Chapter 3
Extending the Number System

In grades K-5, students work from counting numbers to the representation of numbers on the *number line*. In grade 2 they learned how to put counting numbers at equally spaced intervals as tick marks on a straightedge. Then in grade 3 students learned to include fractional numbers p/q on the line, by subdividing each of the unit intervals into q equal line segments and then appending together p copies.

In grade 6 students begin to work with rational numbers (positive and negative), learning how to place them on the number line, and compare them in terms of position on the line. The basic concept is that of the *unit*. The measurement that it represents has to be determined by the size of the numbers in the given context; this we refer to as the *scale* of the model. If the students are asked to plot the numbers 3, 7, 11 at one time, and at another to plot 88, 132 and 157, they understand that the scale should be different in the two problems: in the first, it is alright to have the first hash mark represent 1, but that will not do for the second problem: a choice of 10 (or, even better, 20) for the first tick mark allows us to represent all these numbers on the same page.

In grade 5, students learn to associate whole numbers with points on a ray, and from that, to do the same for fractions (positive rational numbers). The initial point of the ray is zero, and the unit for this model is determined by a designated point along the ray as corresponding to 1 unit. Although the ray can point in any direction, it is customary to visualize this process on a horizontal line, with the origin at the left end of the line. If the content strongly suggests a vertical line (i.e., if we are discussing altitude), the the ray will point upwards.

Any such ray lies on a line, and one of the goals in grade 6 is to make arithmetic sense of the whole line, by associating the points on the other side of the origin to numbers. That is, we make tick marks (for a horizontal model) to the left of the origin by using the unit as the basic measure. These points could be said to represent *left* numbers, whereas the original representation is that of *right* numbers. However, there is good reason to designate the set of points to the right of zero as the *positive* numbers, and those to the left as the *negative* numbers: it makes sense in many contexts, and second, provides a good model for arithmetic operations, as will be learned in grade 7. In grade 6 we concentrate on the use of negative numbers in specific contexts.

On this number line, two different points equally distant from the origin are called *opposites*: given a number n, the number on the other side of 0 and of the same distance from 0 is designated $-n$. Thus 5 is five units to the right of 0, and -5 is five units to the left of zero. Similarly, -3 is 3 units to the the left of 0, and thus $-(-3)$ is three units to the right of 0. But that point already represents the number 3, so we conclude that $-(-3) = 3$.

The distance of a point a from the origin is called the *absolute value* of a, and is designated as $|a|$. A point and its opposite are the same distance from 0 so they have the same absolute value: $|a| = | - a|$.

A source of confusion might result from the double use of the minus sign $(-)$. On the one hand, it is the sign of a negative number, on the other hand, it means "take the opposite." So, the opposite of 3 is -3, and the opposite of -3 is 3, considered as $-(-3)$. This confusion arises when we point out that if a is a negative number, then $|a| = -a$. One might avoid this confusion by avoiding such statements; but one can also confront and discuss this double use of the minus sign $(-)$.

This is compounded by the fact that students already understand the minus sign as the operation of subtraction. So, at this time, it should be made explicit that the symbol for negative number is the same as the symbol for the operation of subtraction and, the reason for this is that these uses are consistent.

This convention is to be made explicit in grade 7, however, the issue may come up and the grade 6 teacher will have to discuss that. The fact that this symbol is used in multiple ways is actually an enhancement of arithmetic, not an impediment, because these multiple uses are consistent with each other. Let's list these interpretations of the negative sign.

- For positive numbers a and b with $a > b$, $a - b$ is the operation of taking b away from a.
- For a positive number a, $-a$ is the point on the left side of the origin of the same distance from 0 as a,
- For any number a, $-a$ is the point on the other side of the origin of the same distance from 0 as a.

At one time textbook writers tried to resolve this confusion by using different symbols for "opposite" and "subtract," but it was quickly found to compound the confusion. This should have been anticipated, because the use of the same sign ("−") for these operations *enhances* arithmetic notation. Once resolved, this confusion becomes a natural piece of arithmetic. The resolution, in grade 7, is the realization that $-a$, in terms of "opposite," is the same as $0 - a$ in terms of subtraction. It follows that $a - b$ is the same as $a + (-b)$: that is: "a subtract b" is the same as "add to a the opposite of b." And this is true for *all* numbers, positive or negative. As this is difficult, but still important, for students to grasp, and as the issue is likely to arise in grade 6, we expose and discuss the issue in this chapter.

We start the discussion with contexts involving increase and decrease in values where the decrease could be larger than the increase, using the concept of symmetry. The next important question is that of scale, which is to be determined by the context in which we are working. This leads directly to a discussion of associating appropriate values to the tick marks on the number line so that our data is sensibly demonstrated.

This leads naturally to the representation of rational numbers on the number line. If we are given a number line with the unit identified, we can now include new tick marks that divide that unit into q equal parts, so that now the rational number p/q is represented by p repetitions of that qth part. If p is positive the repetitions are to the right, of 0 and if p is negative, to the left of 0.

Finally, this construction is brought to the coordinate plane. In grade 5, students plotted pairs of positive numbers in the plane. In grade 6, we extend this to the full plane, observing that the existence of a vertical line and a horizontal line separates the plane into 4 pieces, called quadrants, and pairs of numbers, *positive or negative*, can be represented by points in the plane.

Section 1. The Symmetry of the Number Line

Understand that positive and negative numbers are used together to describe quantities having opposite directions or values (e.g., temperature above/below zero, elevation above/below sea level, credits/debits, positive/negative electric charge); use positive and negative numbers to represent quantities in real-world contexts, explaining the meaning of 0 in each situation. 6.NS.5.

Constructing a Number Line

In grade 5, students began the study of the number line more or less this way: draw a line, select a point on that line and designate it as 0, or the *origin*. Now measure out a line segment from the origin, with the new endpoint designating the number 1, representing the unit for this representation. The size of this segment is to be determined by the given data. Measuring out the same distance adjacent to the right brings us to the point representing 2; going one more such measure brings us to 3, and so forth. Now, by subdividing each such unit interval into q equal parts, we can, in the same way, associate a point to each number of the form p/q, for positive integers p, q.

At this time, one observes that, although the choice of horizontal line was made because it is easiest to work with, there is no need for it to be horizontal; it could be *any* line, although typically it will be horizontal or vertical depending upon the context. In preparation for the extension of the number line, students are asked to locate numbers on the number line, and to identify points on the number line as numbers. Here we shall review that procedure in specific contexts in Example 1, and then, for the extended line, in Example 2.

Example 1. a) In Figure 1 we have a table of the average temperatures (Fahrenheit) during the nine weeks of the months of July and August. Plot them on a number line.

Week	1	2	3	4	5	6	7	8	9
Temp	95	100	98	85	88	72	78	82	88

Figure 1

b) Here is a table showing the altitude of a rocket (measured in yards) in 10 second intervals during the first minute and a half of its flight.

Time (seconds)	10	20	30	40	50	60	70	80	90
Altitude	0	30	100	200	800	1500	2300	3500	6300

Figure 2

Illustrate these data on a number line.

Solution. a) We have a choice: to calibrate the number line by the week, and mark the locations of the temperatures along that line; or the other way around: calibrate the number line by the temperature, and locate the weeks along that line. We have chosen the latter; it could make a good discussion to discuss the appropriateness of that choice.

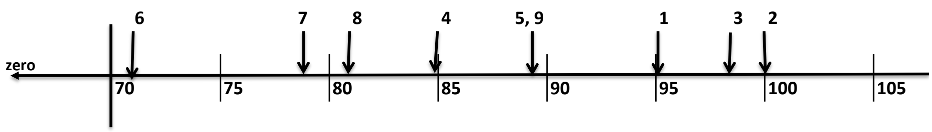
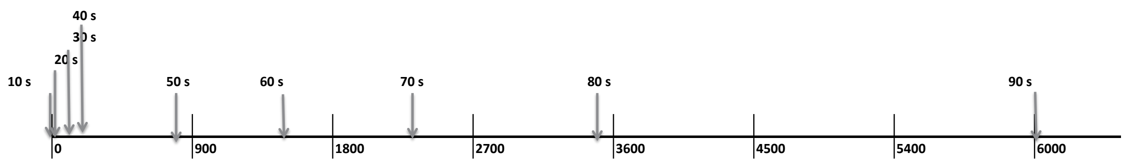


Figure 3

Notice the scale, the beginning point and the endpoint. Since all the temperature data lie between 70 and 100, we started our graph at 70°, ended it at 105°, and put the tick marks at 5° intervals. If we were to draw a conclusion from this graph, we'd probably say that, in our sample, summer temperates are (more or less) equally distributed between 70° and 100°.

b)



In this image, the line is calibrated according to altitude, with an arrow indicating the altitude in each of the first nine 10-second intervals. Here we have visualized that the rocket starts very slowly, but rapidly increases its velocity during the first 90 seconds.

