Chapter 5
Geometry

Geometric and spatial thinking connect mathematics with the physical world and play an important role in modeling phenomena whose origins are not necessarily 'geometrical.' Geometric thinking is also important because it supports the development of number and arithmetic concepts and skills, by providing students with a context for intuitive understanding.

Geometry is the only one of the mathematical strands of the Common Core that exists in every grade. The approach to geometry described above is developed throughout the elementary grades. In the primary grades, students classify shapes, order them (according to size, dimension, complexity, etc.) and get an intuitive introduction to the basic vocabulary of geometry. In third grade, students are introduced to the concept of area (of rectangles) as a model for the operation of multiplication, and through partition of rectangles as a way of understanding fractional expressions. They understand the area of a rectangle as the product of the measures of base and height, and since rotation of the rectangle through 90° doesn't change the area, they visually see the principle of the commutativity of multiplication. Noting that a diagonal of a rectangle partitions the rectangle into two *equal* right triangles, and that any right triangle is exactly half of some rectangle, students see that the area of a right triangle is one half the product of the lengths of the (non-hypotenuse) legs. Finally, in third grade, students worked with slightly more complex polygonal figures to find the area by partition. The first section of this chapter begins with a review of this material, extending it to more complex figures, and finally fully developing the procedure to compute area of general polygonal figures by partition.

Fourth grade is devoted to angles, angle measure, properties of lines and planar figures described by angles, and an introduction of the concepts of *parallel* and *perpendicular*.

An important part of the fifth grade curriculum is the introduction of the coordinate system. In sixth grade, measure is introduced in this context. In the second section of this chapter, we find the perimeter and area of polygons whose sides are horizontal and vertical, and the area of general polygonal figures by partitioning them into such rectangles and triangles whose legs are horizontal and vertical.

Fifth grade takes geometry to three dimensions, introducing students to the concept of volume as multiplicative (the volume of a right rectangular prism is the product of the length, width and height) and as additive (the volume of a solid is the sum of the volumes of its component parts, *no matter how partitioned*).

In Grade 6 we begin the process of turning this intuitive understanding of geometry to a knowledge-based understanding; developing the structure of geometry as a mathematical discipline. This is not to abandon the practice of exploration; until the advent of electronic computation geometry was the main tool of mathematical exploration. In this chapter we develop the arithmetic properties of the concepts of perimeter and area in two dimensions, and volume and surface area in three.

The first section is devoted to the concept of area of polygonal objects; first for the basic configurations of triangles and quadrilaterals, and then for general polygons by partitioning into triangles and rectangles.

In the second section we move to the same calculations, but where the polygons are not described by the lengths of their sides, but by the position of their vertices in a coordinate plane.

The third and fourth sections take us to three dimensions. In the third we start with the concept of volume developed in Grade 5 and put focus on the process of calculating volumes. The final section is about calculating the surface area of polyhedral figures in space, by *appropriately* cutting the polyhedron along its edges so that we can fold it out into a planar polygon whose area we already know how to calculate. This is a fun thing to do, and should be so presented.

The main purpose of the Grade 6 chapter on geometry is *analysis of geometric objects*. Students learn how to calculate perimeter, area and volume, for simple (or *standard*) figures. But the important skill to develop is that of analysis of complex geometric objects, so as to be able to decompose them (appropriate to the context) into a collection of non-overlapping standard figures.

Section 1. Area of Polygons

Find the area of right triangles, other triangles, special quadrilaterals, and polygons by composing into rectangles or decomposing into triangles and other shapes; apply these techniques in the context of solving real world and mathematical problems. 6.G.1.

Throughout this chapter students and teachers use geometric terms and definitions with which they have become familiar: polygons, perimeter and area of two-dimensional objects, and volume and surface area of three-dimensional objects, etc.. Though these terms are not rigorously defined, it is important that they are used correctly and misconceptions are not allowed to develop. We start by reviewing the intuitive understanding of geometric terms.

A *polygon* is a region in the plane bounded by line segments (*sides*). The points where the sides meet are the *vertices* of the polygon. Let's observe:

Example 1. The number of sides of a polygon and the number of vertices of a polygon are equal.

SOLUTION. Note that the endpoints of any side are vertices. Thus every side accounts for two vertices. But since each vertex is an endpoint of two sides, in this count, every vertex is counted twice. So the number of vertices is half the number of side endpoints, and the number of side endpoints is twice the number of sides. That is: there are as many vertices as sides. Alternatively, we can start at a vertex and count sides and vertices as we go around the polygon. Each vertex precedes precisely one side, and each side leads to precisely one vertex. Thus, when we return to the starting vertex, the counts of sides and vertices are the same.

A *triangle* is a polygon with three sides. These are grouped in three categories:

A *right* triangle is a triangle, one of whose angles is a *right* angle.

An *acute* triangle is a triangle, all of whose angles are less than a right angle.

An *obtuse* triangle is a triangle, one of whose angles is greater than a right angle.

In Figure 1 we have exhibited each type. Students will intuitively see that it is impossible for a triangle to have two angles that are right angles (or greater), suggesting that the sum of the angles of a triangle is not greater than a straight angle. If so, it might be useful to discuss this at this time, even though it is a Grade 7 topic.

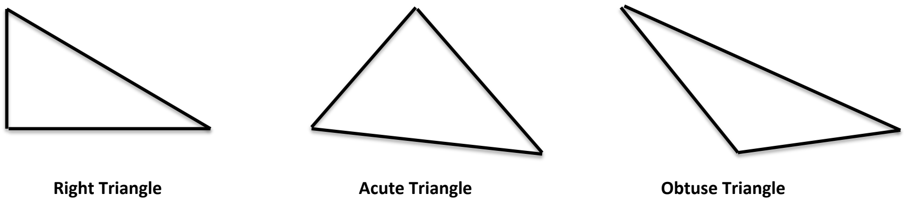


Figure 1

For the purpose of this designation, we note the understanding of *right angle* from fourth grade: If two lines intersect at a point *P*, and the measures of all four angles formed at this intersection are the same, then all four angles are *right* angles.

In addition to classifying triangles by their angles, we can also group them according to the lengths of their sides. A triangle where all three of the sides are of different lengths is called *scalene*. If two (or more) sides are the same length, the triangle is *isosceles*. The final option, all three sides of the same length, is called *equilateral* (*equi-* meaning "equal" and *lateral* meaning "side"). While an equilateral triangle will always be acute, scalene and isosceles triangles can be acute, right, or obtuse. In Figure 2 we show how these classifications relate.

