this section in on divisibility rules, while divisibility rules are not addressed in a specific standard for 6^{th} grade, being able to draw upon them makes working with factors and multiples much easier for students, especially when dealing with larger numbers. They are also very helpful when executing the algorithm for long division. As students investigate how and why the divisibility rules work they not only come away with a quick trick for divisibility but also engage in meaningful discussions about place value and powers of ten. In the next two lessons students learn how to find the greatest common factor (GCF) of two whole numbers less than or equal to 144 and the least common multiple (LCM) of two whole number less than or equal to 12. They also engage in several application problems where finding the LCM and GCF help you reach a desired outcome. In the last few lessons students review the distributive property and how it relates to an area model. These area models are then used to help students express a sum of two whole numbers 1-100 with a common factor as a multiple of a sum of two whole numbers with no common factor.

Concepts and Skills to Master in this Section:

By the end of this section, students should be able to:

1. Determine if a number is divisible by 2, 3, 4, 5, 6, 9, or 10.
2. Find the greatest common factor of two whole numbers less than or equal to 144.
3. Find the least common multiple of two whole numbers less than or equal to 12.
4. Use the distributive property to express a sum of two whole numbers with a common factor as multiple of a sum of two whole numbers with no common factor.

0.2a Class Activity: Divisibility Rules

1. A marching band has 72 members that will march during halftime in a football game. They need to march in rows with the same number of students in each row. How many ways different ways can the band members be arranged? Explain the arrangements.
   The members can be arranged 6 ways; $1 \times 72,\ 2 \times 36,\ 3 \times 24,\ 4 \times 18,\ 6 \times 12,\ 8 \times 9$. Students might aruge that there are 12 arrangements if they account for changing the direction of the row. For example they could have 2 rows with 36 members in each row or 36 rows with 2 members in each row.

Students must find factors of 72 or numbers that "go into" 72 "evenly". Use this problem to review with your student the terms *factor*, *product*, and *divisible by*.

> Two or more numbers that are multiplied to form a **product** are called **factors**.
>
> $$\underbrace{8 \times 9}_{\text{Factor}} = \underbrace{72}_{\text{Product}}$$
>
> This shows that 8 and 9 are factors of 72 because they each divide 72, with no remainder leftover. We say that 72 is **divisible** by 8 and 9.

Often you can test if one number is divisible by another number mentally rather than doing long division to see if there is a remainder. To test a number mentally you can use a Divisibility Rule.

The explanations for each divisibility rule are explored through class discussion. You can refer to the questions prompts given to guide you if you are investigating these rules independently. Before you begin exploring the divisibility rules below it may be helpful to review with your student that the sum of two even numbers is always even and the sum of two odd numbers is always odd.

Divisibility Rule for 2

- Consider the number 2,458. How can you tell if a number is divisible by 2?
  If the number is even it is divisible by 2.
- How can you tell if the number is even?
  If it ends in 0, 2, 4, 6, or 8
- Why does this work? What about the rest of the number?

Write the number in expanded form considering place value.

$$2458 = 2000 + 400 + 50 + 8 = 2 \times 1000 + 4 \times 100 + 5 \times 10 + 8 \times 1$$

Consider each part of the number, the digit of 2 represents 2000, which equals 2×1000, this number is a multiple of 10 or power of 10.

$$2 \times 1000 = 2 \times 10^3.$$

All powers of 10 are even so 2000 is even. This is true for the 400 and 50 as well; they are also powers of 10. That means that $2000 + 400 + 50 = 2450$ is an even number because all of its addends are even. Thus $2450\ + 8$ is even because 2450 is even and 8 is even. This is argument holds true for any number regardless of the number of digits.

- What about $3,578,647$?

$$3{,}578{,}647 = \underbrace{3{,}000{,}000 + 500{,}000 + 70{,}000 + 8000 + 600 + 40}_{\text{Each of these numbers is a power of 10 so they are all even. That means that the sum of all these number is also even.}} + 7$$

This last digit is odd.

$$3{,}578{,}647 = 3{,}570{,}640 + 7$$

$$Even\ Number\ +\ Odd\ Number =\ Odd\ Number$$

2. Explain using words, pictures, examples, or equations why a number is divisible by 2 if the last digit is even and why a number is not divisible by 2 if the last digit is odd.
   Every number that has more than one digit is a sum of a number that is a power of ten and a single digit. Every power of ten is an even number. Since an $even\ + even = even$ and $even + odd = odd$ you only have to consider whether the last digit is even or odd to determine if a number is divisible by 2.

3. Write 5 numbers, each with a different number of digits, which are divisible by 2.
   You may list any 5 numbers that are even, each with a different number of digits.

The same reasoning is used to determine if a number is divisible by 10 and 5.

Divisibility Rule for 10

- Consider the number 2,450. How can you tell if a number is divisible by 10?
  If it ends in zero
- Why does this work, what about the rest of the number?

Write the number in expanded form

$$2450 = 2000 + 400 + 50$$

All of the addends are divisible by 10.

Any number than ends in 0 is a power of 10 regardless of the number of digits.

- What about $3,578,647$?

$$3{,}578{,}647 = \underbrace{3{,}000{,}000 + 500{,}000 + 70{,}000 + 8000 + 600 + 40}_{\text{Each of these numbers are divisible by 10.}} + 7$$

This last digit does not end in zero. It is not a factor of 10.

4. Explain using words, pictures, or equations why you know that a number is divisible by 10 if the last digit 0.
   If the last digit is 0 that mean that the number is a power of 10 because if you write any number in expanded form each of its addends will be divisible by 10 and thus 10 is one of its factors.

5. Write 5 numbers, each with a different number of digits, which are divisible by 10.
   Students may list any 5 numbers that end in 0, each with a different number of digits.

Divisibility Rule for 5

- Consider the number 2,455. How can you tell if a number is divisible by 5?
  If it ends in 0 or 5
- Why does this work, what about the rest of the number?

Write the number in expanded form

$$2455 = 2000 + 400 + 50 + 5$$

The first three addends are powers or multiples of ten and we know that powers of 10 are divisible by 5. The last digit is 5 so it is divisible by 5. Thus if the last digit is 5 it is divisible by 5 and if the last digit is 0 then the number is a power of 10 and it is also divisible by 5.

Any number than ends in 0 or 5 is a power of 5 regardless of the number of digits.

- What about $3,578,647$?

$$3{,}578{,}647 = \underbrace{3{,}000{,}000 + 500{,}000 + 70{,}000 + 8000 + 600 + 40}_{\text{Each of these numbers are divisible by 5 because they are powers of 10.}} + 7$$

This last digit does not end in 0 or 5, it is not divisible by 5

6. Explain in your own words why you know that a number is divisible by 5 if the last digit 0 or 5.
   If the last digit is 0 that means that the number is a power of 10 which is divisible by 5. If the last digit is 5 that means that the number is the sum of a number that is a power of 10 and 5. This makes it divisible by 5.

7. Write 5 numbers, each with a different number of digits, which are divisible by 5
   Students my list any 5 numbers that end in 5 or 0, each with a different number of digits.

Divisibility Rule for 4

- Consider the number 2,358. How can you tell if a number is divisible by 4?
  This one is not as intuitive; but a similar argument can be used using place value and powers of 100.
- Write the number in expanded form

$$2358 = \underbrace{2000 + 300}_{\text{Divisible by 4 because they are multiples of 100 which is divisible by 4}} + 50 + 8$$

Consider each addend; $2000 = 2 \times 1000$ this is divisible by 4 because we know that 1000 is divisible by 4. $300 = 3 \times 100$, this is divisible by 4 because we know that 100 is divisible by 4. In fact, any number that is a multiple of 100 is divisible by 4. Another way to look at the expression is $2358 = 2300 + 50 + 8 = 23 \times 100 + 50 + 8$

Since the first two addends are divisible by 4 then the number is divisible by 4 if the last two addends $(50 + 8)$ form a number that is also divisible by 4. We know that 58 is not divisible by 4. So 2,358 is not divisible by 4. Any number whose last two digits form a number that is divisible by 4 will be divisible by 4 regardless of the number of digits in the entire number.

- What about $3,578,644$?

$$3{,}578{,}644 = \underbrace{3{,}000{,}000 + 500{,}000 + 70{,}000 + 8000 + 600}_{\text{Each of these numbers are divisible by 4 because they are multiples of 100 which is divisible by 4.}} + \underbrace{40 + 4}_{\text{Consider the last 2 digits. 44 is divisible by 4}}$$

This number is divisible by 4 because the last two digits form a number that is divisible by 4.

8. Explain in your own words how you know that a number is divisible by 4 if the last two digits form a number that is divisible by 4.
   If the last two digits are divisible by 4 then the number is divisible by 4 because any digits before the last two represent numbers that are multiples of 100 and all multiples of 100 are divisible by 4.

9. Write 5 numbers, each with a different number of digits, which are divisible by 4.
   Sample answers are given: 16, 216, 2216, 22016, 220016. Students may list any 5 numbers where the last two digits form a number that is divisible by 4. Each number must have a different number of digits.

Divisibility Rule for 3 and 9

- Consider the number 1116. How can you tell if a number is divisible by 3 or 9?
- Write the number in expanded form

$$1116 = 1000 + 100 + 10 + 6$$

We have a sum of several powers of 10 and 6. However 3 and 9 do not divide any power of 10 or 100 or 1000, etc. Thus we cannot use the same reasoning as we have for our previous divisibility rules. But, do note that every power of ten can be expressed as a sum of a number that is divisible by 3 and 9 plus 1.

$$10 = 9 + 1 \quad 100 = 99 + 1 \quad 1000 = 999 + 1 \quad 10{,}000 = 9999 + 1 \quad etc$$

All of the addends that contain only digits of 9 are divisible by 9 and 3.

If we rewrite our number using addends with digits of 9 we get $1116 = (999 + 1) + (99 + 1) + (9 + 1) + 6$. We know that all of the addends that contain only digits of 9 are divisible by 3 and 9 so we only have to look at the addends that are leftover to see if they are divisible by 3 or 9. The leftovers are 1, 1, 1, and 6. Note that these leftover are the actual digits in the number. If we sum these leftovers we get $1 + 1 + 1 + 6 = 9$. Since this sum is divisible by 9 then the entire number is divisible by 9, likewise if this sum is divisible by 3 then the entire number is divisible by 3.

Any number whose digits add to a number that is divisible by 9 is divisible by 9 and any number whose digits sum to a number that is divisible by 3 is divisible by 3 regardless of the number of digits.

- What about 3567?

Consider the sum of the digits. $3 + 5 + 6 + 7 = 21$. 21 is divisible by 3 so this number is divisible by 3. However, 21 is not divisible by 9 so this number is not divisible by 9.

10. Write 3 numbers, each with a different number of digits, which are divisible by 3.
    Sample answers are given: 111, 6222, 93483 Students may list any 3 numbers whose digits sum to a number that is divisible by 3. Each number must have a different number of digits.

11. Write 3 numbers, each with a different number of digits, which are divisible by 9.
    Sample answers are given: 117, 6822, 93483 Students may list any 3 numbers whose digits sum to a number that is divisible by 9. Each number must have a different number of digits.

Divisibility Rule for 6

- Consider the number 2358. How can you tell if it is divisible by 6?
  Since 2 and 3 are both prime factors of 6 if a number is divisible by 2 and 3 it is divisible by 6.
  Use divisibility tests for 2 and 3, if both are satisfied then the number is divisible by 6.
- Is 2358 divisible by 2?
  Yes, it is even
- Is 2358 divisible by 3?
  Yes the sum of its digits 2+3+5+8=18 is divisible by 3.
  Thus 2358 is divisible by 6. Any number is divisible by 6 if it is divisible by 2 and 3 regardless of its number of digits.
- What about 39,441?
  It is not divisible by 2 because it is odd, it is divisible by three because the sum of its digits is 21. However it must by divisible by 2 and 3 in order for it to be divisible by 6, so it is not divisible by 6.

12. Write 3 numbers, each with a different number of digits, which are divisible by 6.
    Sample answers are given: 12, 24, 360
    Students may list any 3 numbers that are divisible by 2 and 3. Each number must have a different number of digits.

13. Summarize the Divisibility Rules below by completing each statement.

A number is divisible by:

- 2 if the last digit is even.
- 3 if the sum of the digits is divisible by 3.
- 4 if the last two digits form a number that is divisible by 4.
- 5 if the last digit is 0 or 5.
- 6 if it is divisible by both 2 and 3.
- 9 if the sum of the digits is divisible by 9.
- 10 if the last digit is 0.

15. Determine if each given number is divisibly by 2, 3, 4, 5, 6, 9, or 10. Justify your answer.
    a. 5040
       2-the number is even; 3-the sum of the digits is divisible by 3; 4-the last two digits form a number that is divisible by 4; 5-the number ends in 0; 6-the number is divisible by 2 and 3; 9-the sum of the digits is divisible by 9; 10-the number ends in 0.
    b. 955
       5-the number ends in 5

16. Circle all the numbers that are factors of 15,033,444.

2 3 4 5 6 9 10

0.2a Homework: Divisibility Rules

Directions: Use divisibility rules to determine whether each number is divisible by 2, 3, 4, 5, 6, 9, or 10. Justify your response.

1. 6,480	2. 135 3, 5, 9	3. 24,640
4. 549	5. 10,523 None	6. 58,762

7. Circle all the numbers that are factors of 6,420.

2 3 4 5 6 9 10

For problems 8-10 find a number that matches each description

8. A 3-digit number that is divisible by 2, 3, and 6. How do you know?	9. A 3-digit number that is divisibly by 3 and 4. How do you know?
10. A 4-digit number that is divisibly by 5 and 9 but not divisibly by 10. How do you know? Any 4-digit number that has a last digit of 5 and the sum of its digits is divisible by 9. Sample Answer: 9945	11. A 4-digit number that is not divisible by 2, 3, 5, or 10. Any 4-digit number that is an odd number that does not have 5 as its last digit and the sum of all of its digits does not equal 3. Sample Answer: 4523

12. Determine whether each statement is true or false. Justify your answer.
    a. If a number is divisibly by 9 it is divisible by 3.

    b. If a number is divisibly by 3 is it divisible by 9.

13. Emily delivers newspapers on Sunday mornings. Each Sunday she must deliver 112 newspapers, she likes to organize them into stacks of equal numbers and deliver one stack at a time. What size of stacks can she make?
    She can make 2 stacks of 56, 4 stacks of 28 or 8 stacks of 14. Or the reverse, 56 stacks of 2, 28 stacks of 4 or 14 stacks of 8.
14. You are the senior member of a team of 9 realtors in an agricultural real estate agency. When a sale is executed all partners receive equal shares, and if there is a remainder, the remainder goes to you as senior partner. In the month of October last year, the following sales were made:
    12,017 acres 77,760 acres 11,010 acres 40,500 acres.
    How many more acres did you get over the rest of the sales team?

0.2b Class Activity: Greatest Common Factor

1. Giada is planting tulip bulbs in a flower garden at her plant nursery. She has 36 tulip bulbs that she would like to plant. She wants to plant them in rows of equal size and use all her bulbs. How many rows of tulips can she plant? List all the possible combinations by organizing your numbers in a table and explain your answer.

Number of Rows	Number of Tulip Bulbs in a Row
1	36
2	18
3	12
4	9
6	6
9	4
12	3
18	2
36	1

Since she wants the same number of tulips per row the number of tulips that go into a row must be a factor of 36. Likewise the number of rows is also a factor of 36.

As students work through the task review important concepts about factors.

- What are factors?
- How do you know if a number is a factor of another number?
- How do you know if you have found all of the factors of a given number?

2. Giada also has 24 daffodil bulbs that she would like to plant. Similarly she wants to plant the daffodils in rows of equal size and use all of her bulbs. How many rows of daffodils can she plant? List all combinations by organizing your numbers in a table and explain your answer.

Number of Rows	Number of daffodils per row
1	24
2	12
3	8
4	6
6	4
8	3
12	2
24	1

Since she wants the same number of daffodils per row the number of daffodils that go into a row must be a factor of 24. Likewise the number of rows is also a factor of 24.

3. Giada decides to just plant the tulip and daffodils next to each other since they bloom around the same time of year and look so pretty together. She would like to use all of her bulbs and wants each row to have an equal number of tulips and an equal number of daffodils. How many rows can she make? List all combinations by organizing your numbers in a table. Justify your answer.