A better way of visualizing the relationship between two quantities, as in the two tables above, is to make a *graph* of the associated numbers in the coordinate plane. Later in this chapter, after discussing the coordinate plane, we will do this (see Example 14).

In both cases we chose the size of the intervals so that we could show all the data in a reasonable way. We will elaborate this in the subsection *Scale*. Although these graphs illustrate positioning on the number line, they do not give us a lot of information (besides that of the spread of data). In the section on the coordinate plane we will graph these (as 2D data) again to reveal more information.

In grade 6, we turn our attention to the part of the number line opposite the positive numbers (typically represented as left of zero). We may need a context to explain what this part of the line represents. For example, the teller at a bank receives and pays out money. If the bank wants to keep track of all transactions, they might record receipts on one side of the line with the notation *R*, and payments on the other side with the symbol *P*. Or, if we want to locate cities on Interstate 70 with Kansas City, MO as the central point, then we would record Denver, CO as W605 mi, and Washington, DC as E1057 mi. If our problem had to do with altitudes, we might use a vertical number line, positioning the origin at sea level.

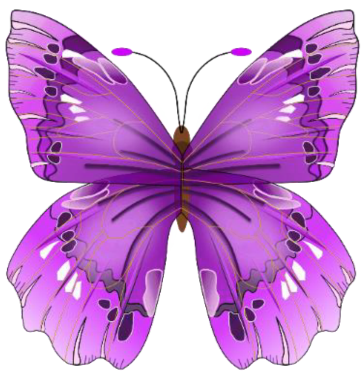
The task, however, for the mathematician is to provide a versatile tool that can be adapted to *any* situation, and to be effective arithmetically. That tool is the *full number line*. As its constuction is dependent upon the concept of symmetry, we begin there.

Symmetry

The concept of *symmetry* was introduced in grade 4 with this standard:

Recognize a line of symmetry for a two-dimensional figure as a line across the figure such that the figure can be folded along the line into matching parts. Identify line-symmetric figures and draw lines of symmetry. 4.G.3.

The image to the left in Figure 1 demonstrates symmetry about a vertical line, but not about any horizontal line. The figure to the right demonstrates symmetry about both a vertical and a horizontal line.



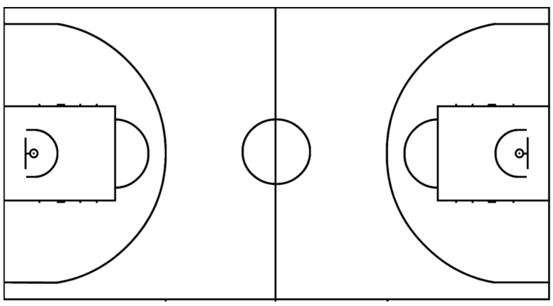


Figure 1a: Symmetry about a vertical line **Figure 1b: Double symmetry**

Let's now work with the idea of symmetry and the rules of arithmetic to create an extension of the number line that identifies points to the left of the origin in a mathematical way. Start with a line, a point designated as 0 and a different point designated as 1. Without any specific context to tell us otherwise, we will take the line to be horizontal, and the points $0, 1$ situated so that 1 is to the right of 0. Now, how do we designate the choice on the other side?

We appeal to arithmetic, since our intent is to extend the arithmetic of the number line. The interval from) to 1 we call the *unit interval*. We now translate that interval so that 0 lands on 1 and designate the point where 1 lands as the number 2. Otherwise put, we have appended a unit interval so that its left endpoint is the point 1, and associated the number 2 to its right endpoint. Continuing in this way we can assign whole numbers to points successively to the right of 0 How do we, from any one point, go one unit to the right? We add one. How do we go one unit to the left? Going one unit to the left undoes the move one unit to the right, which is represented by addition by 1. So going left one unit undoes "addition by 1" so should be represented by "subtraction by 1." Then moving one unit to the left of 0 is $0 - 1$, and moving 2 units to the left is $0 - 2$, and so forth. Now, $0 + 1$ and $0 + 2$ are represented by 1 and 2. Similarly, we represent $0 - 1$ and $0 - 2$ by -1 and -2 and so forth.

Associating the sign of negative numbers through the operation of subtraction could lead to interesting discussions. These will be taken up in grade 7 where we introduce arithmetic operations on the whole real line. This might be a good time to set up that discussion without going into it too deeply. For example: going 7 units to the left from 0 puts us at the point -7. If we go 7 units to the left of 11, we end up at 4. If we go 7 units to the left from 3, where do we end up? Just count the units: 3 units left from 3 we are at 0, and in another 4 units, we are at -4. These moves are exhibited in Figure 2 below. Now, in grade 7, we interpret $a - b$ as the operation of moving b units to the left of a, not just for $a > b$, but for *any* numbers on the number line.

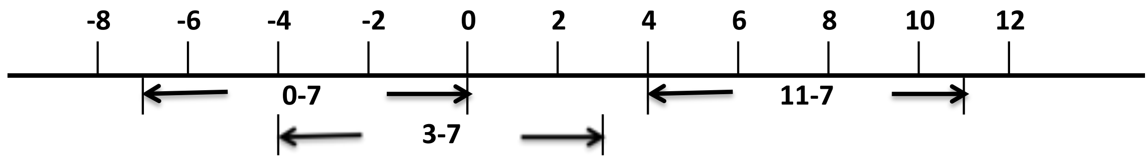


Figure 2

The image of the number line (Figure 3) has a central point designated as the *origin*, marked by a small perpendicular line (called a *tick mark*), and exhibits symmetry about that tick mark. For every positive number d, there are two points that are of distance d from the origin. We designate one as representing the number d, and we define the other point to be the number $-d$. Now, there are certain conventions, that we observe: if the number line is horizontal, then the positive number d is put to the right of the origin, and $-d$ to the left; if the number line is vertical, d is above the origin, and $-d$ is below.

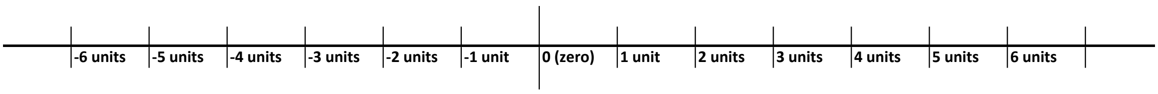


Figure 3. The Number Line

The **number line** consists of a straight line extending indefinitely in two directions from a central point, denoted as 0 (zero), and a line segment starting at 0, denoted as the *unit distance*. For each counting number n, a tick mark is made on the straight line that is n units distance from 0 (the origin). The marks on the right side of 0 are denoted by the counting numbers n, and the marks on the left side by $-n$, called *the opposite of* n.

For any positive integer q we can construct the qth part of the unit interval, and designate it as $1/q$. The point opposite to $1/q$ is designated $-1/q$. Now we can locate, for any integer p, the points p/q and $-p/q$ in the same way.

The word *unit* designates the measure of the interval from 0 to 1. For a problem in a given context, the meaning and the size of that interval is determined by the context. If we are talking about the bank teller, the unit might be dollars; if the question is distance relative to Kansas City it would be miles. Context plays a big role: if the bank is an international business bank, the unit might be thousands of dollars, and if we are flying rather than driving, the unit of distance to Kansas City might be a hundred miles. When we come (see below) to putting fractions p/q on the number line, we do the same thing, but taking the unit as the qth part of the original unit.

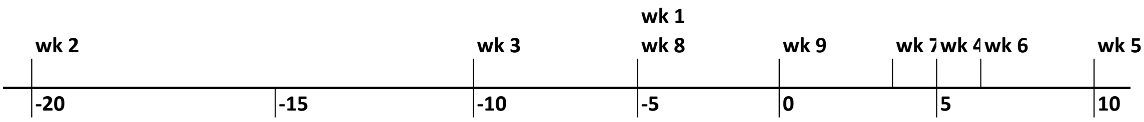
Example 2. a) Here we have a table of the average temperatures (Celsius) during the nine weeks of the months of December and January. Locate the weeks on a number line representing temperature.