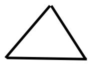
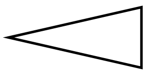
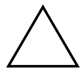
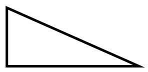
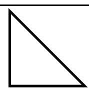
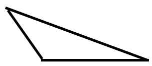
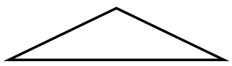
	Scalene	Isosceles	Equilateral
Acute			
Right			----------
Obtuse			----------

Figure 2

A *quadrilateral* is a polygon with four sides. Although we have distinguished triangles by adjectives, we distinguish quadrilaterals by nouns. Here is a list of quadrilaterals with which students are already familiar, starting from the most inclusive:

A *quadrilateral* is a polygon with four sides;

A *trapezoid* is a quadrilateral with at least one one pair of parallel sides;

A *parallelogram* is a quadrilateral with both pairs of opposing sides parallel;

A *rhombus* is a parallelogram with all sides of the same length;

A *kite* is a quadrilateral with two pairs of adjacent sides of the same lengths;

A *rectangle* is a parallelogram with at least one right angle;

A *square* is a rectangle with all sides of the same length.

Note: Unfortunately, there is not universal acceptance of the inclusivity of these definitions (meaning that in the above list, each figure defined is also a figure of the preceding definition). Some define a trapezoid as a quadrilateral that has exactly one pair of parallel sides, meaning that the other pair of sides cannot be parallel (otherwise we'd call this a parallelogram). This is called the *exclusive definition* of the trapezoid as it disallows parallelograms, rectangles, and squares to be special kinds of trapezoids. The definition given above is called the *inclusive* definition and it does allow for parallelograms, rectangles, and squares to be special kinds of trapezoids. We want to make the reader aware of the conflict in an effort to help avoid confusion caused by these conflicting definitions. Students may argue "but that is a rectangle, not a square" or "that is not an isosceles triangle, because it is equilateral". This is legitimate and has to be discussed, with the end result being that it is necessary that there is agreement on definitions, and here the choice is on inclusivity. In our catalogue, squares are particular kinds of rectangles, and rectangles are particular parallelograms and so forth. The important thing is this: when a statement is made, such as: "a parallelogram is a four-sided figure for which opposite sides are parallel," we *include* rectangles and squares.

It is important to also note that there are properties of the various quadrilaterals defined above that are consequences of the above definitions, and sometimes appear as part of the definition. For example,

> a quadrilateral is a polygon with four vertices;
>
> a parallelogram is a quadrilateral for which opposing sides have the same length;
>
> a rectangle is a parallelogram for which all angles are right angles;
>
> an equilateral triangle is a triangle, all of whose angles have the same measure (namely 60°.

There are also some things that are **not** true; for example a square is a rectangle with all sides of the same measure, but a parallelogram with all sides of the same length is not necessarily a square. This later figure is called a rhombus, and a square is a special kinds of rhombus.

In high school mathematics, all of this information will be developed in a logically structured way. In middle school, these, and many other geometric facts may be introduced in an informal way, or may just become part of the students' intuitive understanding. This is in fact how geometry is designed for middle school: to discover geometric facts through exploration, with the emphasis on *exploration*. This activity should be enthusiastically encouraged, with little concern about having "covered" everything. This is true throughout education, but particularly in middle school geometry.

In Figure 3 we illustrate these different categories of quadrilaterals.

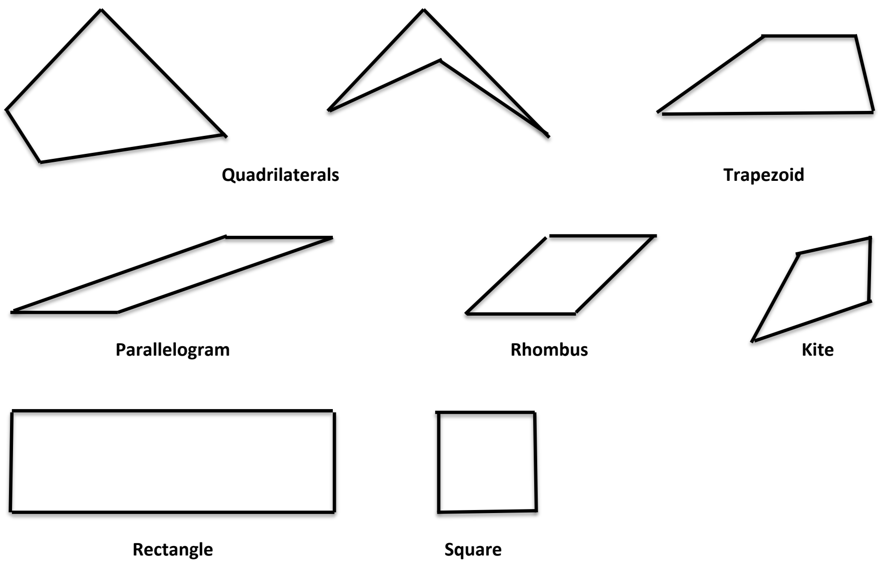
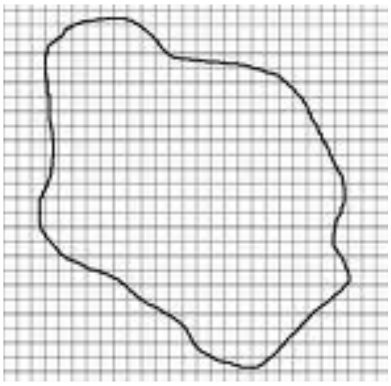


Figure 3

The second image in the first line has one angle that is greater than a straight angle. This figure is, for the most part, ignored in the middle grades - but it may arise in discussions. It is good to keep in mind that most general facts about quadrilaterals apply to this shape as well.

Area

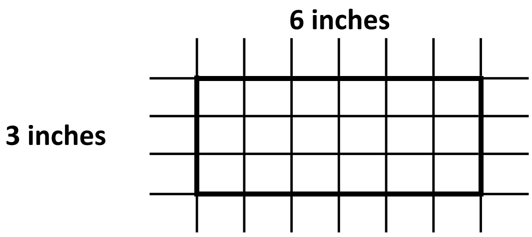


In two-dimensional space, area measures the space enclosed by a figure. The calculation of area of land is one of the oldest problems in mathematics, originating in part from problems of fair apportionment of land inherited by children from their parents. The land itself may include parts of rivers or edges of foothills or swamps, small bodies of water, or hilly regions. One method of finding the area of irregular pieces of land is not very far from the approach taken in calculus: a surveying team will make the relevant linear measurements of the boundary, and then make a scale drawing. Or, the land is photographed from above, thus making it flat. A grid of congruent squares with known dimensions is placed over the drawing or the photo.