Number of Rows	Number of tulips per row	Number of daffodils per row
1	36	24
2	18	12
3	12	8
4	9	6
6	6	4

The number of rows must both be a factor of 24 and 36.
Discuss what common factors are and how to find them.
The common factors show all of the possible row combinations. The table shows these combinations along with the number of each type of flower that would go in each row.

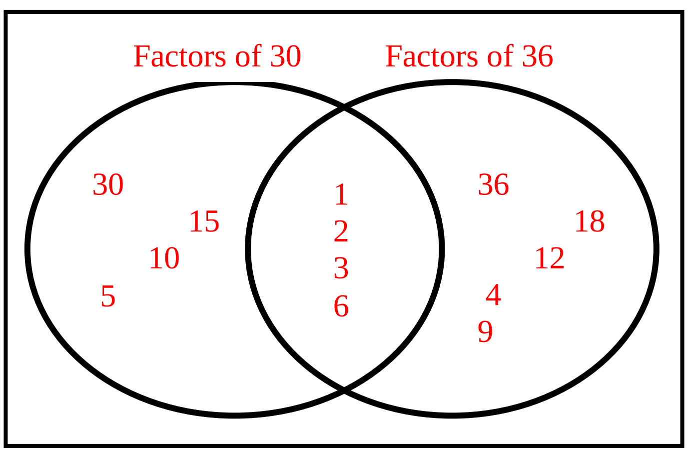
4. What is the greatest number of rows she can make by combining the tulips and daffodils together?
From the table you can see that the greatest number of rows is 6. It will have 6 tulips and 4 daffodils. The greatest number of rows that she can make corresponds to the greatest common factor. Discuss what a greatest common factor (GCF) is and how to find it.

5. In the box write down what a Greatest Common Factor is.

Greatest Common Factor (GCF):
The Greatest Common Factor between two numbers is the greatest of all of their common factors.

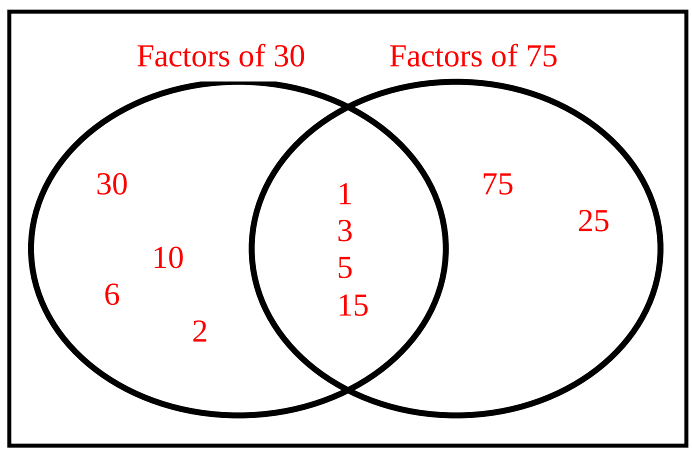
6. Use a Venn Diagram to find the GCF of pair of numbers.

a. 30 and 36



The GCF of 30 and 36 is 6

b. 30 and 75



The GCF of 30 and 75 is 15

Encourage students to use the Divisibility Rules as they find factors of 30, 36, and 75.

7. Make a list of factors to find the GCF of each pair of numbers.

a. 16 and 56
8

b. 21 and 45
3

c. 32 and 54
2

d. 25 and 50
25

e. 51 and 85
17

f. 40 and 63
1

Sometimes it can be time consuming to list all of the factors of a number especially if it is a really big number. Rather than writing out a list of factors for each number you can use the each number's prime factorization to find the greatest common factor.

Review with students what prime and composite numbers are and how you can determine if a number is prime or composite.

To find the ***prime factorization*** for a number write the number as a product of its prime factors. To do this we will make a ***factor tree.***

8. Make a factor tree to write the prime factorization for each number.

a. 60

60
6 10
2 3 2 5

$60 = 2 \cdot 2 \cdot 3 \cdot 5$

b. 88

88
8 11
4 2
2 2

$88 = 2 \cdot 2 \cdot 2 \cdot 5$

c. 136

136
4 34
2 2 2 17

$136 = 2 \cdot 2 \cdot 2 \cdot 17$

d. 96

96
3 32
4 8
2 2 2 4
2 2

$96 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3$

10. Find the prime factorization for each number in a given pair. Then use the prime factorization to find the GCF.

Once students have written out the prime factorization ask them how they think they can find the GFC from the prime factorizations. Some probing questions are below.

- What do we know about a GCF? It must be a factor common to both numbers.
- Can you circle the common prime factors for each number?
- How can use the common prime factors to get the greatest common factor? Multiply all the common prime factors together.
- It may be helpful to once again use a Venn diagram, but with the prime factors.

a. 12 and 56

12
3 4
2 2

$12 = 2 \cdot 2 \cdot 3$

56
7 8
2 4
2 2

$56 = 2 \cdot 2 \cdot 2 \cdot 7$

Common prime factors are 2 and 2

$GCF = 2 \cdot 2 = 4$

b. 27 and 63

27
3 9
3 3

$27 = 3 \cdot 3 \cdot 3$

63
7 9
3 3

$63 = 3 \cdot 3 \cdot 7$

Common prime factors are 3 and 3

$GCF = 3 \cdot 3 = 9$

c. 72 and 84

72
8 9
2 4 3 3
2 2

$72 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3$

84
3 28
4 7
2 2

$84 = 2 \cdot 2 \cdot 3 \cdot 7$

Common prime factors are 2, 2, and 3

$GCF = 2 \cdot 2 \cdot 3 = 12$

d. 112 and 96

112
4 28
2 2 4 7
2 2

$112 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 7$

96
3 32
4 8
2 2 2 4
2 2

$96 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3$

Common prime factors are 2,2,2,2

$GCF = 2 \cdot 2 \cdot 2 \cdot 2 = 16$

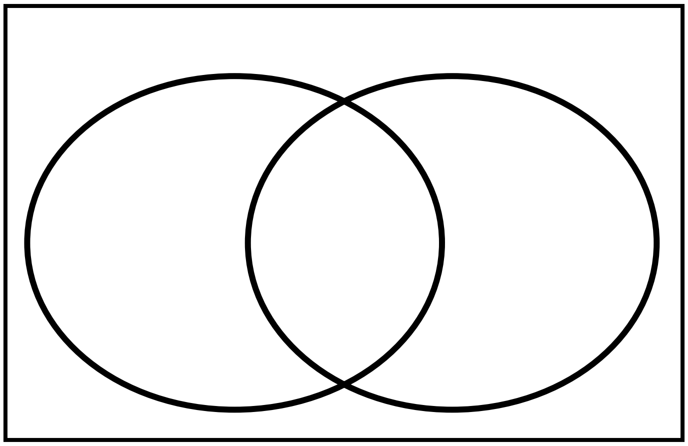
11. Valerie is assembling "goodie" bags for her friends. She has 92 trading cards and 23 mood rings to put into the bags. What is the greatest number of bags that she can assemble with no items left over? How many of each item will be in each bag?
She can assemble 23 bags; each bag will have 23 trading cards and 4 mood rings.

12. There are 60 girls and 48 boys that want to participate in a STEM competition. If each team must have the same ratio of girls to boys what is the greatest number of teams than can participate? How many girls and boys will be on each team?
There will be 6 teams; each team will have 10 girls and 8 boys.

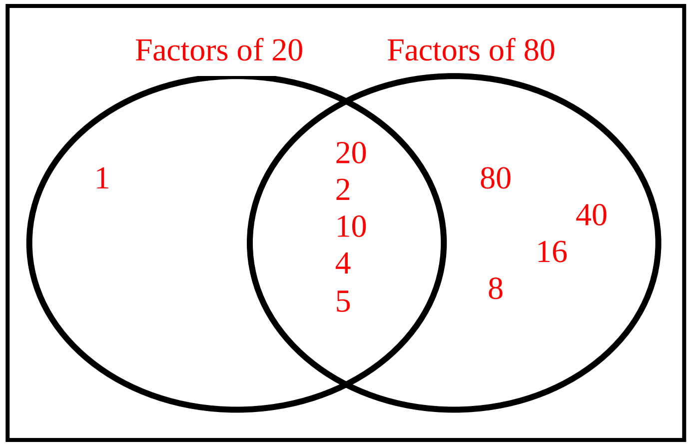
0.2b Homework: Greatest Common Factor

Make a Venn diagram to find the GCF of each pair of numbers.

1. 35 and 40



2. 20 and 80



$GCF = 20$

Make a list to find the GCF of each pair of numbers.

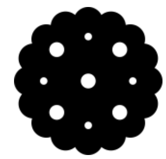
3. 30 and 50	4. 20 and 64 $GCF = 4$
5. 45 and 60	6. 14 and 35 $GCF = 1$

Find the GCF of each pair of numbers by writing each number as its prime factorization.

7. 42 an 70	8. 96 and 144 $GCF = 48$

9. 15 and 75	10. 85 and 70
	$GCF = 5$

11. A caterer has 90 mini macaroons and 120 gingersnaps to arrange on plates. He wants each plate to have the same number of macaroons and each plate to have the same number of gingersnaps.



a. What is the largest number of plates possible?

b. How many macaroons and how many gingersnaps will be on each plate?

12. In a parade, 36 members of a cheerleading squad are to march in front of 120 members of the high school band. Each row is to have the same number of cheerleaders and each row is to have the same number of band members.

a. Find the greatest number of rows possible for the parade?
The greatest number of rows possible is 12.

b. How many cheerleaders and how many band members will be in each row?
There will be 3 cheerleaders and 10 band members per row.

13. Laney is covering the surface of a table with equal-sized tiles. The table is 30 inches long and 24 inches wide.

    a. What is the largest square tile that Laney can use and not have to cut any tiles?

    b. How many tiles will Laney need?

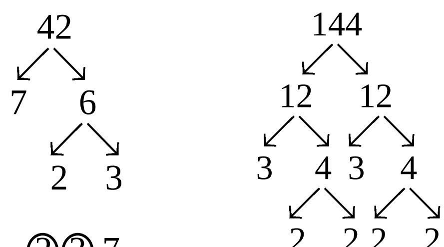
14. Circle the pairs of numbers that have a GCF of 15?

- 30 and 60
- 45 and 75
- 21 and 45
- 10 and 15

17. Write a pair of numbers whose GCF is 10.
    Sample answer: 20 and 30. This problem is intended to started to get students to think about multiples.

18. Write a pair of numbers whose GCF is 8.

19. Find, Fix, and Justify
    Find the error in finding the GCF of 42 and 144 in the problem shown. Explain why it is wrong and fix the mistake.



$42 = 2 \cdot 3 \cdot 7$

$144 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3$

$GCF = 2 \cdot 3 \cdot 2 \cdot 3 = 6 \cdot 6 = 36$

20. True or False. Justify your answer with an example or counter-example.
    a. The GCF of two even numbers is always 2

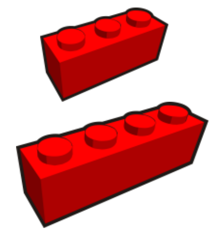
    b. The GCF of two prime numbers is always 1
    True, the GCF of two prime numbers is always one because the only factors of prime numbers are one and the number itself.

    c. Can the GCF of two numbers ever be one of the numbers?
    Yes; the GCF of 6 and 12 is 6.

    d. Can the GCF of two numbers ever be greater than one of the numbers?

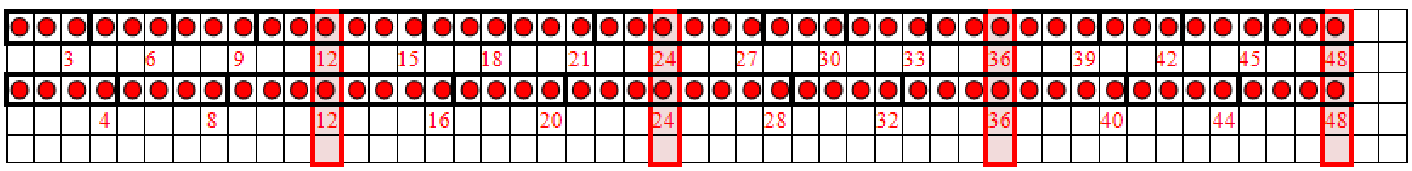
0.2c Class Activity: Least Common Multiple

1. Brooks is making a house out of LEGOS. He snaps two rows of LEGOS down onto his mat. In one row he only uses LEGOS with 3 studs; in another row he only uses LEGOS with 4 studs. His mat is only 50 studs long and he wants to make the rows the same length.



   a. How many of each type of LEGO, 4 stud and 3 stud, can he use in each row, remembering that the rows need to be the same length? If needed use the grid paper below to help you answer.

As students work through this task review what multiples are. Students must recognize that they must identify the multiples of 3 and 4. The grid paper will be helpful for students if they do not recognize this immediately. They might begin by drawing each row of LEGOS and mark off where the rows are equal as shown below. Relate this to multiples of 3 and 4 and then to common multiples of 3 and 4.



Other students might immediately reason that the 3 stud row can have the total stud length be any multiple of 3 that is less than 50: 3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36, 39, 42, 45, 48, 51
Likewise the 4 stud row can have a total stud length be any multiple of 4 that is less than 50: 4, 8, 12, 16, 20, 24, 28, 32, 36, 40, 44, 48, 52

In order for the rows to be the same length he must makes rows that have the number of studs be common multiples of 3 and 4.

Total number of studs per row (common multiples of 3 and 4)	12	24	36	48
Number of 3 studs block per row	4	8	12	16
Number of 4 stud blocks per row	3	6	9	12

   b. What is the smallest possible stud length for the rows? How many of each block will be in these rows.

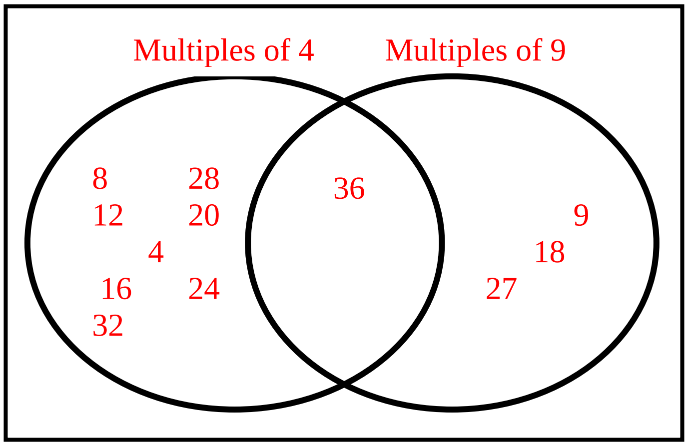
The smallest possible stud length will be the least common multiple of 3 and 4, this is 12. The 3 stud row will have 4 blocks and the 3 stud block will have 3 blocks. Discuss what Least Common Multiple (LCM) is as you answer this question.

2. In the box write down what a Least Common Multiple is.

Least Common Multiple (LCM):
The Least Common Multiple between a pair of numbers is the smallest multiple that the two numbers have in common.

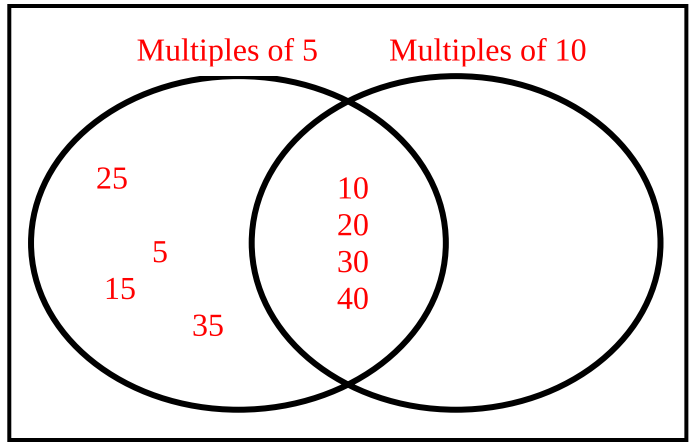
3. Use a Venn diagram to find the LCM of pair of numbers.

a. 4 and 9



The LCM of 4 and 9 is 36

b. 5 and 10



The LCM of 5 and 10 is 10

4. How are factors related to multiples?
It is important for students to understand that factors and multiples share a dual relationship. You can use an example to illustrate this; if you list the multiples of 3 you will note that 3 is a factor of all of its multiples.
Multiples of 3: 3, 6, 9, 12, 15, 18, …
Factors of 3: 3, 1
Factors of 6: 1, 2, 3, 6
Factors of 9: 1, 3, 9
Factors of 12: 1, 2, 3, 4, 6, 12
Factors of 15, 1, 3, 5, 15
Factors of 18: 1, 2, 3, 6, 9, 18

5. Make a list of multiples to find the LCM of each pair of numbers.

a. 9 and 12
36

b. 8 and 6
24

c. 12 and 3
12

d. 4 and 10
20

e. 6 and 11
66

f. 9 and 10
90

Sometimes it can be time consuming to list all of the multiples of a number. Rather than writing out a list of multiples for each number you can use the each number's prime factorization to find the least common multiple.