Week	1	2	3	4	5	6	7	8	9
Temp	5	-20	-10	5	10	6	4	-5	0

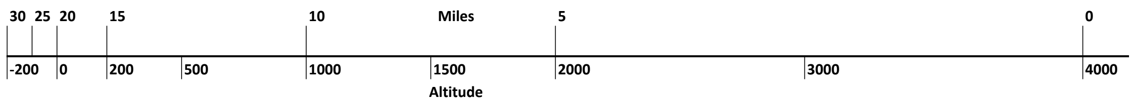
b) This table shows the altitude from the top of a volcanic Indonesian island, to the edge of the continental shelf about 45 miles at sea from the island, given at 5 mile intervals. Locate the mileages on a number line representing altitude.

Miles	0	5	10	15	20	25	30
Altitude	4000	2000	1000	200	0	-100	-200

Solution. a)



b)



These graphics are adequate for showing where the data fall. As for a), we might conclude that temperatures in winter center closely around 0°C, except, occasionally it can get very cold. Similarly, from the graphic for b), we could conclude that, for a small portion of the distance, we are significantly above sea level, but for most of the distance we are at or below sea level. Studying this graph a little more, we note that the mileage is increasing as the altitude decreases. In Example 15, we will graph these data in a way that leads to deeper conclusions more easily.

Opposites

We have already introduced the word *opposite*; here we want to understand it better as required by this standard:

a) Recognize opposites signs of numbers as indicating locations on opposite sides of 0 *on the number line; recognize that the opposite of the opposite of a number is the number itself, e.g.,* $-(-3) = 3$*, and that* 0 *is its own opposite.*

c) Find and position integers and other rational numbers on a horizontal or vertical number line diagram . . .

6.NS.6ac.

This we do, as described above, simply by repeating the techniques learned in grade 5, but this time, using both sides of the origin, designating one side as the negative numbers. When looking at a number line, two numbers are opposites when they are the same distance away from zero, but in opposite directions. Those numbers to the right of 0 are the positive numbers, and those on the left are the negative numbers. For example, "3" represents the point that is 3 units to the right of 0, and "−3" is its opposite, three units to the left of 0. Of two opposites, one will have a positive sign, the other a negative sign. It is customary to not write the plus sign in front of a positive number, so +3 will be denoted by 3.

In Figure 4 below, the point symmetric to a given point X is identified as $-X$. Also, notice that symmetry is a two way concept: if points P and Q are symmetric about 0, then $Q = -P$ and $P = -Q$. It follows that *the point symmetric to the point symmetric to P is P*; in other words, $-(-P) = P$.

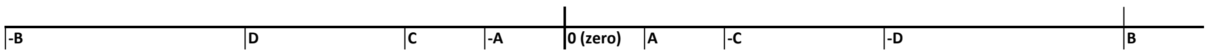


Figure 4

Without a caution, it is natural for students to assume that the negative sign in front of a letter representing an unknown number indicates a point to the left of 0; that is, $-P$ is to the left of 0, wherever P is. But if P is already to the left of 0, since $-P$ is the point symmetric to P, it has to be to the right. This confusion arises because we are using the sign − in two ways: a) to designate negative numbers; b) the point symmetric (about 0) to a given number. For a positive number, say 5, this is correct: −5 is the point on the number line to the left of 0 by 5 units; it is also the point to the left of 0 that is symmetric to 5. The same logic proceeds if we start with a negative number, say −3: −(−3) is the point to the right of 0 that is symmetric to −3, and it is also the point on the number line to the right of 0 by 3 units. We conclude that $-(-3) = 3$.

In grade 2, students learned about the operation of subtraction, designated by the symbol "−." Here that symbol is used to designate the number opposite to a given number. One of the deepest truths about arithmetic is that these two uses are consistent, and supportive of each other. Thus, we can interpret 5 − 7 as 5 *plus* −7, and so give it meaning as the opposite of 2 (namely −2). This issue is confronted in Grade 7, but might arise in this discussion.

For any number a placed on the number line, its distance from the origin is denoted $|a|$, called the *absolute value of a*. We will return to this concept later.

Scale

On a number line, the interval between 0 and 1 is the *unit*. The actual meaning represented by the unit is to be determined by the context: it could be 1 meter, 1 pound, 1 second, etc. It could also be 10 meters, one-quarter pound, 15 seconds, 1/3 of a furlong, etc. Remember, when you draw a number line, you only draw a piece of it, and that piece has to be sufficiently large and sufficiently scaled so as to exhibit the data of the problem being studied.

By *scale* we mean precisely the meaning of the unit. As pointed out above, if we are talking about driving time in a day, the unit might be 100 miles, whereas, if we are talking about flying times, a more appropriate choice would be 500 miles. There is another important concept used in those images, that of *range*: expressed by the lower and upper limit of the measures in the data.

Going back to Examples 1 and 2, we scaled, without saying so, the figures so as to accommodate the data and its range.

Example 3. In Example 1a, the *scale* (that is the measure in terms of the context between two tick marks) is 5°F, and the range is from 70°F to 105°F. What are the scales and ranges for Examples 1b, 2a,2b?

Solution.

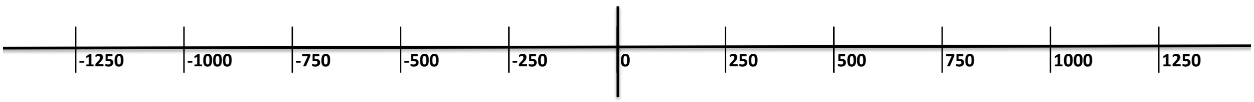
1b) range: 0 yards to 6000 yards scale: 900 yards.

2a) range: -20 to $10°C$ scale: $5°C$.

2b) range: -200 to 4000 feet scale: variable.

In example 2c the author of the image decided that below sea level and above sea level measurements could not show significantly on the same scale. This strategy is often employed when the author feels that some images need more emphasis than others.

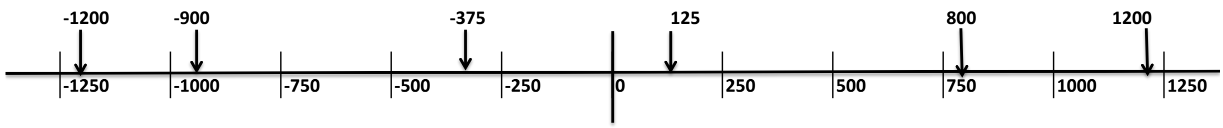
Example 4. In the piece of the number line exhibited in the figure below, what are the scale and range?



Indicate the position of the following numbers on that graphic:

a) -900 b) 1200 c) 125 d) -375 e) -1200 f) 800

Solution. The scale is 250 units and the range is -1250 units to 1250 units.



Rational Numbers on a Number Line

Understand a rational number as a point on the number line. Extend number line diagrams familiar from previous grades to represent points on the line with negative number coordinates. 6.NS.6

This freedom to specify what is meant by the first tick mark gives us a way to work with fractional amounts. In fact, this has already been done in grade 3, where the relevant standard is:

Understand a fraction as a number on the number line; represent fractions on a number line diagram.

a. Represent a fraction $\frac{1}{b}$ on a number line diagram by defining the interval from 0 to 1 as the whole and partitioning it into b equal parts. Recognize that each part has size $\frac{1}{b}$ and that the endpoint of the part based at 0 locates the number $\frac{1}{b}$ on the number line.

b. Represent a fraction $\frac{a}{b}$ on a number line diagram by marking off a lengths $\frac{1}{b}$ from 0. Recognize that the resulting interval has size $\frac{a}{b}$ and that its endpoint locates the number $\frac{a}{b}$ on the number line.3.NF.2

We repeat the grade 3 procedure so as to include negative fractions, by now going to both sides of 0. Start with the image of the number line in Figure 3 (on p. 5), but delete the word *unit*, for now we intend to change the unit.

In Figure 3 in the first line, we have introduced the new unit $\frac{1}{2}$ which is exactly one half of the preceding unit, and then identified the points $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}$ to the right of 0, and the points $-\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, -\frac{7}{2}$ to the left.