The number of squares inside the boundary plus half the number crossed by the boundary provides an estimate for the area. If a more precise estimate is desired, then smaller squares are used. Later, in calculus, students will build upon techniques for finding area of a two-dimensional figure by approximating using smaller and smaller geometric shapes. As the approximating shapes get smaller in area, we have to add more of them to find the total area. This leads to the limit definition of the integral in calculus.

Our approach will involve developing and discussing the area of plane figures, which also applies to the calculation of the surface area of common three-dimensional figures, using nets later in the chapter. That is, we will trace everything ultimately back to the area of squares, because we measure area in square units. In real life problems, the unit square could be one inch, one centimeter, one yard, or even one mile on each side; it depends on what you are measuring.

Now, length is a measure of line segments, and so area is to be thought of as a measure of pieces of a surface, one that gives the planar or surface "content" of a figure.



How do we figure out the area of a polygonal shape? What does it mean to say that the area of a region is 18 square inches? It means that the shape can be covered, without gaps or overlaps, with a total of eighteen 1-inch-by-1-inch squares, allowing for squares to be cut apart and pieces to be moved if necessary. If the figure is a 3×6 rectangle, we can cover it with 1×1 squares, and count the squares: there are 18 of them. In fact, students will recall that this is the geometric intuition that led to the concept of multiplication: 3×6 square units *is* the area of a rectangle with side lengths of 3 units and 6 units.

How about a general polygonal figure? In general, the technique for calculating areas of general polygonal figures, or formulas for specific types of polygons, is based on these principles:

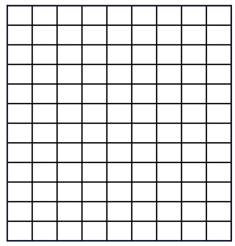
If you move a shape rigidly without stretching it, then its area does not change.

If you combine (a finite number of) shapes without overlapping them, then the area of the resulting shape is the sum of the areas of the individual shapes.

In general a polygonal figure can be partitioned into a combination of simpler shapes (actually, triangles and rectangles) for whose areas formulas have been developed already. Then the area of the original figure is the sum of the areas of the component figures. We will summarize this in the context of figures with which the students are already familiar.

A. Rectangle. If the lengths of the sides of a rectangle are a and b units, then its area is ab square units.

Example 2. In each figure below, each grid square has side length 1 ft. In square yards, what are the areas of the rectangles shown?



A

Solution. Each yard consists of three feet in length. Each square yard, then, consists of nine square feet in area. We have to convert feet to yards, or square feet to square yards. We illustrate both methods.

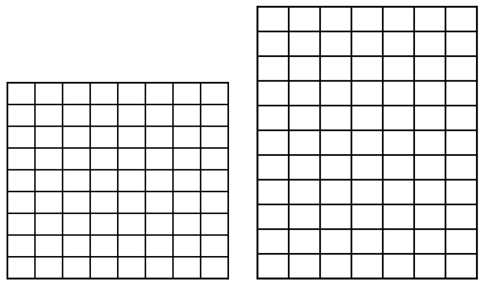
Solution 1. Rectangle A is 9 feet by 12 feet, so in area, consists of 108 square feet. Note that each square yard contains 9 square feet. So, in yards, the area of figure A is $\frac{108}{9} = 12$ square yards.

Rectangle B is 8 feet by 9 feet, so in area, consists of 72 square feet. As above, in yards, the area is $\frac{72}{9} = 8$ square yards.

Rectangle C is 7 feet by 10 feet, so in area, consists of 70 square feet. In yards, the area is $\frac{70}{9} = 7\frac{7}{9}$ square yards.

Solution 2. Here we convert into yards by dividing the foot measurements by 3.

Rectangle A is 3 yards by 4 yards, so has area $3 \cdot 4 = 12$ square yards.



B C

Figure 4

Rectangle B is is 8/3 yards by 3 yards, so has area $\frac{8}{3} \cdot 3 = 8$ square yards.

Rectangle C is 7/3 yards by 10/3 yards, so has area $\frac{7}{3} \cdot \frac{10}{3} = \frac{70}{9} = 7\frac{7}{9}$ square yards.

B. Parallelogram. We have defined a parallelogram as a quadrilateral with opposite sides parallel. It is also true that a parallelogram is characterized as a quadrilateral with opposite sides of equal length; and here we use that description.

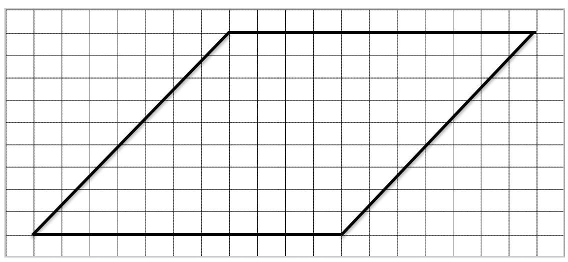


Figure 5

Example 3. In Figure 5, each small square has side length 1 foot. Find the area of the parallelogram in square feet.

SOLUTION. Note that each row consists of 10 complete squares of area 1 square foot, plus two partial squares. But the partial squares in each row can be put together to form a complete square. So, the area of each row is 11 square feet. There are 9 such rows, so the area of the parallelogram is 99 square feet.

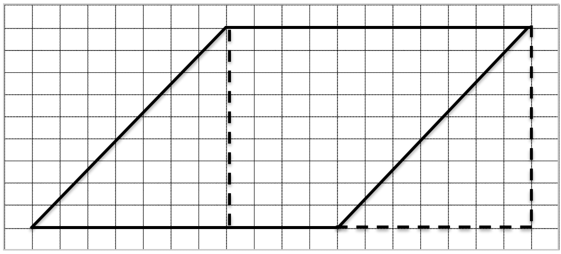


Figure 6

The important thing that we now notice is that what is true for each row is also true for the whole figure. Consider Figure 6, in which we have dropped a perpendicular (dashed line segment) from the upper side to the lower side. Cut the parallelogram along this line, and move the triangle to the left side of the parallelogram so as to form a rectangle (whose vertical sides are the dashed line sgements). Obviously, since we just moved a piece of the parallelogram from one side to the other, the original parallelogram has the same area as the rectangle. Both figures have the same height. They also have the same base length, since all we have done is move the base of the triangle from the left to the right. Thus, the formula: Area = base×height holds for parallelograms as well as rectangles.

Here the base is of measure 11 feet, and the height of measure 9 feet, so the area of the parallelogram is 11×9 square feet.

It is important to make the point that, in calculating areas of parallelograms, one has to make a choice of *base*: one of one of the pairs of parallel sides, Then, the *height* is the **perpendicular** distance between the base and its opposing side. This distinction persists for other figures, to be discussed below. First choose a base, and then the height is the relevant perpendicular distance from the base.