Example

To find the LCM of 10 and 12 we begin by writing each number as its prime factorization.

10
5   2

$10 = 2 \cdot 5$

12
3   4
2   2

$12 = 2 \cdot 2 \cdot 3$

Recall we are looking for a multiple of both 10 and 12. That means that 10 and 12 must both be factors of this number we are looking for. Thus this number's prime factorization must include the prime numbers that are in the prime factorizations of 10 and 12.

$$10 = 2 \cdot 5$$
$$12 = 2 \cdot 2 \cdot 3$$

To be a multiple of 10 its prime factorization must include a 2 and a 5 and to be a multiple of 12 its prime factorization must include 2, 2, and 3. In order to be a multiple common to both 10 and 12 we must meet the factor criteria for both numbers by multiplying these factors together. However we want the least common multiple. That means that we only need to include the 2 two times.

$$2 \cdot 2 \cdot 3 \cdot 5 = 60$$

10

$2 \cdot 2 \cdot 3 \cdot 5 = 60$

12

In other words, the smallest number that meets both of these conditions is $2 \cdot 2 \cdot 3 \cdot 5 = 60$. Students might wonder why we don't include the other two. We don't need to because we have accounted for all the twos needed and we are looking for the "least" common multiple.

6. Find the prime factorization for each number in a given pair. Then use the prime factorization to find the LCM.

a. 9 and 12	b. 8 and 18
12 → 3, 4; 4 → 2, 2 9 → 3, 3 $12 = 2 \cdot 2 \cdot 3$ $9 = 3 \cdot 3$ A multiple of 12 must include factors of $2 \cdot 2 \cdot 3$ and a multiple of 9 must include factors of $3 \cdot 3$. The smallest multiple that meats both criteria is $LCM = 2 \cdot 2 \cdot 3 \cdot 3 = 36$	18 → 2, 9; 9 → 3, 3 8 → 2, 4; 4 → 2, 2 $18 = 2 \cdot 3 \cdot 3$ $8 = 2 \cdot 2 \cdot 2$ A multiple of 18 must include factors of $2 \cdot 3 \cdot 3$ and a multiple of 8 must include factors of $2 \cdot 2 \cdot 2$. The smallest multiple that meats both criteria is $LCM = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 = 72$

c. 20 and 50 $LCM = 100$	d. 12 and 20 $LCM = 60$

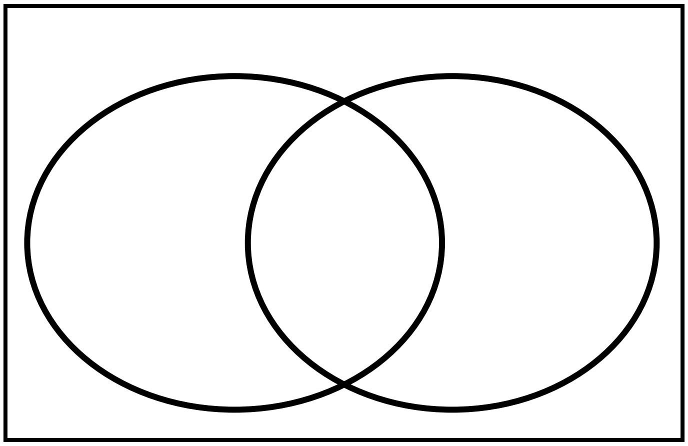
7. Two small gears are aligned by a mark drawn down the center of one gear to the center of the other gear. The first gear has 6 teeth and the second gear has 20 teeth. How many revolutions does the first gear need to make so the center lines match up again?
   The first gear will need to make 60 rotations before the center lines match up again.

8. Plastic forks come in packages of 10 and plastic knives come in packages of 8. If you want one fork for each knife at a party with none leftover what is the least amount of forks and knives that you need to buy? How many packages of forks and knives will you buy?
   All together you will need to buy 40 forks and 40 knives. This will be 4 packages of forks and 5 packages of knives.

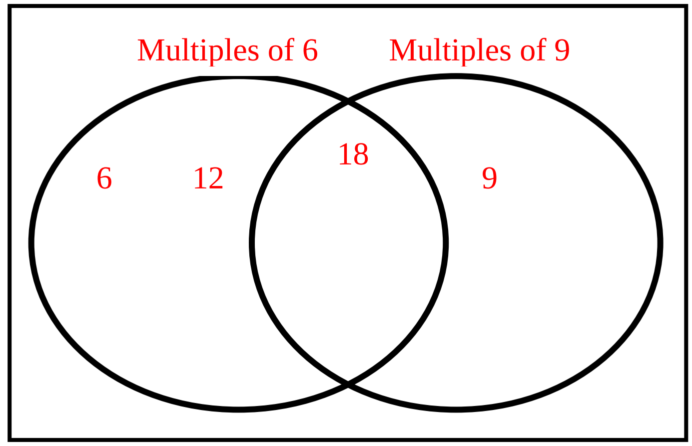
0.2c Homework: Least Common Multiple

Make a Venn diagram to find the LCM of each pair of numbers.

1. 2 and 12



2. 6 and 9



$GCF = 20$

Make a list to find the LCM of each pair of numbers.

3. 6 and 15 $LCM = 30$	4. 12 and 18
5. 5 and 8 $LCM = 40$	6. 20 and 4

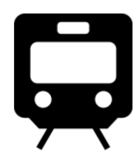
Find the LCM of each pair of numbers by writing each number as its prime factorization.

7. 9 an 60 $LCM = 180$	8. 18 and 45

9. 12 and 18	10. 30 and 50
$LCM = 36$	

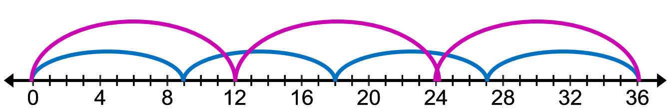
11. One racecar driver can circle a one-mile track in 25 seconds. Another driver takes 20 seconds to circle the same track. If they both start at the same time, in how many seconds will they be together again at the starting line? How many laps will each car have made?

12. Each morning starting at 7:00 am a city train makes a stop at a certain street corner every 15 minutes. On that same corner a city bus makes a stop every 12 minutes. When will the train and the bus be at this corner at the same time?
After 60 minutes a bus and train will be at the corner at the same time. That will be at 8:00 am

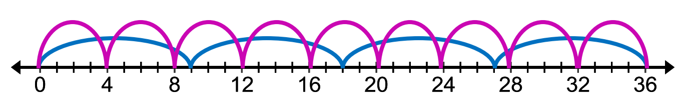


13. Which model represents an LCM that is different than the other 3? Justify your answer.

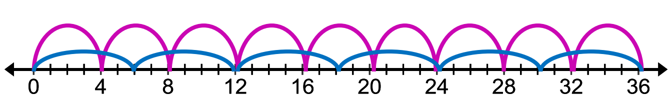
a.



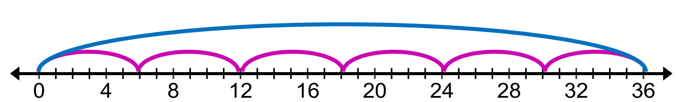
b.



c.



d.



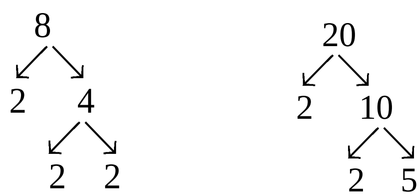
14. Circle the pairs of numbers that have a LCM of 80?

- 16 and 20
- 5 and 16
- 8 and 10
- 20 and 4

15. Write a pair of numbers whose LCM is 15.
Sample answer: 3 and 5

16. Write a pair of numbers whose LCM is 60.

17. Find, Fix, and Justify
Find the error in finding the LCM of 8 and 20 in the problem shown. Explain why it is wrong and fix the mistake.



$8 = 2 \cdot 2 \cdot 2$ $\quad$ $20 = 2 \cdot 2 \cdot 5$

$LCM = 2 \cdot 2 \cdot 5 = 20$

18. True or False. Justify your answer with an example or counter-example.

a. The LCM of two different prime numbers is their product.
True, the prime factorization of a prime number is just the number. Thus the LCM will be the product of the two prime numbers. LCM of 7 and 11 is 77.

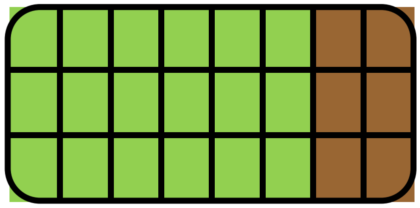
b. The LCM of two numbers will always be bigger than both of the numbers.

c. If two numbers do not contain any factors in common, then the LCM of the two numbers is 1.

d. The LCM of two numbers is greater than the GCF of the numbers.
True, if the GCF is one of the given numbers then the LCM will automatically be the greater number. GCF of 5 and 10 is 5, the LCM of 5 and 10 is 10.

0.2d Class Activity: The Distributive Property

Gordon and Cynthia have made a pan of brownies and have cut them into squares. They have also decided to frost the brownies with mint frosting but have left some of the pieces unfrosted for their friends that don't like frosting as shown.



The purpose of this task is for students to review the distributive property and how it relates to an area model.

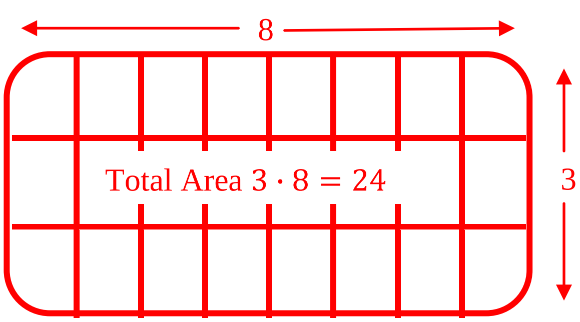
1. How many total brownies are there? Write down a mathematical sentence to show how you arrived at your answer.
   Student answers will vary.
   $3 \times 8 = 24$ This corresponds to the number of rows times the number of columns (Gordon's method)
   $8 + 8 + 8 = 24$ Each row has eight brownies; there are three rows of brownies.
   $3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 = 24$ Each column has 3 brownies; there are 8 columns of brownies
   $3 \times 6 + 3 \times 2 = 18 + 6 = 24$ The number of frosted brownies plus the number of unfrosted brownies. (Cynthia's Method)
2. Gordon states that to find the total number of brownies in the pan he counted the number of rows of brownies and multiplied that number by the number of columns of brownies. Write down a mathematical sentence that represents Gordon's thinking.
   $3 \times 8 = 24$

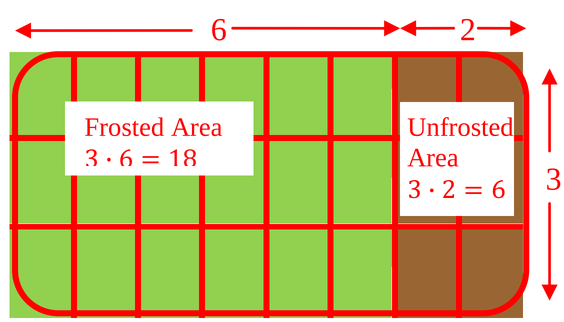
3. Cynthia states that she found the number of frosted brownies first and then she found the number of unfrosted brownies. Once she found the number of each she added them together to find the total. Write down a mathematical sentence that represents Cynthia's thinking.
   $3 \cdot 6 + 3 \cdot 2 = 18 + 6 = 24$

4. Explain why Cynthia and Gordon each arrived at the same answer.

Discuss the connection between the following equations and how they relate to the area model and the distributive property.

$3 \cdot 8 = 24$ $\quad$ $3(6 + 2) = 3 \cdot 8 = 24$ $\quad$ $3(6 + 2) = 3 \cdot 6 + 3 \cdot 2 = 18 + 6 = 24$





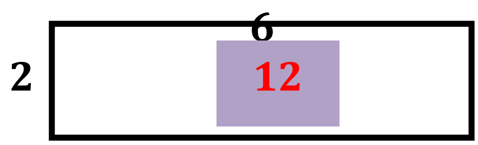
Total Area: $18 + 6 = 24$

Cynthia and Gordon arrived at the same answer because of the Distributive Property. Cynthia's mathematical sentence is equal to Gordon's. $3 \cdot 6 + 3 \cdot 2 = 3(6 + 2) = 3(8) = 24$

Name the factor, product, or addend that is missing from the area model. Then write a mathematical equation that shows the multiplication that the area model represents. Models are not drawn to scale.

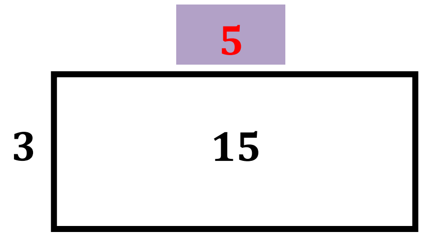
Finding the missing factors and addends in the models below will prepare your students for the next lesson where they will use the distributive property to express a sum of two whole numbers with a common factor as a multiple of a sum of two whole numbers with no common factor. (6.NS.4)

5.



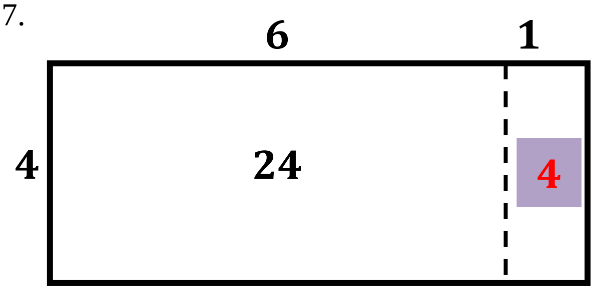
$2 \cdot 6 = 12$

6.



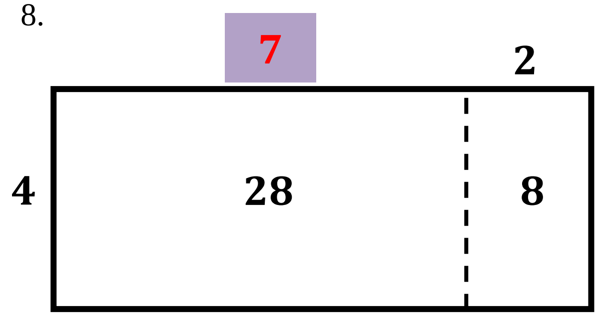
$3 \cdot 5 = 15$

7.



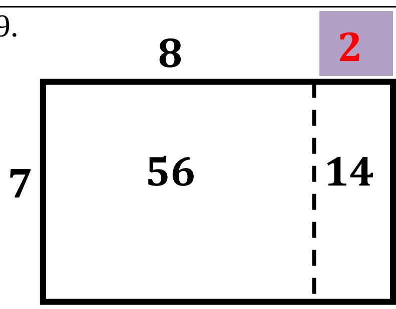
$4 \cdot 6 + 4 \cdot 1 = 24 + 4 = 28$
or $4(6 + 1) = 4(7) = 28$

8.



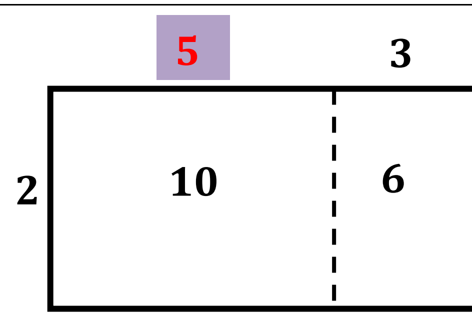
$4 \cdot 7 + 4 \cdot 2 = 28 + 8 = 36$
or $4(7 + 2) = 4(9) = 36$

9.



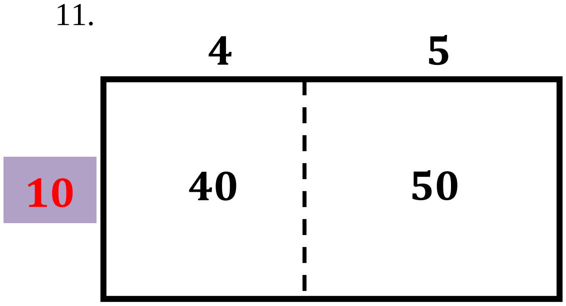
$7 \cdot 8 + 7 \cdot 2 = 56 + 14 = 70$
or $7(8 + 2) = 7(10) = 70$

10.