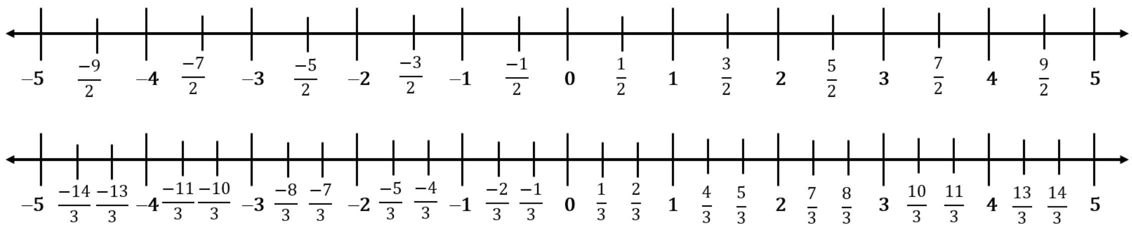


Figure 5. The Number Line with Fractions

In the second line, we have done the same thing, using thirds. This process can be repeated for any positive integer q to represent numbers of the form $\frac{p}{q}$ for integers p that are positive or negative. In particular, the numbers $\frac{p}{q}$, $\frac{-p}{q}$ appear at the same distance from 0, one to the left, and the other to the right.

The Coordinate Plane

Understand a rational number as a point on the number line. Extend coordinate axes familiar from previous grades to represent points in the plane with positive and negative number coordinates.

b. Understand signs of numbers in ordered pairs as indicating locations in quadrants of the coordinate plane; recognize that when two ordered pairs differ only by signs, the locations of the points are related by reflections across one or both axes.

c. Find and position integers and other rational numbers on a horizontal or vertical number line diagram; find and position pairs of integers and other rational numbers on a coordinate plane. 6.NS.6

Recall that in grade 5, pairs of positive numbers were plotted as *coordinates* on a plane, according to the standard:

Use a pair of perpendicular number lines, called axes, to define a coordinate system, with the intersection of the lines (the origin) arranged to coincide with the 0 on each line and a given point in the plane located by using an ordered pair of numbers, called its coordinates. Understand that the first number indicates how far to travel from the origin in the direction of one axis (to the right on the horizontal), and the second number indicates how far to travel in the direction of the second axis (up on the vertical), with the convention that the names of the two axes and the coordinates correspond (e.g., x-axis and x-coordinate, y-axis and y-coordinate). 5.G.1

In this section we extend the coordinate plane so as to include all pairs of numbers, positive as well as negative. First, start as in grade 5: draw a horizontal and a vertical line in a grid, and denote the point of intersection as the *origin*, with the horizontal line the x-axis and the vertical line the y-axis. Now, we plot a pair of numbers (a, b), where a and b are positive integers, by going a units to the right and then b units upwards. That point represents the pair (a, b).

Students may be curious about the rigidity of this prescription. Do the lines have to be horizontal and vertical? Do they have to intersect in a right angle. Do the unit lengths on the two line have to have the same measure? Do the units have to be in the same position relative to each other. From Linear Algebra, we know that the answer is no: all that is needed is two lines that intersect, and a designation and placement of unit length on each line. In fact, when Descartes invented his coordinate system, he did not specify these details, and his contemporaries made choices that suited them best. Fermat liked to have positive values of x go down and positive values of y go to the left. Barrow in England chose another way of representing pairs of numbers. As a result, there was a confusing array of representations. The inventors of the Calculus, Newton and Liebniz, each made a specific choice of coordinatization. We are lucky that they made the same choice, which has become the standard representation, which we now describe:

The Standard Coordinatization of the Plane

Draw a horizontal line and a vertical line, so that the point of intersection is center page. That point is called the *origin* and is denoted $(0, 0)$.

Decide which variable is to be measured along the horizontal line, and which is measured along the vertical line. We shall denote the horizontal variable by x, and the vertical by y.

Select a measure to represent a unit in the x variable, and put a point at that measure to the right of the origin on the horizontal axis. Select a measure to represent a unit in the y variable, and put a point at that measure above the origin on the vertical axis.

Given those units, replicate the number line on both axes. Now, (x, y) is found by moving x units on the horizontal axis, and then y units in the vertical direction.

It is to be understood that x and y can be positive or negative numbers, and that the tick marks are put along the axes as they are on any number line. So, the point $(-5, 0)$ is 5 units to the left of the origin on the horizontal axis, and $(0, -7)$ is 7 units down the vertical axis. We note that the choice of the direction of the y-axis is not accidental, given the choice on the x-axis. It is a replica of the x-axis, rotated in the counterclockwise direction by 90°. It is best to follow the following discussion while looking at Figure 6.

The coordinate grid of grade 5 is just the upper right quarter of the grid in Figure 6, and is referred to as the *first quadrant*, denoted Q1. Starting in Q1, extend every horizontal line to the left of the vertical axis, into what is called the *second quadrant*, Q2, by locating, for every point on that line in Q1, its opposite on that line in Q2. For example, the point (8,3) has an opposite on the line $y = 3$, denoted as the point $(-8, 3)$. In general, the point (a, b) in Q1 has its image, reflected in the y-axis, $(-a, b)$.

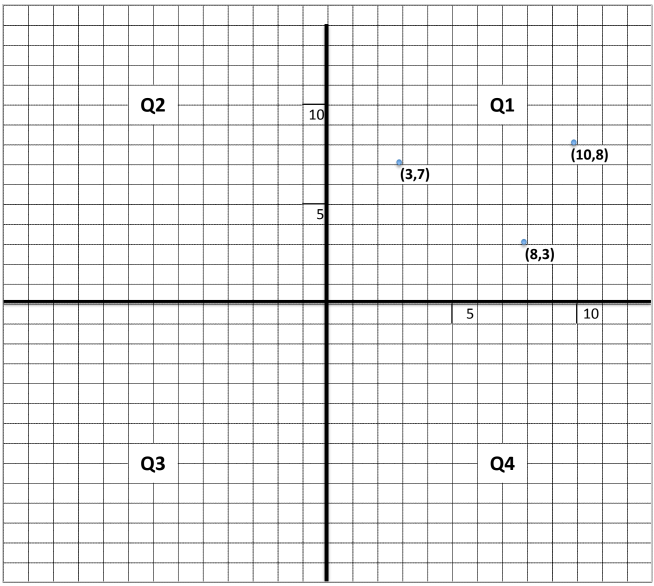


Figure 6

Now we do the same thing across the horizontal axis, by extending every vertical line, and naming all points on those lines across the x-axis as the opposites of the points in Q1 and Q2. The section below the quadrant Q2 is called the *third quadrant*, Q3, and the section below Q1 is called the *fourth quadrant*, Q4. We see that the point vertically symmetric to (8,3) is $(8, -3)$, for it is opposite to $(8, 3)$ on the line $x = 8$ on the y-scale, so it is the y coordinate that changes signs. Similarly, the point vertically symmetric to $(-8, 3)$ is $(-8, -3)$, for it is opposite to $(-8, 3)$ on the line $x = -8$. Note the important fact that this is consistent: the point $(-8, -3)$ (in Q3) is not only vertically opposite to the point $(-8, 3)$ in Q2, but it is horizontally opposite to the point $(8, -3)$ in Q4.

In Figure 7 we have included all points generated by the given points in $Q1$ in Figure 6, by taking opposites in each of the axes.

So, for example, if we start with a point in Q1, say (10,8), the point opposite to it in Q2 is $(-10, 8)$; the point opposite to that in Q3 is $(-10, -8)$; the point opposite to that in Q4 is $(10, -8)$, and the point opposite to that in Q1 is $(10, 8)$, bringing us around full circle.

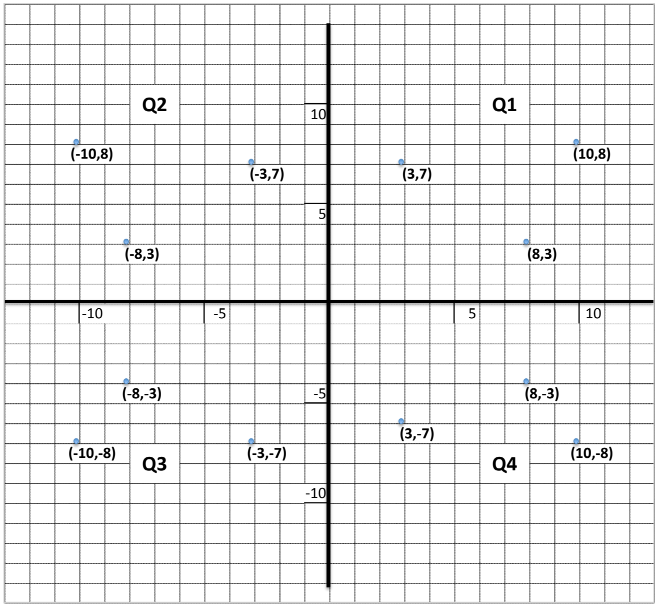


Figure 7

In this description we have used the word "opposite" in two senses, depending upon the quadrant in which we want to find the opposite: that vertically opposed or horizontally opposed. A better word is that of *symmetry*: $(10, 8)$ and $(-10, 8)$ are symmetric to each other in the vertical axis, and $(10, 8)$ and $(10, -8)$ are symmetric to each other across the horizontal axis.