C. Triangle. The key here lies in showing that a triangle is half of a parallelogram. Figure 7 provides directions for doing that.

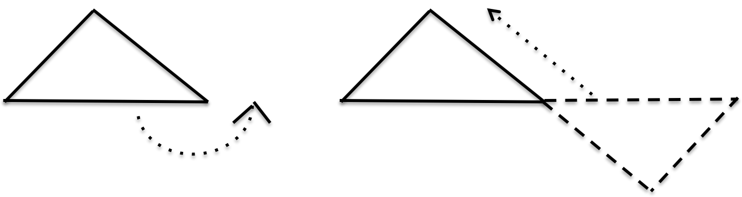


Figure 7

The given triangle is depicted on the left. Rotate a copy of it through a straight angle, getting the figure on the right. Then we slide the dashed triangle upward until we arrive at the parallelogram (the right side of Figure 8). Thus the area of a triangle is half that of a parallelogram of the same base and height (the left side of Figure 8).

If the triangle is a right triangle, the corresponding parallelogram is a rectangle. On the other hand, for an obtuse triangle, the argument is the same and the corresponding figure is still a parallelogram, as shown:

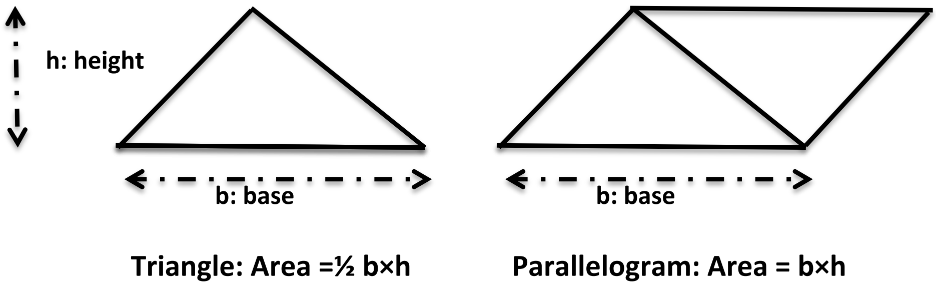
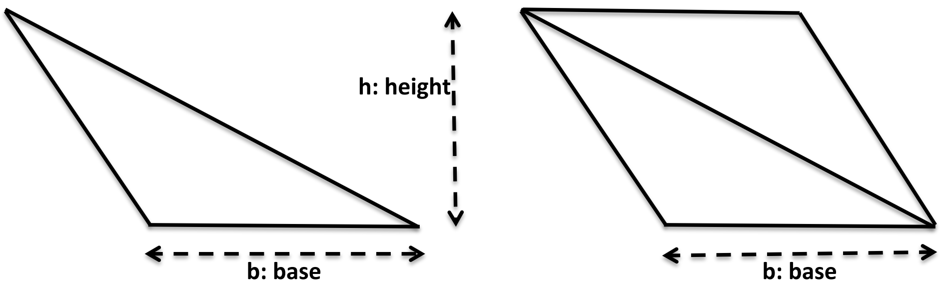


Figure 8



Note that the area of a triangle is defined by the length of one side (the *base*) and the perpendicular distance of that side to its opposing vertex. In particular, all the triangles in Figure 9 have the same area: $\frac{1}{2}bh$.

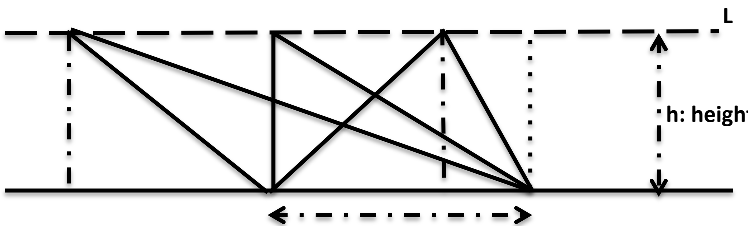


Figure 9: Area = $\frac{1}{2}$(base)×(height)

D. Trapezoid. Let the lengths of the parallel sides be a and b, and the perpendicular distance between them h. Then the area of the trapezoid is $\frac{1}{2}(a + b)h$.

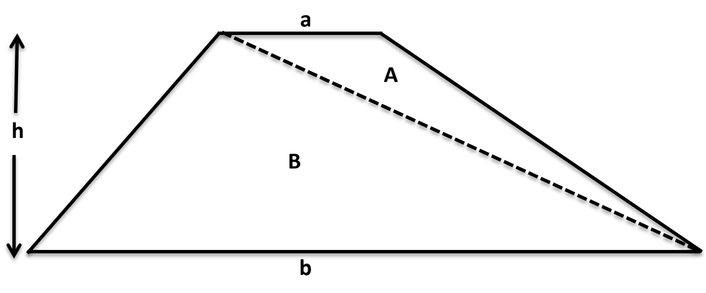


Figure 10

Figure 10 above is an image of the trapezoid, with height h, upper base length a and lower base length b. We have included a dashed line that splits the trapezoid into two triangles. The area of the trapezoid then is the sum of the areas of the triangles. Now, both triangles have the same height, h, but triangle A has base length a and triangle B has base length b. Thus

$$\text{Area of Trapezoid} = \text{Area of } A + \text{Area of } B = \frac{1}{2}ah + \frac{1}{2}bh = \frac{1}{2}(a + b)h.$$

Example 4. The Suderman family bought a triangular piece of property, with 170 feet of roadside frontage, as in Figure11. The distance between grid lines is 10 feet, and thus each grid square has area 100 square feet. The lengths of the other two sides are (roughly) 190 feet (the bottom side of the figure) and 150 feet (the side to the left). Mr. Suderman decides to divide the property into two pieces with the dashed line as shown; on the top piece he will put his house and on the bottom piece he will create a garden. a) What is the area of the house lot? b) What is the area of the garden? (Approximate answers will do).