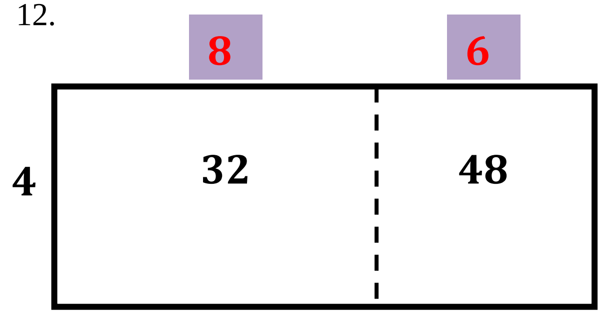
$2 \cdot 5 + 2 \cdot 3 = 10 + 6 = 16$
or $2(5 + 3) = 2(8) = 16$

11.



$10 \cdot 4 + 10 \cdot 5 = 40 + 50 = 90$
or $10(4 + 5) = 10(9) = 90$

12.



$4 \cdot 8 + 4 \cdot 6 = 32 + 24 = 56$
or $4(8 + 6) = 4(14) = 56$

13. Show how the Distributive Property works using the expressions given below.

a. $4(5+7) = 4(5) + 4(7) = 20 + 28 = 48$

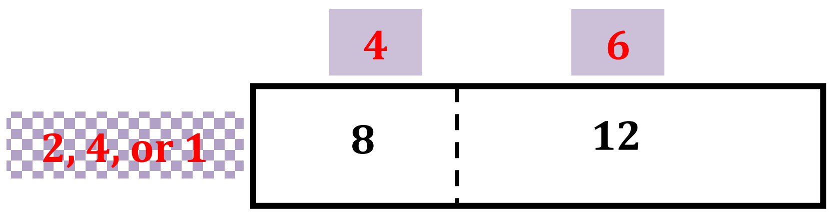
b. $9(7-2) = 9(7) + 9(-2) = 63 + (-18) = 45$

The Distributive Property
To multiply a number by a sum or difference, multiply each number in the sum or difference by the number outside the parentheses.

$$a(b+c) = ab + ac$$
$$a(b-c) = ab - ac$$

For the next few problems students must focus on finding common factors of the two numbers in the given sum. It is okay if students do not immediately identify the greatest common factors at this point. You will notice that there are no problems with subtraction. In 6th grade students are only required to factor a sum of two whole numbers.

14. Lou states that for the area model below the number behind the checkered box is 2.



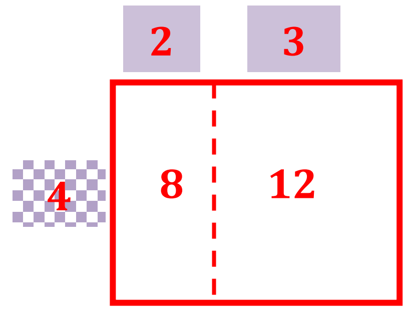
a. Is Lou correct? If so, what must the other missing numbers be? Write a mathematical sentence that describes this area model.
The number behind the checkered box can be 2. If it is, then the number above the 8 must be 4 and the number above the 12 must be 6.
$2(4+6) = 8 + 12 = 20$

b. Are there any other numbers that could be behind the checkered box? If so, what would the other missing numbers be? Write mathematical sentences to describe the area model with these other dimensions as well.
The number behind the checkered box could also be 4. If it is then the number above the 8 must be 2 and the number above the 12 must be 3.
$4(2+3) = 8 + 12 = 20$

The number behind the checkered box could also be 1. If it is then the number above the 8 must be 8 and the number above the 12 must be 12. Be sure to discuss this possibility.
$1(8+12) = 8 + 12 = 20$

c. How do these other numbers affect the length and width of the rectangle? Would the area change? Why or why not?

If the number behind the checkered box is 4 the width of the rectangle would increase (double to be precise) and the length would decrease. The area would stay the same as shown in the model.

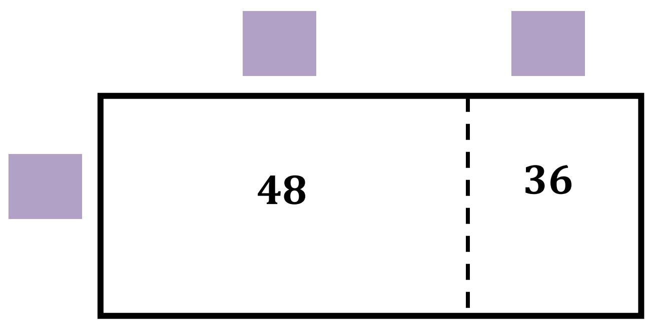


If the number behind the checkered box is 1 the width of the rectangle would decrease and the length would increase. The area would stay the same as shown.

Point out that the area does not change because all of our mathematical expressions are equivalent, in this example they all equal 20.

$$2(4+6) = 4(2+3) = 1(8+12) = 8+12 = 20$$

13. Find the missing numbers for the area problem below. List all possible combinations and write a mathematical sentence for each combination.



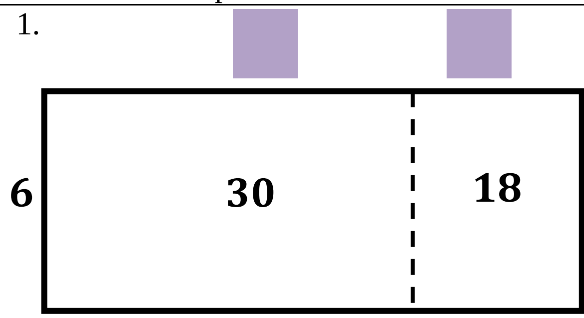
$1(48+36) = 48+36 = 84$
$2(24+18) = 48+36 = 84$
$3(16+12) = 48+36 = 84$
$4(12+9) = 48+36 = 84$
$6(8+6) = 48+36 = 84$
$12(4+3) = 48+36 = 84$

At this point it is okay if students are still experimenting or using “guess and check” to find the missing factors and factor addend. They should begin to recognize that they are looking for common factors between the two addends in the sum.

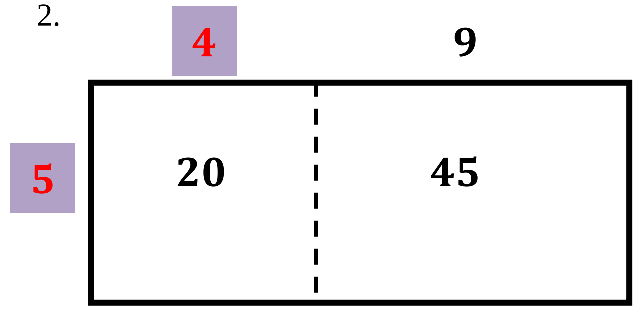
0.2d Homework: The Distributive Property

Name the factor, product, or addend that is missing from the area model. Then write a mathematical equation that shows the multiplication that the area model represents. Models are not drawn to scale.

1.

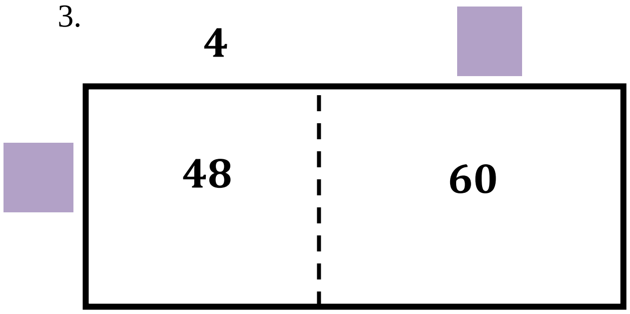


2.

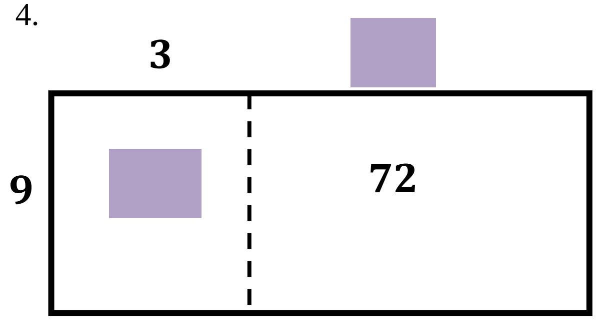


$5(4+9) = 20 + 45 = 65$

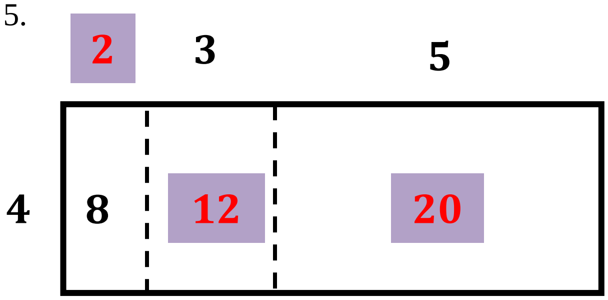
3.



4.

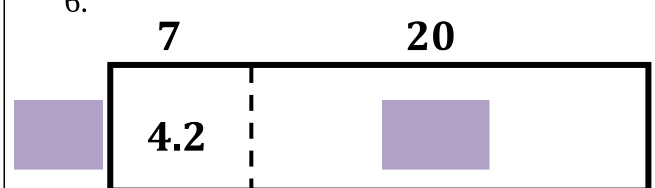


5.



$4(2+3+5) = 8 + 12 + 5 = 25$

6.



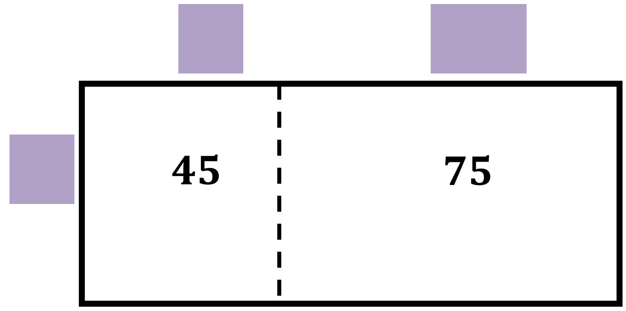
7. Circle all the expression that are equivalent to $12 + 30$

$5(2+6)$	$2(6+15)$	$1(12+30)$	$10(2+3)$
$6(2+5)$	$10(2+3)$	$4(3+8)$	$3(4+10)$

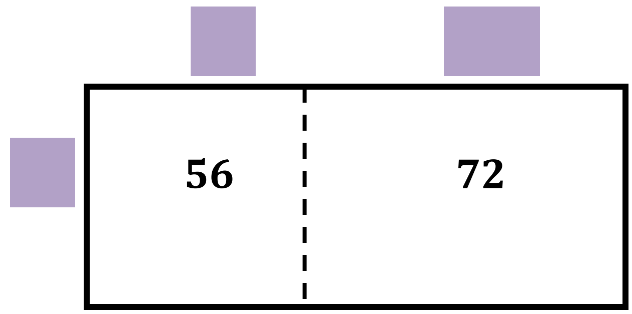
Directions: Find the missing numbers for the area problem below. List all possible combinations and write a mathematical sentence for each combination.

8.

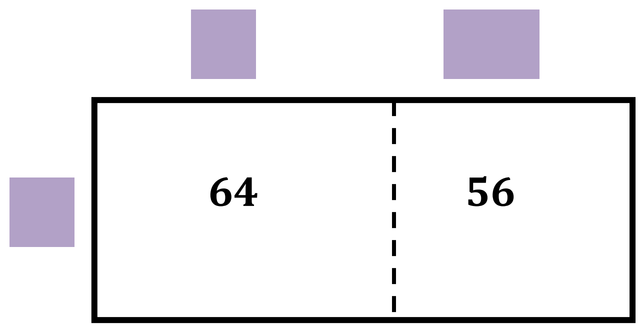
$1(45+75)=45+75=120$
$3(15+25)=45+75=120$
$5(9+15)=45+75=120$
$15(3+5)=45+75=120$



9.

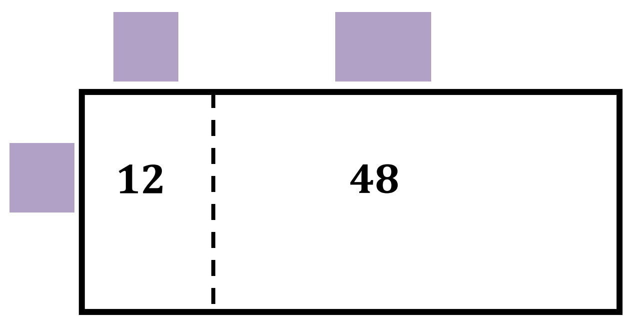


10.

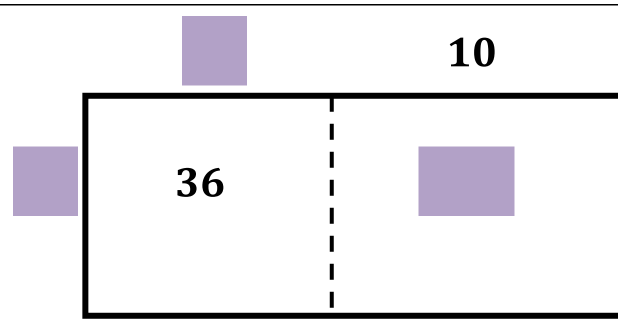


11.

$1(12+48)=12+48=60$
$2(6+24=12+48=60$
$3(4+16)=12+48=60$
$4(3+12)=12+48=60$
$6(8+2)=12+48=60$
$12(3+4)=12+48=60$



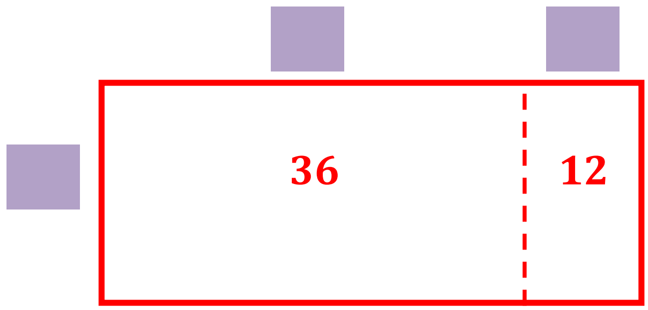
12.



0.2e Class Activity: Using the Distributive Property To Find Equivalent Expressions

In this lesson students begin to make a connection between finding missing numbers in a model to the process of "factoring". In other words; to find the missing numbers they must find a common factor between the two addends in the sum. You can write an equivalent expression to the sum/area as the product of this common factor and a sum of two addends. They will eventually learn that the desired equivalent expression is the one with the GCF as one of the factors, as it is the most useful in the future when rewriting or simplifying expressions or fractions with polynomials.

1. Use the distributive property to write all the equivalent expressions for the sum of $(36 + 12)$. If needed draw a model to reference. Encourage students to try and find the equivalent expressions by just looking at the sum and not using the model. However, the model may be needed to reference so that students clearly understand what terms you are discussing.



Discuss how you can find the equivalent expression without the model. The statements below can aide you in your discussion.

- If you are given just the sum of $36 + 12$ without the model how would you determine what number represents the width on the rectangle (the number outside the parentheses)? This number must be common factor to 36 and 12.
- Use divisibility rules to quickly determine common factors. Start with a common factor of 2. How do you know what addends represent the length of the rectangle (the numbers in the sum inside the parentheses)? These addends are the other factors that multiply with the factor of 2 to get a product of 36 and 12. What number times 2 is 36? What number times 2 is 12? Our addends are 18 and 6. Thus $(36 + 12) = 2(18 + 6) = 50$.
- Move onto the common factor of 3 and use the same reasoning to come up with another equivalent expression. $(36 + 12) = 3(12 + 4) = 50$.
- Repeat the same reasoning with all common factors of 36 and 12.
  $1(36 + 12) = 36 + 12 = 50$
  $2(18 + 6) = 36 + 12 = 50$
  $3(12 + 4) = 36 + 12 = 50$
  $4(9 + 3) = 36 + 12 = 50$
  $6(6 + 2) = 36 + 12 = 50$
  $12(3 + 1) = 36 + 12 = 50$

2. Use the distributive property to find all the equivalent expressions for $24 + 32$

Once again encourage students to use divisibility rules to find common factors of 24 and 32

$1(24 + 32) = 24 + 32 = 56$
$2(12 + 16) = 24 + 32 = 56$
$4(6 + 8) = 24 + 32 = 56$
$8(3 + 4) = 24 + 32 = 56$

3. Examine the equivalent expressions for the sum in number 2 above. Circle the expression that contains a factor that is the GCF of 24 and 32? What is the other factor in this product? How does this factor partner differ from the other factor partners in the other equivalent expressions?