It will seem counter intuitive to students that, to make a point symmetric on the y axis, you take the opposite of the x coordinate; to make a point symmetric on the x axis, you want the opposite of the y. This should be a conversation with students. It may help to emphasize that, in the coordinate grid, the horizontal lines are the lines upon which the x value of the point does not change, and the vertical lines are the lines upon which the x value of the point does not change. Thus, when we seek a pair of points that are symmetric in the y axis, they lie on the same horizontal line, so the y value doesn't change. Thus it must be that the x values are opposites.

Looking further ahead, such a discussion might prepare students for the graphing of transformations on functions. For example, if we reflect the graph of $y = f(x)$ in the y-axis, it is the x value that is replaced by its opposite $(-x)$, so the equation of the reflected graph is $y = f(-x)$. Similarly, if we reflect the graph of $y = f(x)$ in the x-axis, it is the y value that is replaced by its opposite $(-y)$, so the equation of the reflected graph is $-y = f(x)$, or $y = -f(x)$.

There is one more symmetry of which to take notice. Given the point (a, b) consider the line through the origin and (a, b) . The point on that line that is of the same distance as (a, b) from the origin is the point on the line *symmetric to* (a, b) *in the origin*. Its coordinates are: $(-a, -b)$, and we say that $(-a, -b)$ and (a, b) are *symmetric about the origin*.

Symmetry in the coordinate plane:
The points (a, b), $(-a, b)$ are *symmetric* about the y-axis.
The points (a, b), $(a, -b)$ are *symmetric* about the x-axis.
The points (a, b) and $(-a, -b)$ are *symmetric* about the origin.

Look more closely at Figure 7. The point $(-3, -7)$ in Q3 is symmetric in the x-axis to the point $(-3, 7)$ in Q2, which in turn is symmetric in the y-axis to the point $(3, 7)$ in Q1. At the same time the point $(-3, -7)$ is symmetric through the origin to the Q1 point, (3,7). This leads to the rule:

Symmetry and coordinates:
a) to find the point symmetric to a given point in the x-axis, change the y coordinate to its opposite,
b) to find the point symmetric in the y-axis to given point, change the x coordinate to its opposite,
c) to find the point symmetric through the origin, change both coordinates to their opposites.

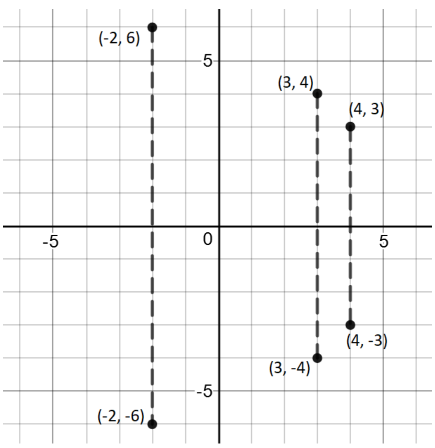
As a *rule* this could be easily misused; for this reason it is important to stress that along horizontal lines the y-value is constant, and along vertical lines the x- value is constant. Another way to say this is that the x-axis is the set of points whose y-coordinate is zero, and the y-axis s the set of points where the x coordinate is zero.

Example 5. The following table gives the coordinates of 8 points in the coordinate plane Create a table that includes the 3 points that are symmetric in the x-axis to the first 3 points, the 3 points that are symmetric in the y-axis to the next 3 points, and the 2 points that are symmetric in the origin to the last 2 points.

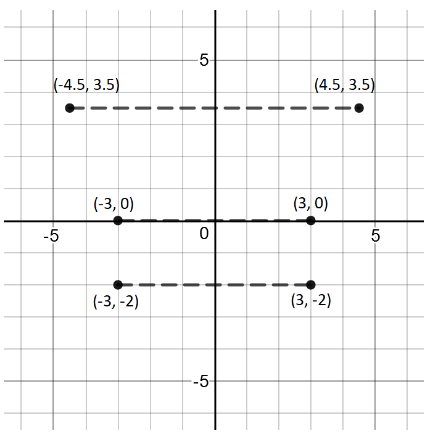
x	4	3	−2	−3	4.5	−3	0	5
y	3	−4	−6	0	3.5	−2	5	−1

Solution.

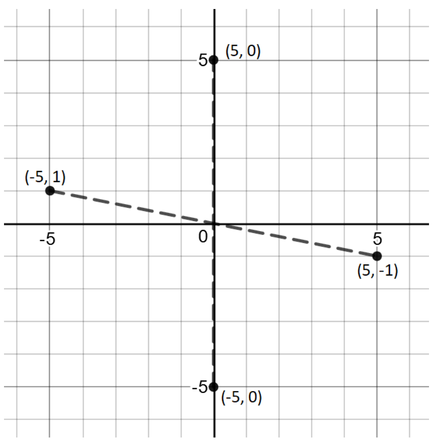
	symmetry in x axis			symmetry in y axis			symmetry in origin	
x	4	3	−2	−3	4.5	−3	0	5
y	3	−4	−6	0	3.5	−2	5	−1
x	4	3	−2	3	−4.5	3	0	−5
y	−3	4	6	0	3.5	2	−5	1



x-axis symmetry



y-axis symmetry



symmetry in the origin

In all of these problems we have not specified the units being used, nor the size in those units of the line segment between two gridlines. The units, of course, are to be chosen in terms of the context, and can be ft., lbs., m., minutes, nanoseconds, etc. Now, for some problems the range of the given data may be such that we cannot put all data points on the same graph, if we choose the distance between grid marks to be one unit. In such a case, we should look for a scale that works for the given data. In particular, the scale in the two directions need not be the same.

Example 6. Plot these points on a coordinate plane:

x	20	0	80	50	-10	−90	20	50
y	−20	10	60	−20	−40	80	20	−80

SOLUTION. First we draw a grid, placing the origin right in the middle of the grid, since there are many negative and positive numbers involved. We then note that all numbers are multiples of 10. Thus it makes sense to designate the edge size of each grid box to be 10 units. To the right is the resulting graphic representation. We have included the grid for greater clarity.

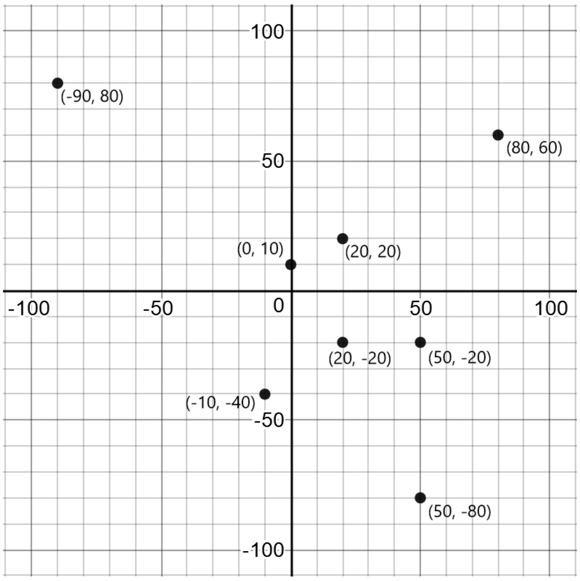


Figure 8

Example 7. a) Plot the points given in the table below.
b) Join the points by line segments in the sequence given in the table.
c) What symmetry does the figure exhibit?

x	0	2	7	3	6	0	-6	−3	−7	−2	0
y	10	4	4	1	-4	0	−4	1	4	4	10

SOLUTION. The points have been plotted in Figure 9, and then joined by line segments sequentially from the table, displaying a 5-point star. The figure is symmetrical about the y-axis.