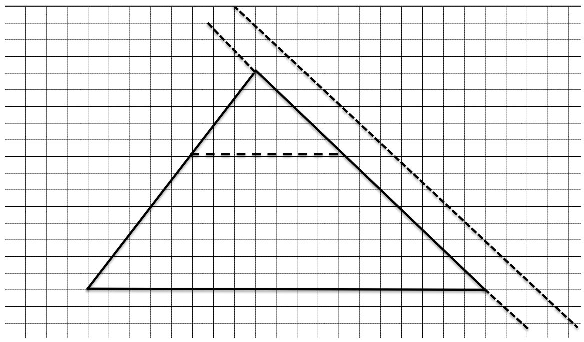


Figure 11

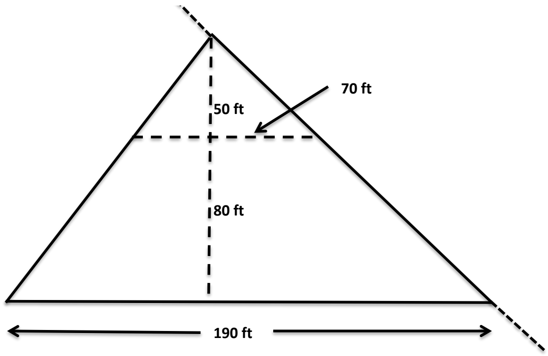


Figure 12

Solution. First of all, the statement of the problem gives us more information than what we need and not enough of what we need. The property has been divided into a triangle and a trapezoid. We have been told the outside dimensions of the whole property, but not the heights and lengths of the bases of the two figures. Fortunately, Figure 11 is on a grid where the distance between two grid lines is 10 feet. Counting squares, we find the height of the triangular house lot is 50 feet, and the base is of length 70 feet. As for the trapezoidal garden, the bases are of 70 feet and 190 feet respectively, and the height is 80 feet. Figure 12 shows the lengths of the relevant segments.

$$\text{Area of house lot} = \frac{1}{2}(70)(50) = 1750 \text{ sq. ft.,} \quad \text{Area of garden} = \frac{1}{2}(70 + 190)(80) = 10400 \text{ sq. ft.}$$

Since the problem asked for an approximate answer, we have estimated lengths to the nearest 10 ft. length. We could have made a rougher estimate, by counting the 100 sq. ft. squares in each lot, using Figure 11. Doing that, we get the estimates: 1900 sq. ft. for the house lot, and 11600 sq. ft. for the garden. (Note: answers vary depending upon how partial squares are evaluated.)

Using Formulas to Calculate Perimeters and Areas

Square. Let s be the length of any of its sides. Then,

$$\textbf{Perimeter} = 4s, \quad \textbf{Area} = s^2.$$

Rectangle. Given a rectangle whose sides are of length w and h units. Then

$$\textbf{Perimeter} = 2w + 2h, \quad \textbf{Area} = wh.$$

Parallelogram. Choose any side of the parallelogram as the *base* and its length b, and let h be the perpendicular distance between the base and the opposite side. Then,

$$\textbf{Area} = bh.$$

Triangle. Choose any side of the triangle as its *base* (its length is b), and let h be the perpendicular distance between the base and its opposing vertex.Then,

$$\textbf{Area} = \frac{1}{2}bh.$$

Trapezoid. Designate the lengths of the parallel sides as a and b, and the perpendicular distance between the parallel sides as h. Then,

$$\textbf{Area} = \frac{1}{2}(a+b)h.$$

Note that we have not included the formula for the perimeter of any figure with a slant side. This is because the length of those slant sides are determined a) by the Pythagorean theorem, or b) the measure of the angle the slant side makes with the horizontal and the height, or c) (approximately) by direct measurement, if possible. Although these concepts will be explored in later grades. it could be useful to have a discussion in Grade 6.

Example 5.

a) What are the perimeter and area of a square of side length 3 inches?

b) Suppose the side length of the square in a) is doubled. By what factors are the perimeter and area enlarged?

c) What are the perimeter and area of a rectangle of base width 4 inches, and height 7 inches?

d) Suppose the width and height of the rectangle in c) is doubled. By what factors are the perimeter and area enlarged?

e) Suppose the width of the rectangle in c) is tripled, but the height remains the same. By what percentage did the perimeter and area increase?

SOLUTION. Solution:

a). $P = 4 \cdot 3 = 12$ in. $A = 3 \cdot 3 = 9$ sq. in.

b). $P = 4 \cdot 6 = 24$ in. The perimeter doubled. $A = 6 \cdot 6 = 36$ sq. in. The area quadrupled.

c). $P = 2 \cdot (4 + 7) = 22$ in. $A = 4 \cdot 7 = 28$ sq. in.

d). $P = 2 \cdot (2 \cdot 4 + 2 \cdot 7) = 44$ in. The perimeter doubled. $A = (2 \cdot 4) \cdot (2 \cdot 7) = 112$ sq. in. The area quadrupled.

e). $P = 2 \cdot (3 \cdot 4 + 7) = 38$ in. $A = (3 \cdot 4) \cdot 7 = 84$ sq. in. The perimeter increased by (approximately) 73% and the area increased by 200%.

As for part e), one must be careful to note that the questions were about the percentage increase, The increase in perimeter is 16 in., and the increase in area is 84 sq. in. If the question were: By what factors were the perimeter and area enlarge, the answer would be, for perimeter: 38/22,or 173%, and for area: 84/28, or 300%.

Example 6. Gabrielle made a rectangular fence on the north side of her property that of height 8 feet and length 72 feet. One winter, the snow was so heavy that the weight of the snow buckled the fence and transformed it into a figure that roughly looked like a parallelogram that was just 7 feet high (see Figure 13).

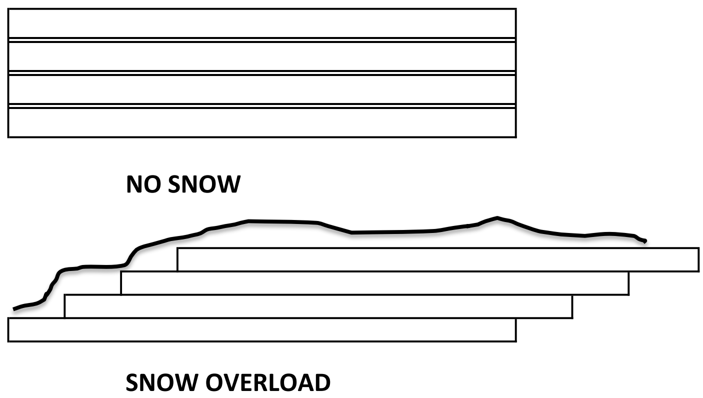


Figure 13

a) What was the area of the original fence?

b) What is now the area of the fence (after the snowfall)?

c) Explain.

SOLUTION. a) Before the snowfall, the fence was a rectangle of side lengths $L = 72$ feet, and height $H = 8$ feet. So the area was 576 square feet.

b) After the snowfall, the fence is a parallelogram of base 72 feet, and height is 7 feet. So the area is now 504 square feet.

c) Originally the fence was a 72×8 foot rectangle, but the snow collapsed it to a 72×7 foot rectangle. But the boards did not shrink in size, so the area should be the same.