Talk about how the expression $8(3 + 4)$ is the only product that contains the GCF of 24 and 32. In this expression the factor partner to 8 is $(3 + 4)$. This is the only factor partner that has a sum that contains two addends with no common factor.

4. Examine the equivalent expressions for the sum in number 1 as well. Which expression contains a factor that is the GCF of 36 and 12? How does its factor partner differ from the other factor partners in the other equivalent expressions?

$12(3 + 1) = 36 + 12 = 50$ The same is true; the expression that contains a factor that is the GCF has a factor partner that has a sum that contains two addends with no common factor.

5. Use the distributive property to find all the equivalent expressions for each sum given. Circle the expression that contains a factor that is the GCF of the two addends in the original sum. Check and see if this expression follows the same principle as the expressions with the GCF from numbers 1 and 2 above.

a. $45 + 60$	b. $42 + 70$	c. $20 + 60$
$1(45 + 60) = 45 + 60 = 105$	$1(42 + 70) = 42 + 70 = 132$	$1(20 + 60) = 20 + 60 = 80$
$3(15 + 20) = 45 + 60 = 105$	$2(21 + 35) = 45 + 60 = 105$	$2(10 + 6) = 20 + 60 = 80$
$5(9 + 12) = 45 + 60 = 105$	$7(6 + 10) = 45 + 60 = 105$	$4(5 + 15) = 20 + 60 = 80$
$15(3 + 4) = 45 + 60 = 105$	$14(3 + 5) = 45 + 60 = 105$	$5(4 + 12) = 20 + 60 = 80$
		$10(2 + 6) = 20 + 60 = 80$
		$20(1 + 3) = 20 + 60 = 80$

All of the expressions that contain the GCF have a factor partner that is a sum of two numbers with no common factor.

Discuss with students that finding the equivalent expression that contains the GCF is often the most desirable expression because it will help them to rewrite or simplify expressions that contain variables (monomials or polynomials) or fractions in the future.

6. Find the GCF of the two numbers in each given sum. Use the distributive property to write an equivalent expression to the sum that contains the GCF as one of its factors. How do you know that you found the correct equivalent expression?

a. $42 + 14$ $14(3 + 2)$	b. $36 + 27$ $9(4 + 3)$
c. $55 + 44$ $11(5 + 4)$	d. $16 + 72$ $8(2 + 9)$

You know that you have found an expression that contains the GCF if its factor partner is a sum that contains two numbers that have no common factor.

Factoring out the GCF from a given sum can be viewed from the perspective of multiples as well. If you use the distributive property to write an equivalent expression for a sum by finding the GCF of the two original addends then these two original addends are multiples of the GCF and their sum is also a multiple of its factor partner. We know that the factor partner is a sum of two whole numbers with no common factor. Thus the original sum can be expressed as a multiple of a sum to two whole numbers with no common factor.

For example $36 + 12$ can be written as the equivalent expression of $12(3 + 1)$ by factoring out the GCF of 12. Since 12 is a factor of 36 and 12 then 36 and 12 are both multiples of 12. Also the sum of 36 and 12 is a multiple of the factor partner $(3 + 1)$. The distributive property shows that when you add these two numbers, that are both multiples of 12, their sum (50) is also a multiple of 12. $36 + 12 = 12(3 + 1) = 12(4) = 50$. The following task will help students make this connection.

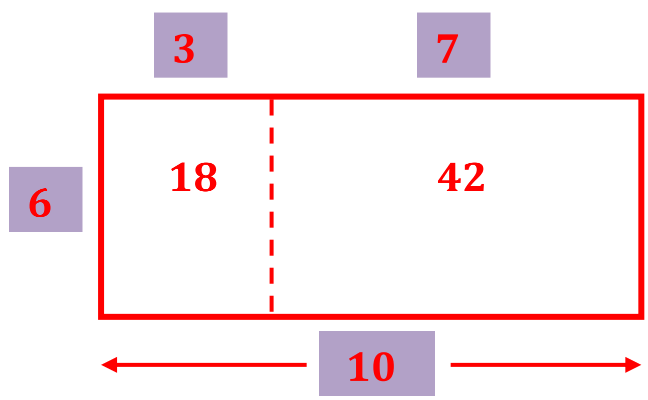
7. Nina was finding multiples of 6. She states,

"18 and 42 are both multiples of 6, and when I add them, I also get a multiple of 6."

$$18 + 42 = 60$$

Explain to Nina why adding two multiples of 6 will always result in another multiple of 6.

18 and 42 are both multiples of 6. Which means that 6 is a factors of both numbers. If you add both numbers you can use the distribute property to show that their sum is also a multiple of 6.
$18 + 42 = 6(3 + 7) = 6 \cdot (10)$



If you have any two multiples of 6 you can use the distributive property to show that the sum of these two numbers is also a multiple of 6. To express this with symbols you can write any two multiples of 6 as

$$6 \cdot a = 6a$$
$$6 \cdot b = 6b$$

where a and b are any whole number. If you add them you get $6a + 6b$. Using the distributive property you can rewrite this expression as $6a + 6b = 6(a + b) = 6 \cdot (a + b)$. This shows that the sum of any two multiples of 6 is always a multiple of 6.

*[*This is an Illustrative Mathematics Task]*

0.2e Homework: Using the Distributive Property To Find Equivalent Expressions

Directions: Use the distributive property and the GCF to write an equivalent expression for each given sum.

1. List the factors of 24:

   List the factors of 60:

   What is the GCF of 24 and 60:

   Use the GCF to write an equivalent expression for $24 + 60$

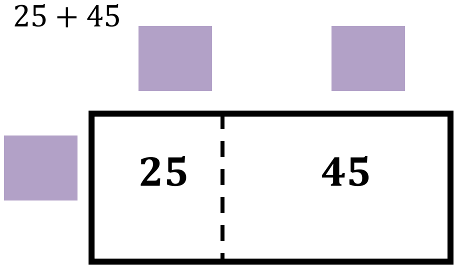
2. List the factors of 42:
   $1, 2, 3, 6, 7, 14, 21$

   List the factors of 49:
   $1, 7$

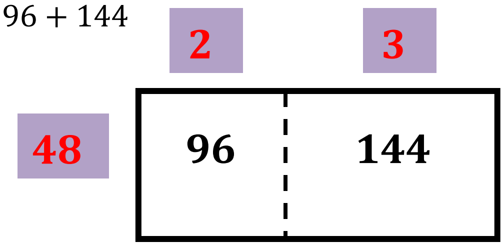
   What is the GCF of 42 and 49:
   7

   Use the GCF to write an equivalent expression for $42 + 49$
   $42 + 49 = 7(6 + 7)$

3. $25 + 45$



4. $96 + 144$



   $96 + 144 = 48(2 + 3)$

5. $16 + 36$

6. $54 + 81$

7. $72 + 32$

8. $34 + 17$

9. $35 + 75$

10. $13 + 15$

    $13 + 15 = 1(13 + 15)$

11. Create your own example that uses the distributive property to rewrite a sum as an equivalent expression using the GCF. Choose numbers for a, b, and n, where n is the GCF of a and b.

$$a + b = n(a) + n(b) = n(a + b).$$

Section 0.3: Arithmetic Operations with Decimals

Section Overview:
The last section of this appendix returns to work that students have done in previous grades with arithmetic operations of decimals. In this section students build upon their knowledge of operations with multi-digit whole numbers and extend similar reasoning to multi-digit decimals. They connect the modeling done in previous grades with decimals addition and subtraction and extend this to algorithms for adding and subtracting decimals. Next they do the same thing with multiplying multi-digit decimals. They draw upon the modeling and work done in previous grades and connect them to an algorithm for decimal multiplication. Finally they deal with multi-digit decimal division, similarly connecting the work done in previous grades with modeling to an algorithm. Throughout this section students also work on achieving fluency with these arithmetic operations by estimating, using mental math, converting between fractions and decimals, etc.

Concepts and Skills to Master in this Section:

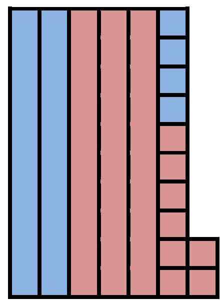
By the end of this section, students should be able to:

1. Fluently add multi-digit decimals using the standard algorithm.
2. Fluently subtract multi-digit decimals using the standard algorithm.
3. Fluently multiply multi-digit decimals using the standard algorithm.
4. Fluently divide multi-digit decimals using the standard algorithm.

0.3a Class Activity: Adding and Subtracting Multi-Digit Decimals

Marta has created the model below. She claims that this model can be used to represent the sum of 24 and 38.

You may need to point out that the first addend (24) is shaded blue and the second addend (38) is shaded pink. The sum is shown by combining all the rods and squares together. If needed, you can use base-ten blocks to model this as well.



In previous grades students used base-ten blocks to model addition for whole numbers and numbers with decimals up to hundredths. The blocks and model help students to understand why we “carry” as we bundle into groups of ten when using the addition algorithm. The purpose of this task is to extend that understanding to adding decimals greater than hundredths. Similarly, we can use models to carry and bundle into groups of ten and relate this to the addition algorithm with decimals.

4. If Marta’s claim is true, what is the value of the small square?
   A small square has a value of 1 unit.

5. What is the value of a rod (long rectangle)?
   A rod has a value of 10

6. Find the sum of 24 and 38 using the addition algorithm and discuss how this relates to the model above.

$$\begin{array}{r} {}^{1} \\ 24 \\ +38 \\ \hline 62 \end{array}$$

   Review with your class how the algorithm relates to the model and place value. i.e. That “carrying” the one represents the grouping of the 10 ones into a rod of 10, which is then combined with the other rods of 10.

Using the same model now suppose that the small square represents $\mathbf{0.1}$.

7. What would the value of the rod be?
   The rod would equal 1

8. What would the sum be equal to?
   The sum would equal 6.2

This requires students to really make sense of the manipulative has they change their thinking and they use blocks to represent decimals or parts less than 1.

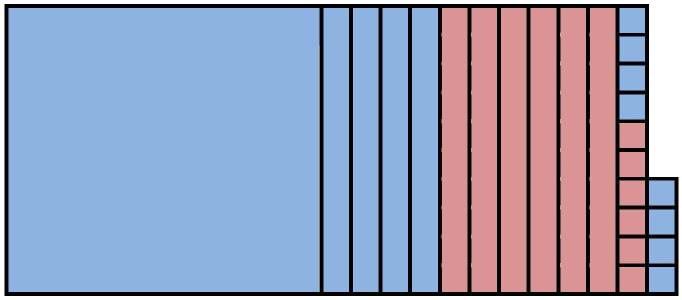
9. Find the sum supposing that the small square represents 0.1 using the addition algorithm and discuss how this relates to the model.

$$\begin{array}{r} {}^{1} \\ 2.4 \\ +3.8 \\ \hline 6.2 \end{array}$$

   The model shows that the algorithm still holds true for decimals of tenths.

10. What other sums might be represented with this model? Find at least two and for each sum identify what a square represents and what a rod represents.
    Sample answer: $\mathbf{0.24 + 0.38}$, for this sum the value of a square is $\mathbf{0.01}$ and the value of a rod would be $\mathbf{0.1}$.
    Sample answer: $\mathbf{240 + 380}$, for this sum the value of a square is $\mathbf{10}$ and the value of a rod is $\mathbf{100}$.
    Other sums might be $\mathbf{2400 + 380}$, $\mathbf{0.024 + 0.038}$, or even $\mathbf{48 + 76}$. For the last sum the value of small square would be $\mathbf{2}$ and the value of a rod would be $\mathbf{20}$, however, for our purposes we are only going to investigate units with a base of 10.

Study the new model below.



11. Irina claims that the model represents the sum of $\mathbf{1480}$ and $\mathbf{660}$. For this to be true what is the value of a small square, a rod, and large square?
    The small square equals 10
    The rod equals 100
    The large square equals 1000

12. What does the sum equal altogether? Use the addition algorithm to find the sum as well.

$$\begin{array}{r} {\scriptstyle 1\,1} \\ 1480 \\ +\ \ 660 \\ \hline 2140 \end{array}$$

13. Carly states that when she looks at the model she sees the sum of $0.\mathbf{148} + \mathbf{0.066}$. For Carly's statement to be true what is the value of a small square, a rod, and a large square?
    The value of a small square is 0.001
    The value of a rod is 0.01
    The value of a large square is 0.1

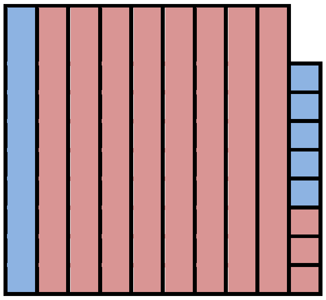
14. What does Carly's sum equal altogether? Use the addition algorithm to find the sum as well. Discuss how the algorithm relates to the model.

$$\begin{array}{r} {\scriptstyle 1\,1} \\ 0.148 \\ +0.066 \\ \hline 0.214 \end{array}$$

The model shows that the algorithm still holds true for decimals of thousandths as well. Students should begin to understand that the addition algorithm works for all decimals because we "regroup" bundles of ten has we "carry" numbers in the algorithm.

15. If needed draw a model to find the following sums. Name what the small squares represent and then find each sum using the addition algorithm. You may want to provide graph paper for students to draw their models.

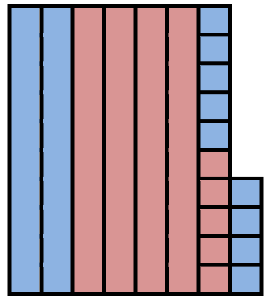
a. $\mathbf{0.015 + 0.083 = 0.098}$



$$\begin{array}{r} 0.015 \\ +0.083 \\ \hline 0.098 \end{array}$$

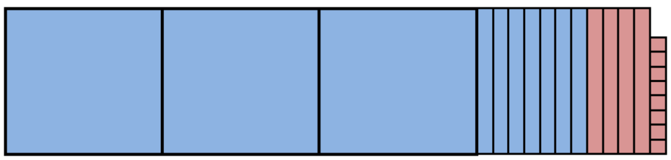
The small square = 0.001

b. $\mathbf{0.029 + 0.045 = 0.074}$



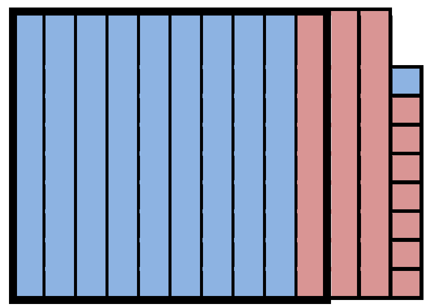
The small square = 0.001

c. $\mathbf{0.37 + 0.048 = 0.419}$



The small square = 0.001

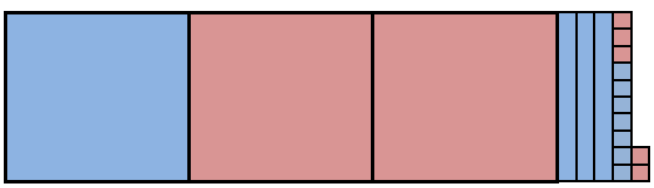
d. $\mathbf{0.091 + 0.037 = 0.128}$



$$\begin{array}{r} 1 \\ 0.091 \\ +0.037 \\ \hline 0.128 \end{array}$$

The small square = 0.001

e. $\mathbf{1.37 + 2.05} = 3.42$



The small square = 0.01

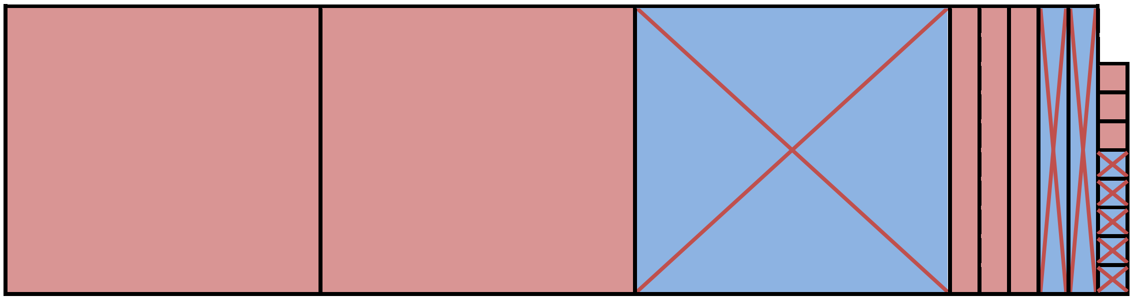
f. $\mathbf{0.103 + 0.0091 = 0.1221}$

The model for this sum would require either really big 10 x 10 squares or really small squares that represent 0.0001. Encourage students to use the algorithm.