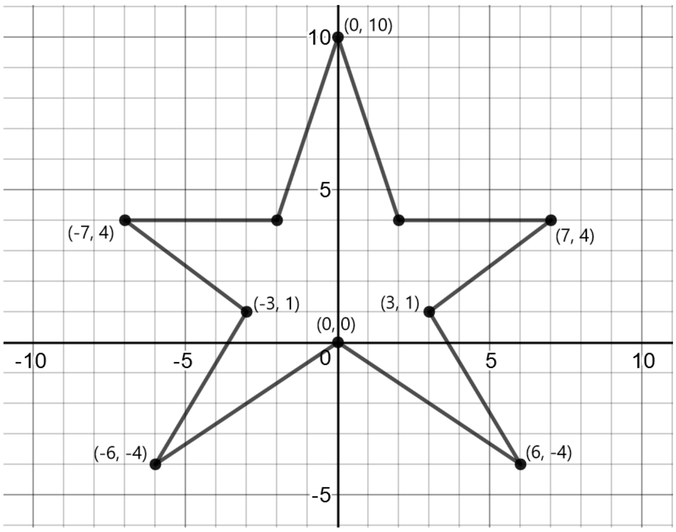


Figure 9

Section 2. Order in the Number Line

Understand ordering and absolute value of rational numbers.

a) Interpret statements of inequality as statements about the relative position of two numbers on a number line diagram. For example, interpret $-3 > -7$ *as a statement that* -3 *is located to the right of* -7 *on a number line oriented from left to right.*

b) Write, interpret, and explain statements of order for rational numbers in real-world contexts. For example, write $-3°C > -7°C$ *to express the fact that* $-3°C$ *is warmer than* $-7°C$*.*

c) Understand the absolute value of a rational number as its distance from 0 on the number line; interpret absolute value as magnitude for a positive or negative quantity in a real-world situation. For example, for an account balance of -30 *dollars, write* $|-30| = 30$ *to describe the size of the debt in dollars.*

d) Distinguish comparisons of absolute value from statements about order. For example, recognize that an account balance less than -30 *dollars represents a debt greater than 30 dollars. 6.NS.7.*

Our students already have a sense that, given two different positive numbers, one is bigger than the other. In particular, as we move to the right along the number line, the numbers get bigger. This property of positive numbers persists in the full number line, with a little bit of care in language. Is 3 bigger than -7? To move from positive numbers to all integers (in fact all numbers) we should reinterpret the statement $a < b$ to mean that a is to the left of b. Clearly, given any two different points on the number line, one is left of the other. Unfortunately, we do not have the luxury of changing usage, so the symbol "<" will continue to be read as "less than," but should be thought of as "to the left of."

With this understanding of the symbols "<" and ">", we can state:

> The rational numbers are ordered: we say *a is less than b*, and write $a < b$, whenever b is to the right of a on the number line. This is the same as saying that a is to the left of b.
>
> Given two numbers a and b one of these statements is true: $a = b$, $a < b$, $a > b$.
>
> The absolute value $|a|$ of an rational number a is the distance from the point on the line to 0. A number and its opposite have the same absolute value.
>
> Every rational number has an opposite, or additive inverse. 0 is its own opposite.

Example 8. Compare the two given numbers. The choices are: $<$, $=$, $>$.

$$\text{a) } -\frac{3}{7} \text{ and } \frac{1}{5} \quad \text{b) } \frac{2}{3} \text{ and } \frac{3}{2} \quad \text{c) } -\frac{2}{3} \text{ and } -\frac{6}{4} \quad \text{d) } -\frac{12}{3} \text{ and } -\frac{9}{4} \quad \text{e) } -\frac{12}{3} \text{ and } \frac{9}{4}$$

SOLUTION.

a) $-\frac{3}{7} < \frac{1}{5}$ because the first term is negative (left of zero) and the second is positive (right of zero). By the same reasoning, any positive number is greater than any negative number.

b) $\frac{2}{3}$ is less than one, and $\frac{3}{2}$ is bigger than 1. So $\frac{2}{3} < \frac{3}{2}$. Another way to reason is this: the first fraction expresses 2 parts of one-third, and the second expresses 3 parts of one-half. We need the same unit to compare, so we go to *sixths*. The first is four sixths, and the second is nine sixths, and thus the second is the greater amount.

c) Just above, we concluded that $\frac{6}{4}$ is to the right of $\frac{2}{3}$. Taking opposites, $-\frac{6}{4}$ is left of $-\frac{2}{3}$, so $-\frac{2}{3} > -\frac{6}{4}$.

d) We can do the same analysis, or we can simplify. $-4 < -2\frac{1}{4}$.

e) $-12/3 < 9/4$ by the same reasoning as a).

The next figure shows all of these rational numbers on a number line.

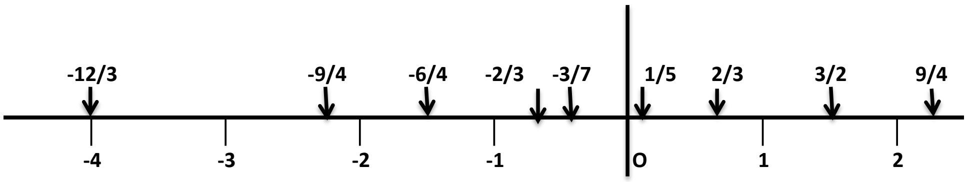


Figure 10

Example 9. We are given two rational numbers, a and b and we are told that $a < b$. Which of the following statements are true, which false, and which are sometimes true and sometimes false? Explain.

a) $-a < -b$ b) $|a| < |b|$ c) $a < |b|$ d) $a + b \leq |a + b|$ e) $b > -a$.

SOLUTION. The assertion states that a is to the left of b on the number line. Keeping that thought in mind, we can answer:
a) Since taking opposites reverses direction, and we are told that $a < b$, then $-a > -b$, so this statement is false.

b) If a and b are nonnegative, the statements $a < b$ and $|a| < |b|$ are the same, so in this case the statement is true. If a and b are both negative, then the statement $a < b$ tells us that a is farther from the origin than b, so $|a| > |b|$, so b) is false. If $a < 0$, and $b > 0$, then it depends upon which is further from the origin. These are all the cases.

c) This is correct. For any number b, we know that $b \leq |b|$, so if $a < b$, then surely $a < |b|$.

d) This problem is interesting because any number is always less than or equal to its absolute value, so this is true *no matter* the values of a and b.

e) If b is closer to the origin than a (for example, if $a = -10$ and $b = 1$), the statement is false. Otherwise it is true.

Section 3. Negative Numbers in the Real World

Solve real-world and mathematical problems by graphing points in all four quadrants of the coordinate plane. Include use of coordinates and absolute value to find distances between points with the same first coordinate or the same second coordinate. (6.NS.8)

First, let us look at various ways that negative numbers come up in the real world. Of course, negative numbers come up naturally in Celsius measurement of temperature, where negative temperatures measure degrees below freezing, and they come up in accounting as designation of a debit. But there are also situations where measure is two sided, as in the following problems, where the use of the arithmetic of the integers plays an important role.

Example 10. To the right is a table of distances from NYC (in this sense) for cities that lie directly north and south of New York, starting with the furthest north and ending with the furthest south.

City	Miles
Great Falls	201
Albany	150
Peekskill	48
Yonkers	20
New York	0
Newark	15
Princeton	50
Philadelphia	96
Baltimore	191

Note: Here, city locations are identified by their centers. In this particular case, the center of New York City is Columbus Circle, so what we are being told is that the Town Hall of Yonkers is 20 miles north of Columbus Circle.

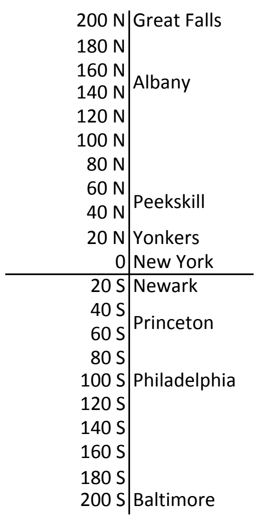
Fill in the blanks in this table:

	From	To	Miles
a)	Great Falls	Peekskill	
b)	Albany	Peekskill	
c)	Albany	Princeton	
d)	Great Falls	Philadelphia	
e)	Peekskill	Baltimore	
f)	Great Falls	Baltimore	
g)	Princeton	Baltimore	

Solution. Here is the completed table:

	From	To	Miles
a)	Great Falls	Peekskill	153
b)	Albany	Peekskill	102
c)	Albany	Princeton	200
d)	Great Falls	Philadelphia	297
e)	Peekskill	Baltimore	239
f)	Great Falls	Baltimore	392
g)	Princeton	Baltimore	141

Notice that when the two cities are on the same side of New York, we subtract the smaller distance from the larger, and when they are on opposite sides of New York, we add them. If we plot the cities on a vertical line, as in the figure to the right, we see why this is the case.