The explanation is suggested by Figure 13, which shows spaces between the boards before the snow but no spaces between boards after the snow. Thus, what has happened is that the spaces have disappeared. According to our diagram, there are three spaces and the fence collapsed by a foot, so they must have been 4 inches in each gap between boards.

Now, if fences were made so that there were no such gaps, then there would have been no "give" to accommodate the weight of the snow, and the fence might have just fallen forward, especially if there were a back wind, as is usually the case.

Example 7. Find the area of the trapezoid of Figure 14.

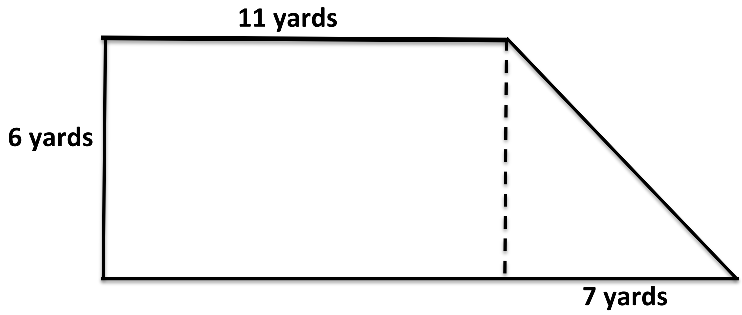


Figure 14

Solution. These are the dimensions our surveyor gave us. The trapezoid consists of a rectangle that is 11 by 6 yards, and a right triangle of height 6 yards and length 7 yards. So, the area is

$$\text{Area } = 6 \times 11 + \frac{1}{2}(6 \times 7) = 66 + 21 = 87 \text{ square yards.}$$

Since the topic here is of using formulas, we should calculate that the length of the bottom base is 18 yards, and use the formula

$$\text{Area } = \frac{1}{2}(11 + 18) \times 6 = \frac{1}{2}(174) = 87 \text{ square yards.}$$

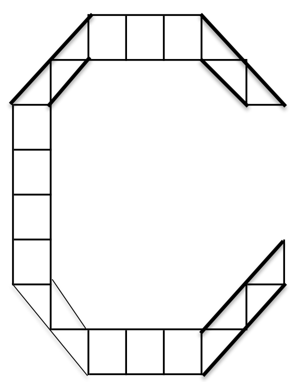
The point of this example is to demonstrate that formulas work to give the correct answer, but applying the formula without understanding is not necessarily the way to proceed. In this case, the trapezoid is clearly the sum of a rectangle and a triangle, and the use of that fact gives a valid way to calculate areas. In the next section we consider figures where there is no single formula to use, and for which it is necessary to partition the figure into pieces whose areas are computable.

Area of Irregular Figures

The way we calculated the area of the trapezoid (partitioning it into figure whose area we already know how to compute) generalizes to general polygons, as is illustrated in the following examples.

Example 8. In the diagram below right, we have the letter "C" ready to filled in with black ink and printed in 1250 copies.

The side of each square measures 1 cm, and the cost for these copies is to be \$0.005 per square cm. of ink. How many square centimeters on each page are to be inked? What will be the cost per page? What will be the total cost?



Solution. Let us take a moment to analyze the figure. It is composed of squares and triangles. Looking more closely we see that (fortunately) each triangle is exactly half a square. Therefore, we can conclude that if S is the number of squares and T is the number of triangles, the area is

$$\text{Area } = S + \frac{1}{2}T \text{ sq. cm.}$$

So, let's count: there are 3 squares at the top, 4 on the left side and 3 at the bottom. There are 3 triangles top left, top right, bottom left and 4 triangles bottom right. Therefore

$$\text{Area } = (3 + 4 + 3) + \frac{1}{2}(3 \times 3 + 4) = 10 + 6.5 = 16.5 \text{ sq. cm.}$$

This is how much area has to be inked; the cost of which is \$0.005 per square cm. So, for one page the cost is $16.5 \cdot 0.005 = 0.0825$ dollars (or 8.25 cents). The total cost for 1250 copies is \$103.125., which is \$103.13 rounded to nearest cent.

Example 9. Find the areas of the polygons in Figure 15, given that the length of the side of the square in the grid is 1 cm.

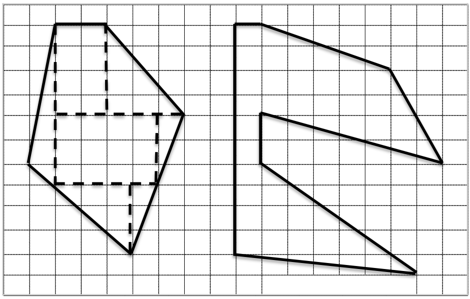


Figure 15

Solution. Left polygon: we have partitioned the figure on the left into rectangles and triangles so that we can use the formulas we know (see Figure 15a). Note that there is more than one way to partition the figure so that we can calculate; for example, we needn't have drawn a line between triangles 4 and 5.

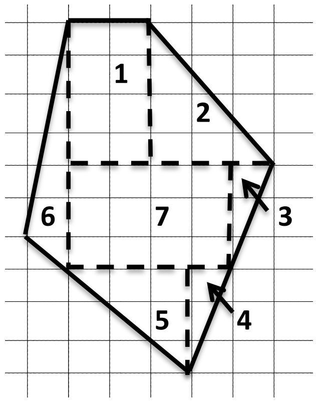


Figure 15a

Now let us use this partition to calculate the areas. Region 1 is a rectangle of side lengths 2 and 4, so has area 8 cm^2. Region 2 is a triangle with leg lengths 4 and 3, so its area is $\frac{1}{2}(4 \cdot 3) = 6$ cm^2. Here is a table of all the calculations: The area of the whole region is the sum:

Region	1	2	3	4	5	6	7
Area (cm^2)	8	6	1.5	1.5	4.5	3.5	12

$$\text{Area} = 8 + 6 + 1.5 + 1.5 + 4.5 + 3.5 + 12 = 37 \text{ cm}^2.$$

Right polygon: We employ the same procedure, noting that there are very many ways to partition the figure into a collection of rectangles and triangles. Figure 15b shows one way. Let's calculate the component areas. Region 1 is a triangle of base length 5 and height 2, so has area 5 cm^2. Region 2 is a rectangle of the same dimensions, so has area 10 cm^2. Here is a table of all the calculations:

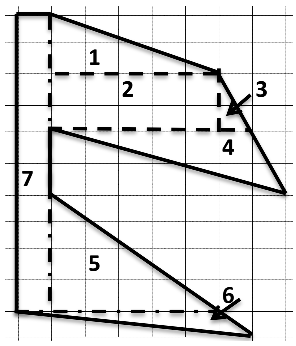


Figure 15b

Region	1	2	3	4	5	6	7
Area (cm^2)	5	10	1	6	10	3	10

The area of the whole region is the sum of these areas:

$$\text{Area} = 5 + 10 + 1 + 6 + 10 + 3 + 10 = 45 \text{ cm}^2.$$

Section 2. Polygons in the Coordinate Plane

Draw polygons in the coordinate plane given coordinates for the vertices; use coordinates to find the length of a side joining points with the same first coordinate or the same second coordinate. Apply these techniques in the context of solving real world and mathematical problems. 6.G.3.