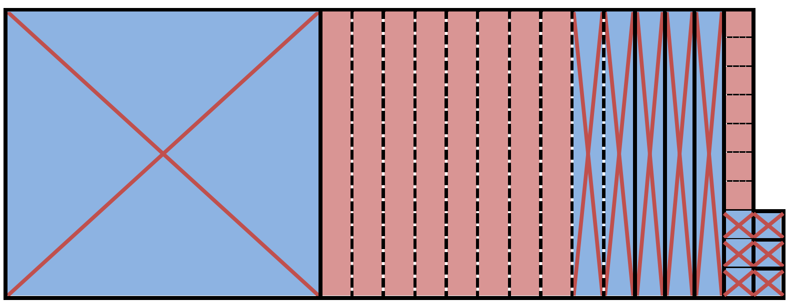
13. Draw models to find the following differences. Then find each difference using the subtraction algorithm.

The purpose of the problems below is for students to see that the same principles hold true for the subtraction algorithm as well. Place value allows us to regroup and unbundle in order to “borrow”. This will work for any decimals. The entire model is the minuend and the subtrahend is shown in blue. The final difference is what is leftover (shown in pink)

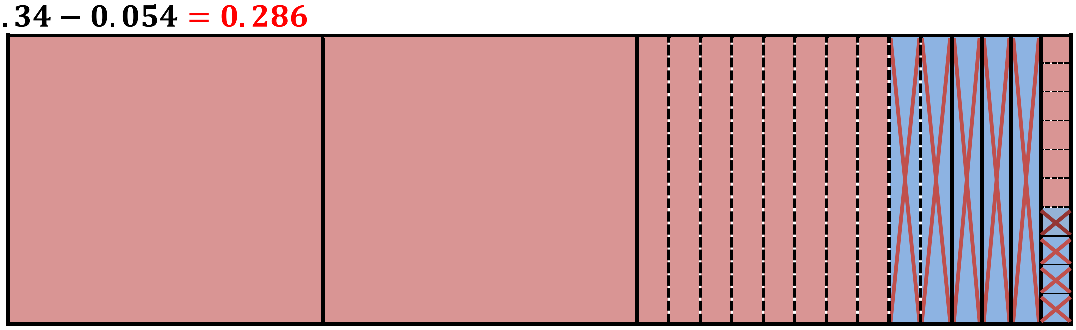
a. $\mathbf{0.358 - 0.125} = \mathbf{0.233}$



b. $\mathbf{0.243 - 0.156} = \mathbf{0.087}$



c. $\mathbf{0.34 - 0.054} = \mathbf{0.286}$



14. Bev has $35.65. She buys a candy bar for $0.89 and package of pens for $3.56.

a. How much money did Bev spend?
Bev spent $4.45

b. How much money does she have leftover?
Bev has $31.20

15. Peyton is driving to college, her college is $345\frac{11}{50}$ miles away from her parent's home. She has already driven $24\frac{9}{10}$ miles. How much further does she need to drive? Estimate your answer before calculating.
$345\frac{11}{50} - 24\frac{9}{10} = 345.21 - 24.9 = 10.31$

This problem is a good example of how changing mixed fraction into decimals is sometimes easier than keeping the numbers as fractions and finding a common denominator. Be sure to solidify the addition and subtraction algorithm in the problems above. Students will have the opportunity to practice fluency in adding and subtracting decimals in their homework assignment for this section.

0.3a Homework: Adding and Subtracting Multi-Digit Decimals

Directions: Find each sum or difference.

1. $4.398 + 70.04$ 74.438	2. The three sides of a triangle have the measurements of 2.32 cm, 15.4 cm and 112.09 cm. What is the perimeter of the triangle?	3. $\frac{677}{1000} + \frac{3}{10} + 10\frac{6}{10}$
4. Portia is painting a room. She needs $4\frac{4}{5}$ of a gallon of white paint, $2\frac{1}{10}$ of a gallon of light blue paint, and $1\frac{9}{25}$ of a gallon of grey paint. How many gallons of paint does she need altogether? 8.26	5. $13\frac{21}{50} + 14.0389 + 13.08$	6. $54.07 - 8.3955$
7. $66.7 - 0.392$ 66.308	8. Carlotta has a piece of wood that measures 0.8392 meters; she cuts off a piece that is 0.05 meters. How much wood does she have left?	9. $42\frac{3}{4} - 11\frac{1}{2}$
10. $1.7 + 4\frac{3}{100} - 2.303$ 3.427	11. Malone got 8 out of 10 on his last spelling test. On the next test he got 76%. What is the sum of the two test percentages?	12. Write three decimals that have a sum of 114.056.

0.3b Class Activity: Multiplying Muli-Digit Decimals

In previous grades, students learned to multiply multi-digit numbers with decimal factors limited to hundredths and decimal products limited to thousandths. This lesson will help students extend their previous understanding of decimal multiplication to numbers that have decimals beyond hundredths.

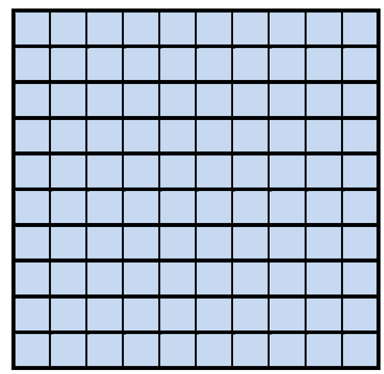
Part 1

Use the model given to discuss the following questions.

1. Assume the side length of the large square is 10.
   What is the area of the large square? What is the side length of a small square? What is the area of the small square?

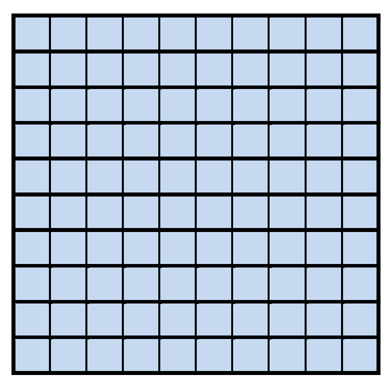
Encourage students to label the model with their findings and to justify their reasoning. Possible arguments are given below.

Since the side length of the large square is 10 units long then the area of the large square is $10 \times 10 = 100$. The side length of the large square is 10 units long and there are 10 small squares on each side length; this means that the side length of a small square is $10 \div 10 = 1$. Since the side length of the small square is 1 then the area of the small square is $1 \times 1 = 1$. Students might also argue that the area of the small square is 0.01 or $\frac{1}{100}$ of the area of the large square. $0.01 \times 100 = 1$ *or* $100 \div 100 = 1$.

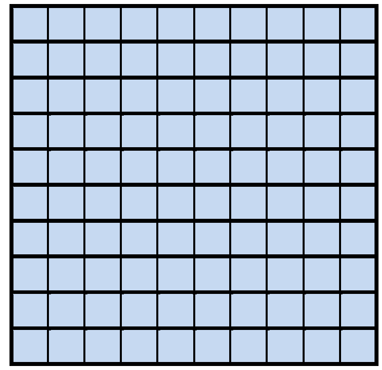


2. Now assume the side length of the large square is 100.
   What is the area of the large square? What is the side length of a small square? What is the area the small square?

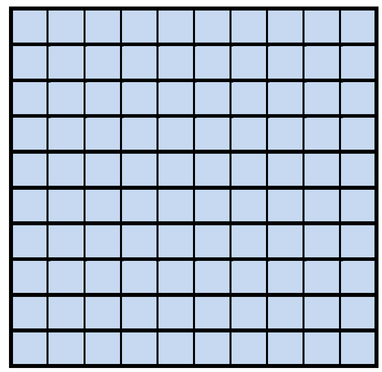
The area of the large square is 10,000 because the side length of the large square is 100 units long and $100 \times 100 = 10{,}000$. The side length of the small square is 10. Since there are 10 small squares on each side length then the side length of a small square is $100 \div 10 = 10$. Since the side length of the small square is 10 then the area of the small square is $10 \times 10 = 100$. Students might also reason that the area of the small square is 0.01 or $\frac{1}{100}$ of the area of the large square. $0.01 \times 10{,}000 = 10$ *or* $10{,}000 \div 100 = 10$.



3. Now assume the side length of the large square is 1.
What is the area of the large square? What is the side length of a small square? What is the area the small square?
The area of the large square is 1. Since the side length of the large square is 1 unit long then $1 \times 1 = 1$. Since the side length of the large square is 1 unit long and there are 10 small squares on each side length then the side length of a small square is $1 \div 10 = 0.1$ The area of the small square is 0.01, this is because the side length of the small square is 0.1 and $0.1 \times 0.1 = 0.01$. If students do not have a strong understanding of decimal multiplication from previous grades they might reason that the area of the small square is 0.01 or $\frac{1}{100}$ of the area of the large square. $0.01 \times 1 = 0.01$ *or* $1 \div 100 = 0.01$.



4. Assume the side length of the large square is 0.1.
What is the area of the large square? What is the side length of a small square? What is the area the small square?
The area of the large square is 0.01 because the side length of the large square is 0.1 units long and $0.1 \times 0.1 = 0.001$. The side length of the small square is 0.01 because the length of the large square is 0.1 units long and there are 10 small squares on each side. $0.1 \div 10 = 0.01$ or 0.1 of 0.1. Since the side length of
of
the small square is 0.01 then the area is $0.01 \times 0.01 = 0.0001$. Most students will not know how to multiply 0.01 and 0.01 since they only produced products up to thousandths in previous grades. However, they can reason that the area of the small square is 0.01 or $\frac{1}{100}$ of the area of the large square. $0.01 \times 0.1 = 0.0001$ *or* $0.01 \div 100 = 0.0001$.

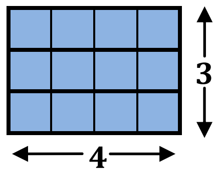


5. Summarize your thinking by stating the following products
   a. $10 \times 10 =$ 100
   b. $1 \times 1 =$ 1
   c. $0.1 \times 0.1 =$ 0.01
   d. $0.01 \times 0.01 =$ 0.0001

At this point students should be able to give an argument for why $0.1 \times 0.1 = 0.01$ and $0.01 \times 0.01 = 0.0001$ from a geometric perspective as related to the base ten grids used above. Also they should be able to tell you why the product is smaller than the two factors. They might also begin reasoning about how the placement of the decimal point in the product is related to the total number of decimal places to the right of zero in the factors. This will be solidified in the next part of this section.

Part 2

1. Use the model below to answer the questions that follow.



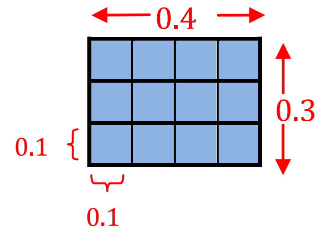
a. What is the length of each unit?
Each unit has a length of 1

b. What is the area of each small square?
The area of each small square is 1.

c. What is the area of the rectangle?
The area of the rectangle is 12.

d. Write a multiplication equation that represents the dimensions and area of this rectangle.
$4 \times 3 = 12$

2. Now suppose the length of each unit is 0.01.



a. Label the length of each unit (small square) on the rectangle and label the side lengths of the rectangle.

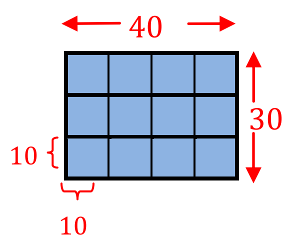
b. What is the area of each small square?
The area of each small square is 0.01.

c. What is the area of the rectangle?
0.12

d. Write a multiplication equation that represents the area of the rectangle above.
$0.4 \times 0.3 = 0.12$

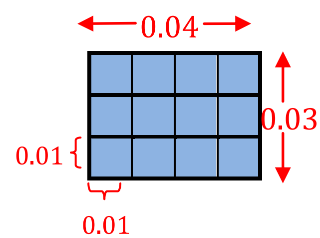
For problems 3 through 8 label each rectangle with the given dimensions so that it represents the multiplication equation. Label the dimensions of each unit (small square). Find the area of each small square and the area of each rectangle that represents the solution to each equation. Then write the solution to the equation.

3. $40 \times 30 =$ 1200



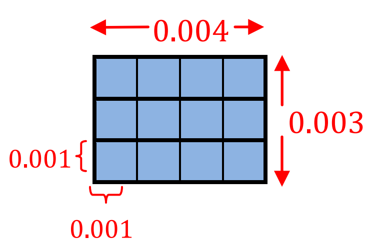
Area of the small square:100
Area of the rectangle:1200

4. $0.04 \times 0.03 =$ 0.0012



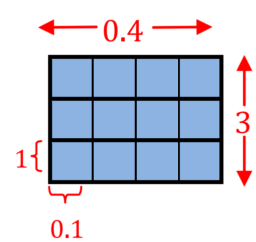
Area of the small square: 0.0001
Area of the rectangle:0.0012

5. $0.003 \times 0.004 =$ 0.000012



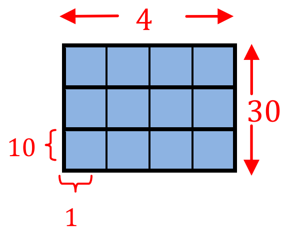
Area of the small square:0.000001
Area of the rectangle:0.000012

6. $0.4 \times 3 =$ 0.12



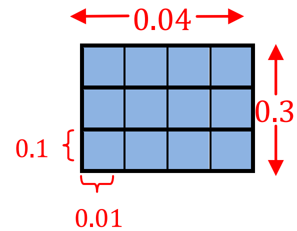
Area of the small square: 0.1
Area of the rectangle:0.12

7. $4 \times 30 = 120$



Area of the small square: 10
Area of the rectangle: 120

8. $0.04 \times 0.3 = 0.012$



Area of the small square: 0.001
Area of the rectangle: 0.012

The purpose of this task is for students to think about the placement of the decimal in the product when multiplying decimals. Students may use a variety of strategies to obtain the products above.

- They may use reasoning used in previous grades about patterns in the number of zeros that occur when multiplying whole numbers by powers of ten to also reason about the placement of the decimal point when a decimal is multiplied by a power of ten.
- They might also arrive at the final product by finding the area of each unit (small square) as learned in Part 1 of this section and then add the area of all the small squares to get the final product.
- Encourage them to check their answer using estimation reasoning as well. For example, if you are multiplying 0.04×0.3 you are taking four hundredths of three tenths therefore your product is going to go be in the thousandths.

A common misconception may arise if students infer that the decimal point is placed in regards to the number of zeros and not the number of places to the right of the decimal point. Also watch out for problems that might occur if students are unclear on the placement of the decimal point, for example some students might state that $0.5 \times 0.02 = 0.001$ rather than the correct product of 0.01 which arises from 0.010.

Be sure to clearly illustrate the connection between the total number of decimals places to the right of 0 in the factors to the number of places to the right of 0 in the product.

For example $\underbrace{0.04 \times 0.3}_{\text{Total of 3 places to the right of the decimal point}} = \underbrace{0.012}_{\text{3 places in the product}}$

Part 3

1. Use the fact that $16 \times 12 = 192$ to find each product. Justify your answer using an estimation argument and by discussing the placement of the decimal point or the expected number of zeros in the product.