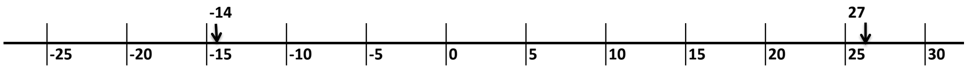


This is precisely what we get if we consider this as a vertical number line with New York as the origin, if we delete the "N" above 0, and replace the "S" in numbers below zero by the negative sign.

Example 11.

On a cold winter's day, the high temperature for the day was 27°F and the low was −14°F. What was the total spread of the temperature for the day?

Solution. For this problem we draw a temperature scale, with 0 in the middle and temperature increasing to the right:



Points to the left of 0 represent temperatures below zero, and that representation is expressed by the minus sign: $-14°$F means "fourteen degrees below zero Fahrenheit." The drop in temperature from the day's high is then the sum of the drop to zero from 27°F plus the drop from zero to 14°F below zero: thus the spread in temperature is $27 + 14 = 41°$F.

Example 12. Marigold started the week with \$143 in her account. During the week she made these bank transactions (payments by check and deposited income):

Monday: purchased groceries for \$39.

Tuesday: received \$10 from her Mom, and paid off a loan at \$90.

Wednesday: went to dinner and a movie with a friend. Cost: \$41.

Thursday: paid the paper delivery service \$27.

Friday: received a pay envelope containing \$232.

What is the balance in her account at the end of the work week?

Solution. First, we represent income by positive numbers, and for expenses, we put a minus sign in front of the number. Then we try to tabulate the data day-by-day so as to determine her ending balance. One way is illustrated in Figure 11. Each column is a day of the week, in sequence. In the first row, we insert Marigold's starting balance, in the second row, that day's change, and in the third row the end-of-day balance. We do this by first filling in the given data for Monday, and then for the last calculate the result of that change. That number is Tuesday's starting balance, and now we proceed in the same way. So, Marigold starts out with \$143, expends \$39, so ends the day with $143 - 39 = 104$ dollars. That is Tuesday's starting balance.

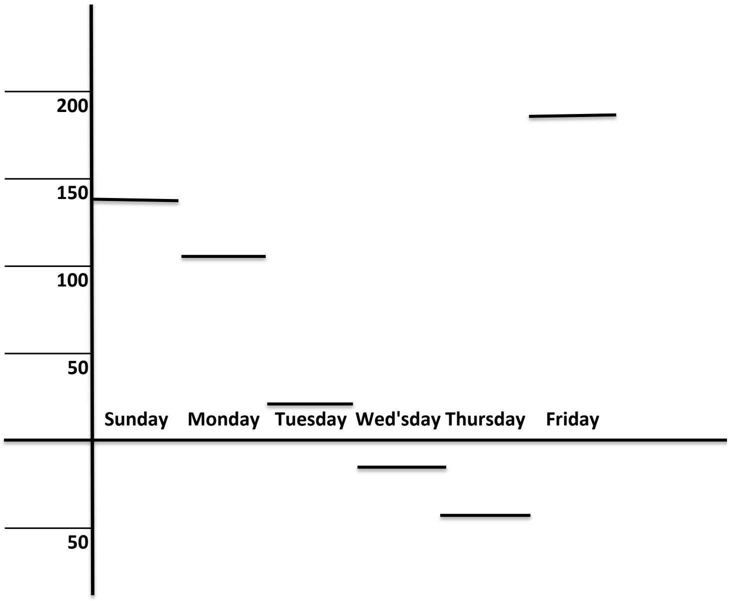
Day	Mon	Tue	Wed	Thu	Fri
Balance	143	104	24	−17	−44
Change	−39	10, −90	−41	−27	+232
New Balance	104	24	−17	−44	188

Figure 11

On Tuesday Marigold had an income of \$10 and an expense of \$90. So her end-of-day balance is $104+10-90 = 24$ dollars. Continuing in this way through Friday (payday) we end up with a final balance of \$188.

Example 13. Is there a graphic for Example 12 that better describes Marigold's financial activity for the week?

Solution. We have, for each day, three numbers: beginning balance, change and new balance. Since the "change" number is the difference between the first two, we can put that aside, and graph the points whose coordinates are (Balance, New Balance). That graph doesn't show much, so now we try anotherrepresentation. with the horizontal axis representing the day of the week, and the vertical axis representing the balance (beginning or ending). In Figure 12, we have chosen "end balance" for the y-axis, including the starting balance of \$143 as Sunday's ending balance.



The graph shows us clearly that Marigold's balance goes down, more or less steadily, until she receives her pay on Friday. It also shows each day's change as the difference of the position of two success horizontal lines.

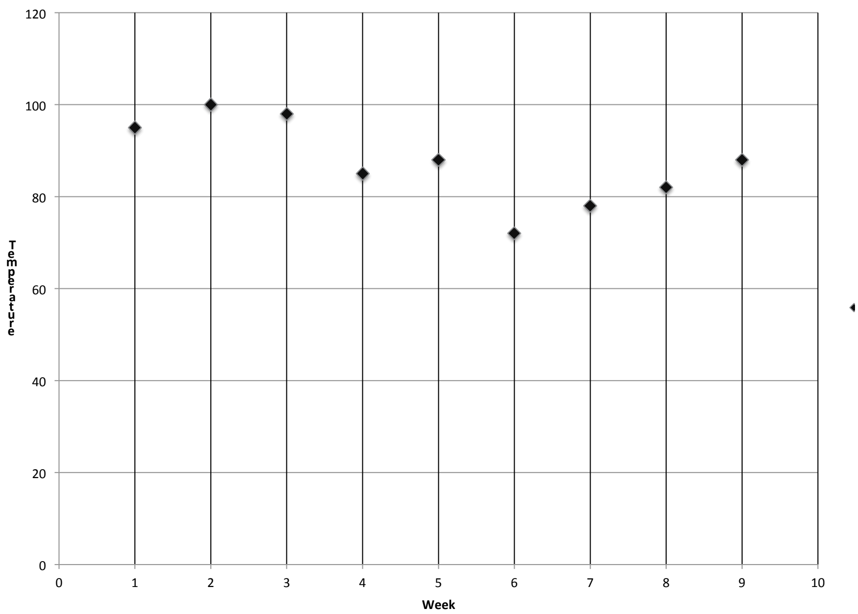
Example 14. Let us take another look at the data sets of Example 1.

Week	1	2	3	4	5	6	7	8	9
Temp	95	100	98	85	88	72	78	82	88

Time (seconds)	10	20	30	40	50	60	70	80	90
Altitude	0	30	100	200	800	1500	2300	3500	6300

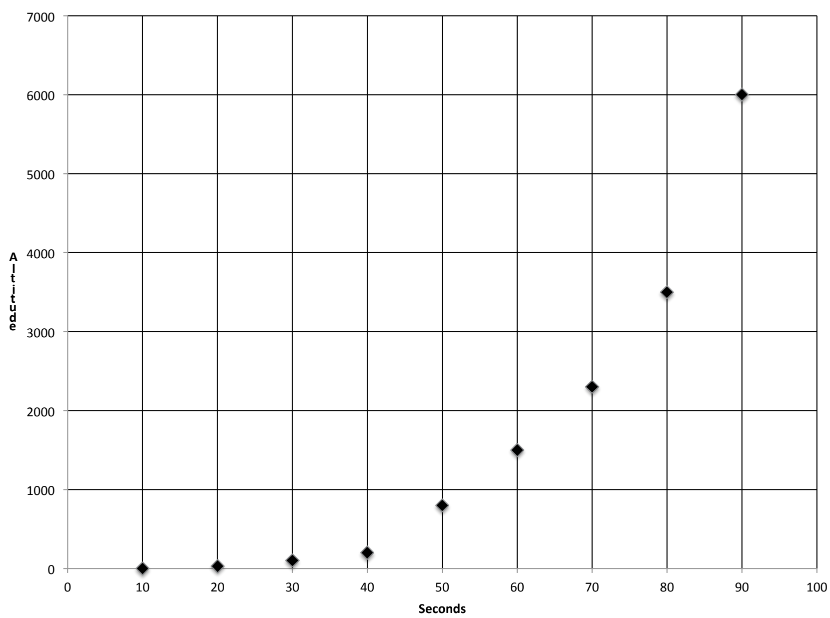
Draw graphs of those data; that is, choose the variables along each axis and plot the relevant points. Do the graphs give us more information than the table, or, at least, is there information that is more clearly exhibited?