The Coordinate Plane

In Example 9 of the previous section, we calculated the area of two polygons drawn on a rectangular grid. Often one needs to calculate the area of a polygonal figure on a coordinate plane with vertices at the coordinate points. We begin this discussion of figures in the coordinate plane where we left off in Chapter 3 at the end of section 1. First, we need to be sure that students know how to create a polygonal figure, given the coordinates of the vertices, and then we go into the calculation of area in terms of the coordinates of the vertices.

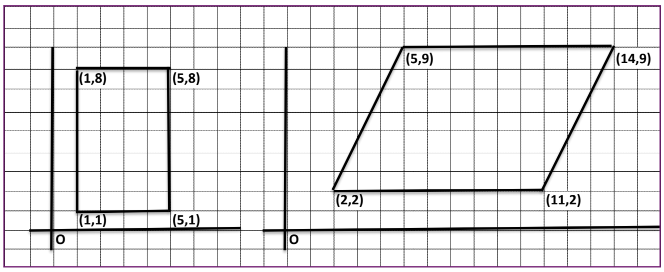
Example 10. In the tables below, we have given the coordinates of the vertices of four polygons in the plane. Draw the polygons. Describe them.

a)	x	1	5	5	1		*b)*	x	5	14	11	2
	y	8	8	1	1			y	9	9	2	2

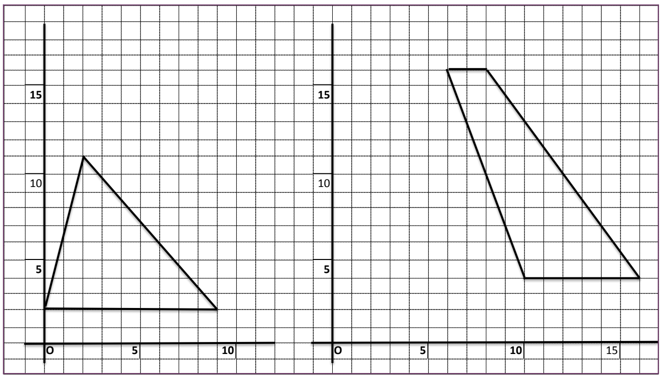
c)	x	2	9	0		*d)*	x	7	9	16	11
	y	11	2	2			y	16	16	4	4

Solution.

a) b)



a) rectangle b) parallelogram



c) triangle d) trapezoid

When a polygonal figure is drawn in a coordinate plane with sides that are either horizontal or vertical, we can use the coordinates to measure the sides of the figure, and then determine its area. If the distance between consecutive grid lines is 1 unit (that is, represents 1 cm or 1 in or 1 mi), then the length of a horizontal or vertical line segment is the number of grid lines traversed. So, for example, if two points are on the same horizontal grid line, their y-coordinates are the same, and the number of squares between two points is the *difference* in the x-coordinates, taking the larger one first. To illustrate, the distance between (2, 5) and (9, 5) is $9 - 2 = 7$ units. Similarly, for two points on the same vertical grid line, their x-coordinates are the same, and the number of squares between two points is the *difference* in the y-coordinates, taking the larger one first. So, the distance between (2, 5) and (2, 1) is $5 - 1 = 4$ units.

Example 11. Find the area of polygons a) and b) in Example 10.

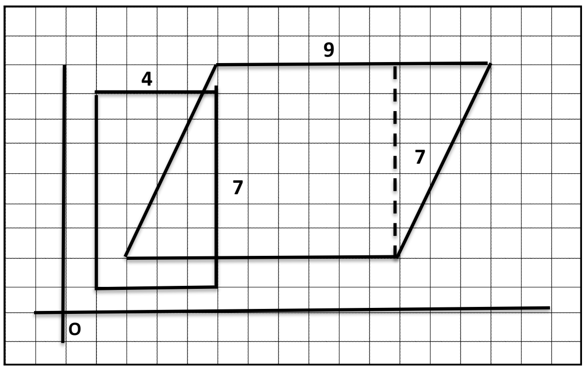


Figure 16

Solution. We have used the coordinates to find the lengths of the bases and the heights of the figures, as shown in the image above.

The area of the rectangle is the product of the base and the (perpendicular) height. The base of the rectangle is the line segment joining (1,1) and (5,1), so its length is $5 - 1 = 4$ units. The height is the length of either vertical leg, so is $8 - 1 = 7$ units. Thus the area is $4 \cdot 7 = 28$ square units.

The area of the parallelogram is the product of the base and its height. The base is the line segment joining (2, 2) and (11, 2), so its length is $11 - 2 = 9$ units. The height is the distance between the two horizontal sides. One such side is on the line $y = 2$ and the other on the line $y = 9$, so the height is $9 - 2 = 7$ and the area is $9 \cdot 7 = 63$ square units.

Example 12. Find the area of polygons c) and d) in Example 10.

SOLUTION. In this image it is easier to count boxes to determine lengths of relevant horizontal and vertical lines. The result is the shown in the image to the right.

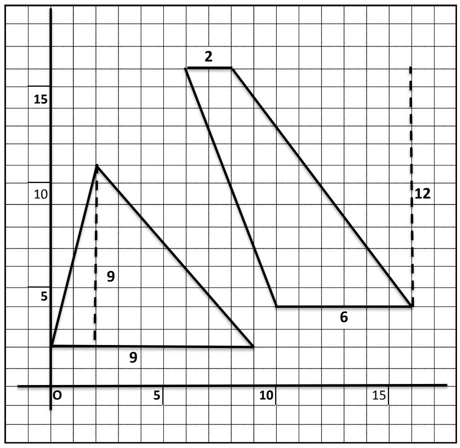


Figure 17

The area of the triangle is one half the product of the measure of its base and its height. The base measures 9 boxes across, and the height (shown by the dashed line measures 9 boxes high. Thus the area of the triangle is $\frac{1}{2}(9 \cdot 9) = 40.5$ square units.