Possible estimation arguments are given.

a. $16 \times 1.2 = 19.2$
1.2 is a little more than 1 so the product of 16 and 1.2 should be more than the product of 16 and 1. 19.2 is a reasonable estimate for this. Or 1.2 is ten times smaller than 12 so our product should be 10 times smaller than 192. This can be obtained by moving the decimal one place to the left. 19.2 is ten times less than 192. In other words, one of the factors has a decimal point that is one place to the left of 12, thus our product should have a decimal that is moved once place to the left.

b. $160 \times 12 = 1920$
Since 160 is ten times as large as 16 then the product should be ten times as large at 192. One of the factors has an additional zero so that product should have an additional zero.

c. $160 \times 120 = 19200$
Since the two factors are ten times as large as the original factors then the product should be 100 times bigger than the original product. In other words, each factor has an additional zero so the product should have 2 additional zeros.

d. $1600 \times 12 = 19200$
Since 1600 is 100 times as big as 16 then our product should be 100 times bigger. In other words there are two additional zeros in the first factor so our product will have two additional zeros. Relate this to the problem above.

e. $1.6 \times 1.2 = 1.92$
1.6 is close to 2 and 1.2 is close to 1. So the product should be to 2. 1.92 is a reasonable estimate for this. 1.6 is ten times smaller than 16 and 1.2 is ten times smaller than 12. Thus our product should be 100 times smaller 192. This can be obtained by moving the decimal point two place to the left of the original product, giving us 1.92. In other words each factor has a decimal point the is 1 moved one place to the left of the original factors. Making a total of two places to the left for the factors, this means our product must have a decimal point that is moved two places to the left.

f. $0.16 \times 1.2 = 0.192$
This product will be less than 1 because we are taking 0.16 of 1.2. Since 0.16 is one hundred times smaller than 16 and 1.2 is ten times smaller than 12. Then our product will be 1000 times smaller than the original product. This can be obtained by moving the decimal point three spaces to the left in the original product yielding 0.192. In other words the first factor has a decimal point that has been moved two places to the left, the second factor has a decimal point that has been moved one place to the left. So we must move the decimal point in the product a total of three places to the left.

g. $0.16 \times 0.12 = 0.0192$
This product will be much less than one since we are taking 0.16 of 0.12. Since 0.16 is one hundred times smaller than 15 and 0.12 is also 100 times smaller than 12. Then our product will be 10,000 times smaller than 192. This can be obtained by moving the decimal in the original product 4 places to the left, yielding 0.0192. In other words the first factors has a decimal point that has been moved two place to the left, the second factor has a decimal point that has been moved two places to the left. So we must move the decimal point the product a total of four places to the left.

2. Dallin has begun to do the following multiplication problem. His work is shown below; he does not know where to place the decimal point in the product. Correctly place the decimal point for him and justify your answer.

$$\begin{array}{r} 1 \\ \cancel{1}\ \cancel{3} \\ 4.\underline{37} \\ \times\quad 1.\underline{25} \\ \hline 2185 \\ +\ \ 8740 \\ \hline 10925 \\ +43700 \\ \hline 5.\underline{4625} \end{array}$$

The decimal must be placed 4 places to the left in the product because the first factor has the decimal placed two places to the left and the second factor has the decimal placed two places to the left, making a total of 4 places to the left. Also if we estimate the product by rounding the products we know that $4 \times 1 = 4$ so our final product should be close to 4. Thus we must place the decimal point between 5 and 4.

Directions: For each problem estimate the product. Then find the product by changing the decimals to fractions. Check your answer by using the multiplication algorithm.

3. 25.62×11.7

Estimation: You can round 25.62 to 30 and 11.7 to 10. This gives us $30 \times 10 = 300$. This product can be estimated by reasoning that $3 \times 1 = 3$ and each factor has one zero so our product will have two zeros, this yields 300.

Convert to Fractions: $25.62 = 25\frac{62}{100}$ and $11.7 = 11\frac{7}{10}$. Thus $25\frac{62}{100} \times 11\frac{7}{10} = \frac{2562}{100} \times \frac{117}{10} = \frac{299754}{1000}$. Since the denominator is 1000 the last digit should be in the thousandths place when writing the decimal. Thus $\frac{299754}{1000} = 299.754$

Standard Algorithm: The placement of the decimal point can be obtained several ways. The estimation above yielded a product close to 300, which would place the decimal between the 9 and 7. Or, since 25.63 is 100 times smaller than 2562 and 11.7 is ten times smaller than 117 than our final product must be $100 \times 10 = 1000$ times smaller than 299,754. We can do this by moving the decimal point to the left three spaces. They could also reason that there are 2 decimal digits in the first factor and 1 decimal digit in the second factor; this will make 3 decimal digits in the product.

$$\begin{array}{r} \cancel{3}\,\cancel{4}\,\cancel{1} \\ 25.\underline{62} \\ \times \quad 11.\underline{7} \\ \hline 17934 \\ 25620 \\ +\ 256200 \\ \hline 299.\underline{754} \end{array}$$

4. You work part time at a book store and get paid \$12.05 per hour. In the entire month of March you worked 78.25 hours. How much money did you make in March?

Estimation: You can round 12.05 to 12 and 78.25 to 80. This gives us $12 \times 80 = 960$. You can estimate this product by reasoning that $12 \times 8 = 96$ and the second factor has one zero so our product will have one zero. This yields about \$960 made in March.

Convert to Fractions: $12.05 = 12\frac{5}{100}$ and $78.25 = 78\frac{25}{100}$. Thus $12\frac{5}{100} \times 78\frac{25}{100} = \frac{1205}{100} \times \frac{7825}{100} = \frac{9429125}{10000}$. The denominator is 10,000 so the last digit should be in the ten thousandths place when writing the decimal. Thus $\frac{9429125}{10000} = 942.9125$

Standard Algorithm: The placement of the decimal point can be obtained several ways. The estimation above yielded a product close to 300, which would place the decimal between the 9 and 7. Or, since 25.63 is 100 times smaller than 2,562 and 11.7 is ten times smaller than 117 than our final product must be $100 \times 10 = 1000$ times smaller than 299754. We can do this by moving the decimal point to the left three spaces. They could also reason that there are 2 decimal digits in the first factor and 1 decimal digit in the second factor; this will make 3 decimal digits in the product.

$$\begin{array}{r} \cancel{1}\ \ \cancel{3} \\ \cancel{1}\ \ \cancel{4} \\ \cancel{1} \\ \cancel{1}\ \ \cancel{2} \\ 12.\underline{05} \\ 78.\underline{25} \\ \hline 6025 \\ 24100 \\ 964000 \\ \underline{8435000} \\ 942.\underline{9125} \end{array}$$

0.3b Homework: Multiplying Multi-Digit Decimals

Directions: Solve each problem below; be sure to estimate your answer first. (Hint: If desired and appropriate change your decimals to fractions in order to evaluate.)

1. 423.56×63.72 $26{,}989.2432$	2. Sienna spends $27.50 on coffee and a Danish for breakfast each week for an entire semester. How much money does she spend on breakfast for the semester if there are 16.25 weeks in a semester?
3. Flora is designing a triangular structural beam. The base of the triangular beam measures 14.002 meters and the height is 3.85 meters. Find the area of the triangular beam?	4. 0.000125×0.005
5. $1.037 \times 5 - 0.68$ 4.505	6. 0.0021×14.2
7. A person's weight on the moon is about 0.167 of their weight on Earth. How much does a 168 pound astronaut weigh on the moon? $28.056\ pounds$	8. $5.32(4.2 + 1.85)$

9. Describe each pattern given below, then find the next two terms.
   a. $1, 0.3, 0.09, 0.027$ ...

   b. $17, 0.17, 0.0017$...

   c. $10, 15, 22.5, 33.75$ ...
   To obtain the next term in the pattern multiply by 1.5; 50.625, 75.9375

10. Andrei has enough ceramic tiles to cover a rectangular patio area of 60.75 ft^2. He has made a list of possible dimensions he could use for the rectangular patio. Some of these dimensions are wrong. Circle the dimensions that will **not** work for an area of 60.75 ft^2. Explain why these dimensions will not work.
    - 27 ft × 2.25 ft
    - 2.7 ft × 22.5 ft
    - 0.27 ft × 22.5 ft
    - 27 ft × 22.5 ft
    - 27 ft × 22.50 ft
    - 0.27 ft × 225 ft
    - 270 ft × 0.225 ft

11. You and two friends are going out for lunch at a restaurant. The menu for the restaurant is shown. Use the menu to choose lunch items for you and your friends then complete the following.
    a. Write the item and its price on the check provided.
    b. Find the subtotal for your total bill.
    c. Multiply the subtotal by 0.07 to find the sales tax and then find the total bill with tax.
    d. Find 20% of the total bill to determine how much money you should leave as a tip.

GUEST CHECK

Date	Invoice # 568345	
	Subtotal:	
	Tax:	
Tip:	Total:	

Lunch Menu

Sandwiches
Turkey or Ham.......$5.40
Tuna......................$4.75
Chicken..................$6.20

Pizza by the slice
Pepperoni...................$3.75
Hawaiian..................$4.30
Cheese.......................$ 3.50

Salads
Caesar.........................$8.30
BBQ Chicken.............$9.45
Spinach Strawberry...$7.60

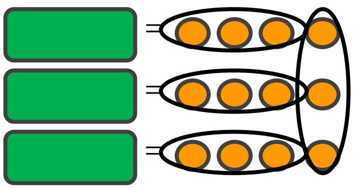
Drinks
Soda...........................$1.75
Lemonade.................$2.05
Bottled Water...........$1.60

0.3c Class Activity: Dividing Muli-Digit Decimals

Roxy has bins of loose candy at her shop. She sells Creamy Dreamies for $0.75 each, Sour Powers for $0.25 each and Gummy Yummies for $0.05 each. She has also decided to sell bags of each different kind of candy for $3.00 a bag. How many of each candy will be in a $3.00 bag? Draw a model to find your answers.

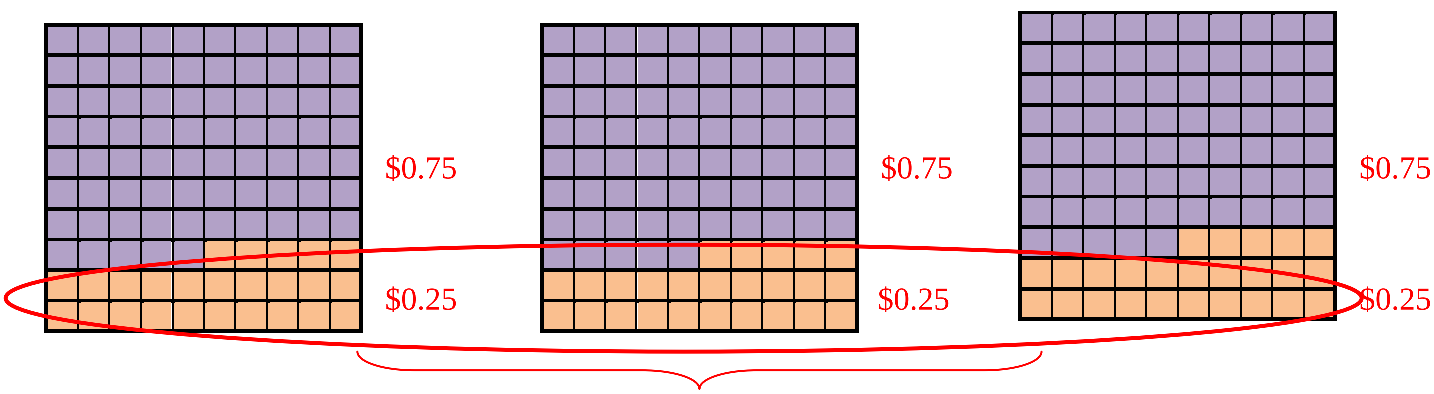
Creamy Dreamies

Students must first recognize that this problem can be solved by dividing the total cost of the bag of candy by the individual cost of each candy. However, they may not know how to divide when the divisor is a decimal. Ask students to see if they can figure out the number of each kind of candy by drawing a model.



This shows that if bag of Creamy Dreamies cost $3.00, and an individual Creamy Dreamie cost $0.75 we can break up each dollar into quarters and see that there are 4 groups of $0.75 in $3.00.

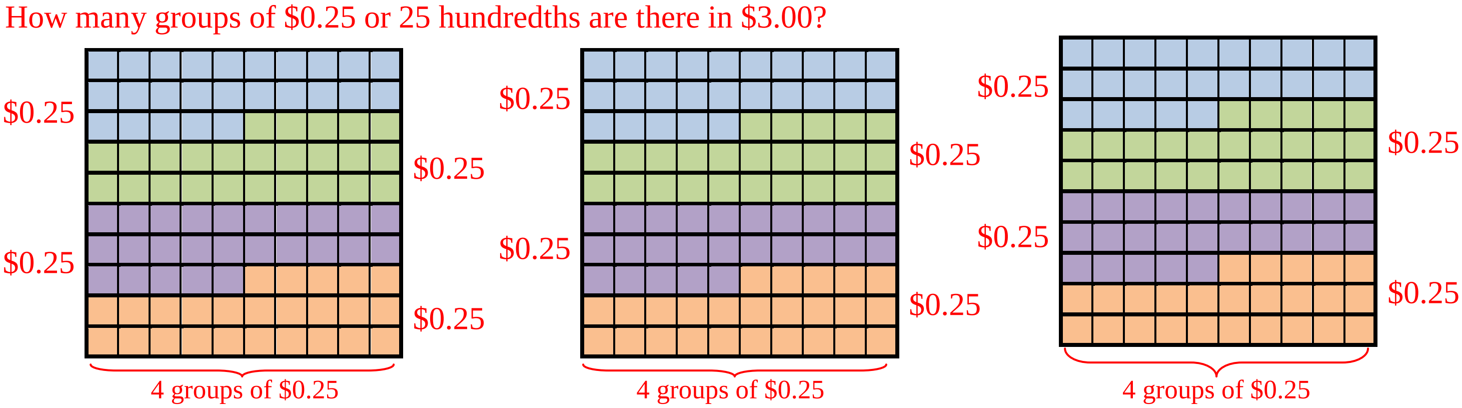
Another way to model this problem is by using base-ten models. Using a base-ten model will help them when they get to the bag of Gummy Yummies because they have to split their dollars into pennies. You could provide students with base-ten blocks or graph paper to draw their models. We can think of the 10 x 10 model as $1.00 or 100 hundredths. How many groups of $0.75 or 75 hundredths are in $3.00?



$0.25 + 0.25 + 0.25 = 0.75$

Sour Pours

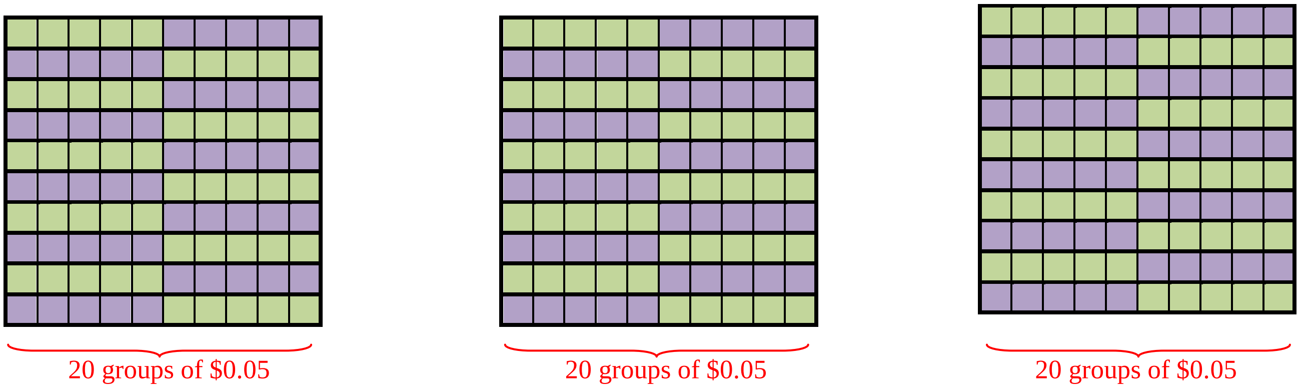
How many groups of $0.25 or 25 hundredths are there in $3.00?



Each $1.00 has 4 groups of $0.25 so altogether there are 12 groups of $0.25. Thus there are 12 Sour Powers in $4.00 bag.

Gummie Yummies

How many groups of \$0.05 or 5 hundredths are there in \$3.00?



Each \$1.00 has 20 groups of \$0.05 so altogether there are 60 groups of \$0.05.

Is there a way that you can find the number of Gummy Yummies in a bag without drawing model?

The following discussion questions can help students understand why they can multiply the dividend and divisor by the same power of ten and then use the division algorithm to arrive at their quotient.

What if you thought of each dollar as 100 hundredths, how many one hundredths would 3 dollars be?

$3 \times 100 = 300$. There would be 300 hundredths.

Similarly how many one hundredths are there in 5 hundredths? There are $0.05 \times 100 = 5$.

So we would be dividing 300 by 5 and $300 \div 5 = 60$.