Solution. a) We have a choice as to which variable is the horizontal (x) variable, and which is vertical (y). We suggest choosing "Week" for x and "Temp" for y, since our interest is in the variation of the temperature as time goes by. This is the graph:



What the graph shows us is that there is not much variation in temperature; the summer started out hot, in the middle a cooling spell and after that a warming trend.

b)



We have chosen to put “Time” on the horizontal axis, and “Altitude” on the vertical. The image for Example 14b shows us a clustering of low altitudes and a spreading of high altitudes. But now we see how altitude behaves according to time: the rocket’s altitude increases and as time goes by it increases more and more rapidly.

Example 15. Let us take another look at the data in Example 2.

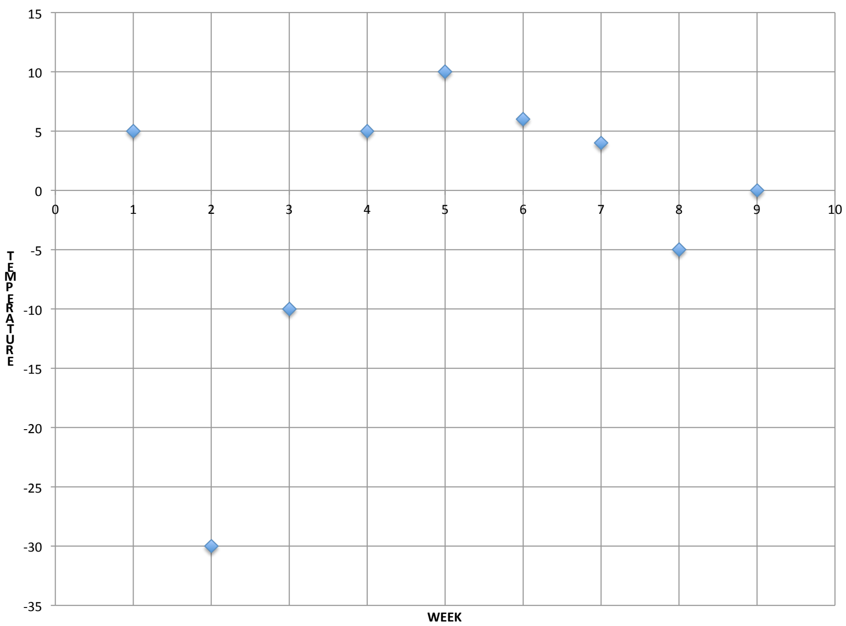
Week	1	2	3	4	5	6	7	8	9
Temp	5	−30	−10	5	10	6	4	−5	0

Miles	0	5	10	15	20	25	30
Altitude	4000	2000	1000	200	0	−100	−200

Draw a graph of those data (choose the variables along each axis and plot the relevant points). Does the graph give us more information than the table, or, at least, is there information that is more clearly exhibited?

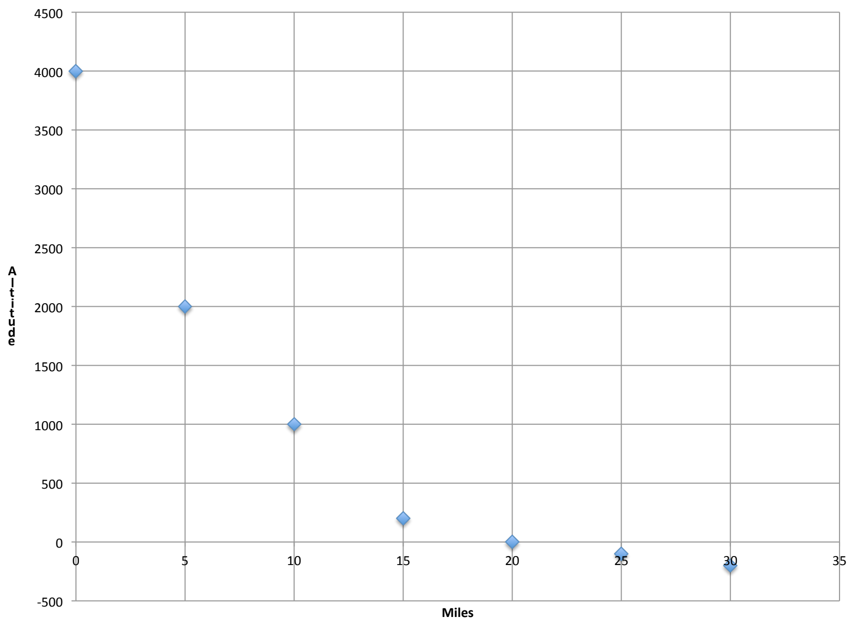
Solution. This problem is the same as the preceding one, but now we have both positive and negative numbers. We will select "week" for the horizontal axis in the first graph, and "miles" in the second. Here are the graphs.

a)



Here we can deduce that the temperature hovers around 0°C, except for a bitter cold spell in the second week.

b)



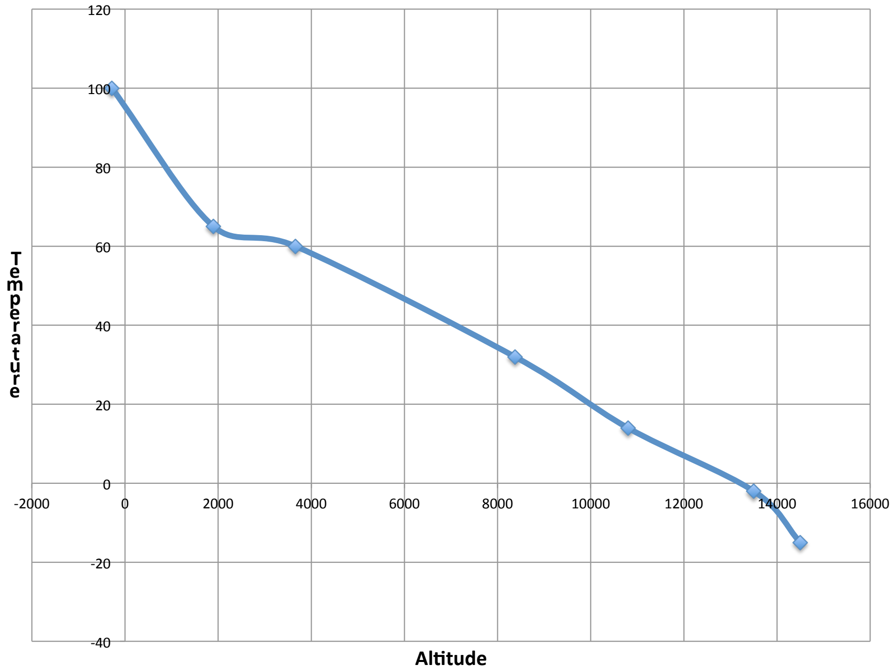
We now see that, as we move out to see from the mountain, the altitude drops steeply over the first 15 or so miles, and when we hit sea level, the water remains relatively shallow until we are about another 20 miles out to sea.

Example 16. An ambitious group of hikers trekked from the lowest point in the continental United States to the highest point. That would be from Badwater Basin in Death Valley to the summit of Mt. Whitney. This is a trip of 135 miles by foot, from an altitude of 280 feet below sea level to 14,500 feet above sea level, that takes almost a week. This table lists all the points at which the team spent the night, the altitude of that location, and the temperature at dawn when they left that location. The last entry gives the conditions at the summit, which was reached at noon on Saturday.

Day	Mon	Tue	Wed	Thu	Fri	Sat	Noon
Location	Badwater	Para Spgs	Swansea	Whitney Portal	Midway	Summit Camp	Summit
Altitude	−280	1900	3660	8375	10800	13500	14500
Temp	100	65	60	32	14	−2	−15

Note: This hike was actually taken in October, 2014; see
http://www.thehikinglife.com/2014/10/lowest-to-highest-death-valley-to-mt-whitney-ca-usa-2014/
However, our data are fictitious.

SOLUTION. Plot these data in a way that exhibits the relation between altitude and temperature. Pick the size of the squares in your grid appropriately. What conclusions are put in evidence in the plot?



We see that the temperature decreases as the altitude increases. Of course the table shows this as clearly, but what the graph shows us is that, for most of the climb, the temperature decreases by about 14 degrees for every 2000 feet climbed.

In this section we have indicated some of the purposes of graphing data; most importantly of discerning trends in the change of the data. That is something that is often hard to recognize in the table, but easy to see in the image. In the next chapter, we will illustrate the use of graphs in the statistical (rather than behavioral) study of data - and these themes will continue to play a significant role throughout secondary education.