In other words when you multiply the dividend and the divisor both by 100 and then divide with the long division algorithm you get the same quotient.

$$3 \times 100 = 300$$
$$0.05 \times 100 = 5$$
$$300 \div 5 = 60$$

Another way to look at this is by writing the divisor and dividend as fractions. For example $3 \div 0.05 = \frac{3}{0.05}$, now what fraction can you multiply this by so that the denominator is a whole number? This fraction is $\frac{100}{100}$. Thus $\frac{3}{0.05} \cdot \frac{100}{100} = \frac{300}{5} = 60$.

Try your method with Creamie Dreamies and the Sour Powers to see if you get the same answer.

Ask students to multiply the divisor and dividend each by 100 or change it to a fraction and multiply by 100 and then divide.

<u>Creamie Dreamies</u>

$3 \times 100 = 300$

$0.75 \times 100 = 75$

$300 \div 75 = 4$

$3 \div 0.75 = \frac{3}{0.75} \times \frac{100}{100} = \frac{300}{75} = 4$

<u>Sour Powers</u>

$3 \times 100 = 300$

$0.25 \times 100 = 25$

$300 \div 25 = 12$

$3 \div 0.25 = \frac{3}{0.25} \times \frac{100}{100} = \frac{300}{25} = 12$

It might help students to better understand why they are getting the same quotient even though they changed the numbers by illustrating with whole numbers. For example if you have 8 pieces of candy to distribute amongst 4 friends, each friend will get 2 pieces of candy. If you now multiply 8 and 4 by the same number, say 10. $8 \times 10 = 80$ *and* $4 \times 10 = 40$, this would be as though you had 80 pieces of candy to distribute amongst 40 friends. How much would each friend get? $80 \div 40 = 2$. Each person would still get two pieces of candy. When we are dividing we are trying to determine the number of groups, so you will get the same answer if you multiply the dividend and divisor by the same amount. This is true for decimals as well as illustrated with the \$3.00 bags of candy.

Directions: For each problem determine what number you can multiply the dividend and divisor by so that you get a whole number divisor. If needed re-write each division problem as a fraction. Then re-write each quotient with a whole number divisor.

As you work through these problems help students to remember how multiplying by a power of ten moves the decimal point. Some discussion questions are below.

When you multiply the dividend and divisor each by ten how many spaces does the decimal point get moved? 1 space to the right.

When you multiply the dividend and divisor each by 100 how many spaces does the decimal point get moved? 2 spaces to the right

When you multiply the dividend and divisor each by 1000 how many spaces does the decimal point get moved? 3 spaces to the right.

1. $63 \div 3.5 =$ $\frac{63}{3.5} \times \frac{10}{10} = \frac{630}{35} = 630 \div 35$	2. $603 \div 50.25 =$ $\frac{603}{50.25} \times \frac{100}{100} = \frac{60300}{5025}$ $= 13000 \div 5025$	3. $18.2 \div 1.4 =$ $\frac{18.2}{1.4} \times \frac{10}{10} = \frac{182}{14} = 182 \div 14$
4. $0.75 \div 0.15 =$ $\frac{0.75}{0.15} \times \frac{100}{100} = \frac{75}{15} = 75 \div 15$	5. $1{,}488 \div 0.024 =$ $\frac{1488}{0.024} \times \frac{1000}{1000} = \frac{1488000}{24} =$ $1488000 \div 24$	6. $36.47 \div 0.7 =$ $\frac{36.47}{0.7} \times \frac{10}{10} = \frac{364.7}{70} =$ $364.7 \div 70$
7. $0.52 \div 0.001 =$ $\frac{0.52}{0.0013} \times \frac{1000}{1000} = \frac{5200}{13} =$ $5200 \div 13$	8. $0.987 \div 12.3 =$ $\frac{0.987}{12.3} \times \frac{10}{10} = \frac{9.87}{230} =$ $9.87 \div 230$	9. $4.23 \div 0.012 =$ $\frac{4.23}{0.012} \times \frac{1000}{1000} = \frac{4230}{12} =$ $4230 \div 12$

Now re-write each problem as a quotient with a whole number divisor by moving the decimal points. Check to see if it matches the quotient you wrote above. Then calculate each quotient.

10. $63 \div 3.5$ $63.0 \div 3.5$ $= 630 \div 35$ $= 18$	11. $603 \div 50.25$ $603.00 \div 50.25$ $= 60300 \div 5025$ $= 12$	12. $18.2 \div 1.4$ $18.2 \div 1.4$ $= 182 \div 14$ $= 13$

13. 0.75 ÷ 0.15 $0.75 \div 0.15$ $= 75 \div 15$ $= 5$	14. 1,488 ÷ 0.024 $1488.000 \div 0.024 =$ $1488000 \div 24 =$ 6200	15. 36.47 ÷ 0.7 $36.47 \div 0.7 =$ $364.7 \div 7 =$ 52.1
16. 0.52 ÷ 0.001 $0.520 \div 0.001$ $= 520 \div 1$ $= 52$	17. 0.987 ÷ 10.5 $0.987 \div 10.5 =$ $9.87 \div 105 =$ 0.094	18. 4.23 ÷ 0.012 $4.230 \div 0.012 =$ $4230 \div 12 =$ 352.5

Explain in your own words how to divide decimals by decimals

To divide a decimal by a decimal
Multiply the divisor and dividend by the same power of ten to make the divisor a whole number. Then place the decimal point in the quotient and divide as you would with whole numbers. Continue until there is no remainder.

0.3c Homework: Dividing Multi-Digit Decimals

Directions: Solve each problem below; estimate your answer first. Round your answer to the nearest thousandth unless otherwise specified.

1. $2143 \div 2.3$ 931.739	2. $0.408 \div 0.51$
3. A car travels 284.49 miles in 4.35 hours. What is its unit rate?	4. $2.3835 \div 0.0007$
5. Sunny Skies Sunscreen comes in a 4 oz tube and costs \$8.49. Beach Balance Sunscreen comes in 5.5 oz tubes and costs \$10.29. Which sunscreen is the better deal? Justify your answer. Sunny Skies Sunscreen cost 2.13 per ounce and Beach Balance Sunscreen cost \$1.88 per ounce. That means that Beach Balance is the better deal?	6. $0.18 \div 0.0027$
7. Place the decimal in the correct location for each quotient. a. $14.6 \div 4 = 365$ b. $6.38 \div 11 = 58$ c. $88.27 \div 7 = 1261$ d. $15.38 \div 0.05 = 3076$	8. The thickness in centimeters of four different coins are given below. Find the average mean thickness of all four coins. Penny- 0.152 cm Nickel- 0.195 cm Dime- 0.135 cm Quarter- 0.175 cm
9. The area of a rectangular painting is 720.5 inches. The length is 36.0625 in. Find the width of the painting. Round your answer to the nearest hundredth.	10. Without calculating fill in the blank with $<, >,$ or $=$. a. $33.6 \div 0.7$_ $>$ _$3.36 \div 7$ b. $152.52 \div 16.4$_ $>$ _$152.52 \div 1.64$ c. $100.8 \div 1.6$_ $<$ __$10.08 \div 0.0016$ d. $5.92 \div 0.74$_ $=$ __$59.2 \div 7.4$

0.3d Class Activity: Solving Problems with Multi-Digit Decimals

Directions: Perform each indicated operation.

1. $0.5 + 1.674$ 2.174	2. $4.192 - 1.255$ 5.447	3. $14.9(0.56)$ 8.344	4. $2.92 \div 0.002$ 1460
5. Add 26.59, 1.80, and 13. 41.39	6. Find the difference of 42.05 and 11.621 30.429	7. Find the product of 13.6 and 901.15. 124.44	8. What is the quotient of 72.05 and 0.11? 655

Use mental math to perform each indicated operation. Be ready to discuss your mental math strategies.
Discuss student's mental math strategies, sample strategies are given. Encourage students to get in the habit of considering a mental math strategy before jumping to an algorithm because often times they are much easier and less time consuming.

9. $99 + 36$
   135, give 1 from the 36 to the 99 and you get $100 + 35 = 135$
10. $42 - 29$
   Add 1 to each number and you get $43 - 30 = 13$. It is okay to change the value of each number by the same number or scale factor because we are looking for a difference. This stratgey would not work for addition or multiplication.
11. 50×8
   Double the 50 and cut the 8 in half and you get $100 \times 4 = 400$. Since we doubled one number and cut the other number in half we are not changing the overall value of the product.
12. $120 \div 5$
   Doube both numbers and you get $240 \div 10 = 24$. Once again this works because you are changing both numbers by the same scale factor and you are dividing, this is similar to changing each number by the same power of 10.

Perform each indicated operation using two different methods. Be ready to discuss your prefered method. Round each answer to the nearest thousandth.

13. $0.5 \div 0.1$

Method 1:	Method 2:
Multiply each number by the same power of ten to get a whole number divisor (move both decimal points one place to the right). $0.5 \div 0.1 = 5 \div 1 = 5$	Change each decimal to a fraction first and then divide. $0.5 \div 0.1 = \frac{5}{10} \div \frac{1}{10} = \frac{5}{10} \times \frac{10}{1} = 50 = 10 = 5$

14. 7.05×24.25

Method 1:	Method 2:
Multiply using the standard algorithm $7.05 \times 24.25 = 7050 \div 24205$ $\begin{array}{r} {\scriptstyle 2\ 1\ 3} \\ {\scriptstyle \cancel{2}\ \cancel{1}\ \cancel{2}} \\ 24.25 \\ \times 7.05 \\ \hline 12125 \\ 00000 \\ +1697500 \\ \hline 170.9625 \end{array}$	Change each decimal to a fraction first and then divide. $7.05 \times 24.25 = 7\frac{5}{100} \times 24\frac{25}{100} = \frac{705}{100} \times \frac{2425}{100} = \frac{1709625}{10000} = 170.9625.$

Solve each problem

**The following three probems are Illustrative Mathematics Tasks.*

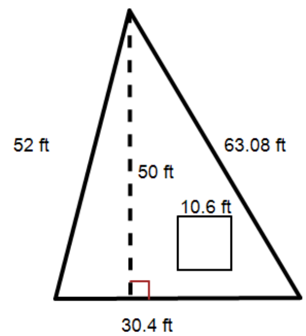
Tell students to be mindful about the method in which they choose to perform each operation.

15. A group of 10 scientists won a $1,000,000 prize for a discovery they made. They will share the prize equally. How much money will each person get?
    Each scientist will get $100,000
    Students can definitely find the answer to this problem with long division. However it is much easier and demonstrates a greater level of fluency if a student uses mental math to solve this problem.

16. Two cousins shared 0.006 kilograms of gold equally. How many kilograms of gold did each cousin get?
    Each cousin gets 0.003 kilograms of gold.
    Once again students can use long division to obtain their answer. However it is much easier to use mental math to solve this problem.

17. A barrel contained 160 liters of oil that costs $51.20. What is the cost for one liter? How many liters can you buy for $1.00?
    One liter costs of oil costs $0.32. You can buy 3.125 liters of oil for $1.00.
    For this problem students will most likely use a standard algorithm or change the decimals to fractions.

18. A preschool is putting new fence up in their triangular play yard and they are planting grass. Use the picture below to answer the questions that follow.



a. If the fence is to go around the perimeter of the triangular yard how much fencing will they need?
They will need 145.48 feet of fence.

b. The grass will be planted everywhere in the yard except in the square sandbox. How many square feet of grass will they need to plant.
They will need 647.64 square feet of grass.

c. Grass costs \$0.35 per square foot. How much money will they spend on the grass?
The grass will cost \$226.68 for the grass.

d. The fencing comes in panels of 9.25 ft. How many panels of fencing do they need to order?
They will need to order 16 panels of fencing.

19. Hallie is in 6th grade and she can buy movie tickets for \$8.25. Hallie's father was in 6th grade in 1987 when movie tickets cost \$3.75.
*[*This is an Illustrative Mathematics Task]*

a. When he turned 12, Hallie's father was given \$20.00 so he could take some friends to the movies. How many movie tickets could he buy with this money? How much money would he have leftover?
Hallie's father could buy $5.\overline{3}$ movie tickets in 1987. However, given this context it does not make sense for him to purchase a partial movie ticket so it is sufficient to say that he could buy 5 movie tickets. He would have \$1.25 leftover.

b. How many movie tickets can Hallie buy for \$20.00? How much money will she have leftover.
Hallie can buy $2.\overline{42}$ movie tickets. Once again, given this context she we would say that she can buy 2 movie tickets. She will have \$3.50 leftover.

c. On Hallie's 12th birthday, her father said,

When I turned 12, my dad gave me \$20 so I could go with three of my friends to the movies and buy a large popcorn. I'm going to give you some money so you can take three of your friends to the movies and buy a large popcorn.

How much money do you think her father should give her?
Since $4 \times 3.75 = 15$, a large popcorn had to cost \$5.00 or less if her father bought it with the change from buying the tickets. Hallie's movie tickets cost $8.25 \div 3.75 = 2.2$ times as much as movie tickets cost in 1987. Assuming the price of popcorn increased at the same rate, and since $2.2 \times 5 = 11$, she should be able to buy a large popcorn for \$11.00. Four tickets cost $4 \times 8.25 = 33$ dollars. With these assumptions, Hallie's father should give her at least \$44.00.

0.3d Homework: Solving Problems Multi-Digit Decimals

Directions: Perform each indicated operation.

1. $0.6 + 2.633$	2. $5.13 - 1.356$	3. $17.4(0.23)$ 4.002	4. $4.45 \div 0.005$
5. Add 26.53, 1.6, and 340	6. Find the difference of 324.05 and 9.623 314.427	7. Find the product of 1.2 and 822.15	8. What is the quotient of 42.05 and 0.022? 1911.36…
9. $9.43 - 0.18$ 9.25	10. $1.01(6.2)$	11. 0.47×3.01 1.4147	12. $4.192 + 1.255$
13. $9.2 \div 0.375$ $24.5\overline{3}$	14. $500 \div 3.2$	15. $41.3 + 0.28 - 14.005$	16. $2500 - 2350.006$

Solve each problem

Problems 19 and 20 are Illustrative Mathematics Tasks

19. Jayden has $20.56. He buys an apple for 79 cents and a granola bar for $1.76.

    a. How much money did Jayden spend?

    b. How much money does Jayden have now?

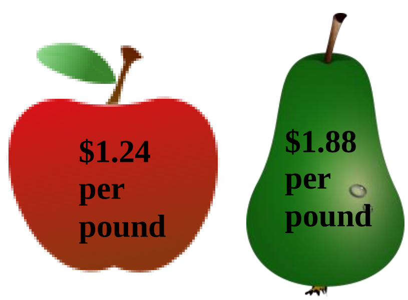
20. Seth wants to buy a new skateboard that costs $167. He has $88 in the bank.

    a. If he earns $7.25 an hour pulling weeds, how many hours will Seth have to work to earn the rest of the money needed to buy the skateboard?
    Seth will have to work about 10.9 hours to earn enough money. It is appropriate if a student rounds their answer to $11 given the context.

    b. Seth wants to buy a helmet as well. A new helmet costs $46.50. Seth thinks he can work 6 hours on Saturday to earn enough money to buy the helmet. Is he correct?
    Seth is not correct. He will only earn $43.50. He will need 3 more dollars to buy the helmet.

    c. Seth's third goal is to join some friends on a trip to see a skateboarding show. The cost of the trip is about $350. How many hours will Seth need to work to afford the trip?
    Seth will need to work about 48.28 hours to afford the trip. It is appropriate if a student rounds their answer to 49 hours given the context.

21. You buy 2.8 pounds of apples and 1.375 pounds of pears. You hand the cashier a $20 bill. How much change will you receive?



22. A box company makes a certain box in two sizes. The material used to make both boxes costs $1.50 per square foot. Use the information below to answer each question

	Length	Width	Height
Box 1	15.5 cm	4.06 cm	3 cm
Box 2	10.85 cm	5 cm	3.48 cm

a. What is the volume of each box?

b. How much does a person save by choosing to make Box 2 instead of Box 1.

23. A car can travel 50.7 miles on 2 gallons of gasoline.

a. How far can the car travel on 9.5 gallons of gasoline?
The car can travel 240.825 miles on 9.5 gallons of gasoline.

b. A hybrid car can travel 68.4 miles on 2 gallons of gasoline. How much farther can the hybrid car travel on 9.5 gallons of gasoline?
The hybrid car can travel 84.075 more miles than the regular car on 9.5 gallons of gasoline.

24. Tickets to the school play cost $5.25. The amount received from tickets sales is $640.50. How many tickets were sold?