As for the trapezoid, our count produces lengths of the upper base of 2 units, and of the lower base, 6 units. The height (shown by the dashed line) is 12 units. The area then is

$$\frac{1}{2}(2 + 6) \cdot 12 = 48 \text{ square units.}$$

More General Area Considerations

Recall that in fourth grade, students found the areas of rectangles with whole number side lengths, and in fifth grades students worked with rectangles of fractional length. The relevant standard is:

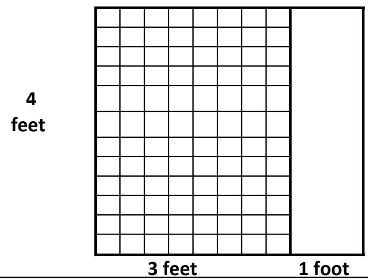
Find the area of a rectangle with fractional side lengths by tiling it with unit squares of the appropriate unit fraction side lengths, and show that the area is the same as would be found by multiplying the side lengths. Multiply fractional side lengths to find areas of rectangles, and represent fraction products as rectangular areas. 5.NF.4b

Let us quickly review this, as a preliminary to extending the discussion to three dimensions. In two dimensions we start the measurement discussion with the square: say its side length is 1 cm. We say that the area of the square is 1 cm^2 (1 square centimeter). If the side length is 5 cm, then the area of the square is $5\text{ cm} \cdot 5\text{ cm} = 25\text{ cm}^2$. In general with a side length of s cm, the area is $s\text{ cm} \cdot s\text{ cm} = s^2\text{ cm}^2$. Notice the term s^2 for $s \cdot s$; this is called *exponential* notation. The *exponent* in s^n tells how many times the number s is multiplied. Thus $s^2 = s \cdot s$, $s^3 = s \cdot s \cdot s$, and so forth. In general, if the linear unit of measure is *foot, meter*, the unit for area measure is *square foot, square meter*.

For a general rectangle, since the sides may be of different lengths, we introduced the terms *length* and *width*. If the length is L centimeters, and the width is W centimeters, then we can fill the rectangle with L columns of 1 cm^2 squares, each column containing W squares. The total number of 1 cm^2 squares comprising the rectangle is $L \cdot W$. We say that the rectangle contains $LW\text{ cm}^2$. Notationally, this is represented by the equation: Area = $LW\text{ cm}^2$. We note that, in general, if the linear measure is in *unit*, the area measure is in *square units* or unit^2. So: linear *foot* leads to area *square foot*; linear *km* leads to area km^2.

Now, given a rectangle of dimensions, say, 3 feet of width and 4 feet of length, and therefore area 12 square feet, supposing that we increase the width by 1 foot, by how much do we increase the area of of the rectangle? This situation is depicted in Figure 18. The grid-marked rectangle is the original and the add-on is to its right. We have added a rectangle of area 4 square feet, so have increased the size of the rectangle by this area, which is 33% of the area of the original rectangle. In earlier grades, an example like this was used to illustrate the distributive property:

Figure 18



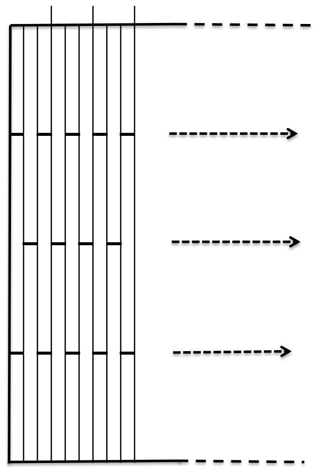
$(3 + 1) \cdot 4 = 3 \cdot 4 + 1 \cdot 4 = 12 + 4 = 16.$

This of course holds for fractional lengths; let's consider various such situations in the following problems.

Example 13. In a house being built the living room floor is to be made of wide maple boards, which come in widths of 4 inches and lengths of 12 feet. The floor is 24 feet long and 22 feet and 8 inches in width. How many boards will be used to cover the floor?

SOLUTION. A. The dimensions of the floor can be written at 24 feet long and $22\frac{2}{3}$ feet wide. Thus the area is:

$$24 \times (22 + \frac{2}{3}) = 24 \times 22 + 24 \times \frac{2}{3} = 528 + 16 = 544 \text{ ft}^2 .$$



Each board is of width $\frac{1}{3}$ ft and length 12 feet, and so covers

$$\frac{1}{3} \times 12 = 4 \text{ ft}^2 .$$

Therefore, the number of boards needed is $544 \div 4 = 136$.

B. Since the room is twenty-four feet long, the craftsman knows that he can cover the length with two boards, short end to short end. That covers $\frac{1}{3}$ of 1 foot of width. Three such pairs (6 boards) laid side-by-side will cover 1 foot of width and all the length. Therefore, to get 22 feet, he needs $22 \cdot 6 = 132$ boards. To cover the last 2/3 foot of width, he will need 2 pairs (4 more boards) for a total of 136 boards.

In both cases, note that the figure shows that, to maintain strength of the floor, the board lengths are staggered: one segment of lengths 6 ft - 12 ft - 6 ft is followed by a segment of lengths 12 ft - 12 ft.

Example 14. A rectangular platter measures $8\frac{1}{4}$ inches by 9 inches. What is the area of the platter?

SOLUTION. Figure 19 illustrates the solution. A rectangular plate of dimensions 8 inches wide by 9 inches long has an area of 72 in^2. The additional 1/4 inch in width is reproduced along the length of 9 inches, so adds to this area $\frac{1}{4} \cdot 9 = 2.25$ in^2. Thus the answer is 74.25 in^2. Another way to see this is to use the distributive property of multiplication over addition:

$$(8 + \frac{1}{4}) \cdot 9 = 8 \cdot 9 + \frac{1}{4} \cdot 9 = 72 + 2.25 = 74.25 \text{ cm}^2.$$

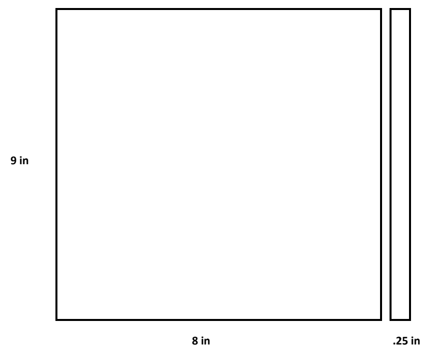


Figure 19

Example 15. A farmer wishes to plant corn in a rectangular plot that is $50\frac{1}{4}$ yards wide and $36\frac{1}{2}$ yards long. Estimate to the nearest square yard the area of the plot.

SOLUTION. If we just round the dimensions to 50 yards wide by 36 yards long, and multiply, we get an estimate of 1800 square yards. Figure 20 shows that this computation neglects two rather large strips and one small rectangle. The vertical strip has $36 \cdot \frac{1}{4} = 9$ square yards, and the horizontal $50 \cdot \frac{1}{2} = 25$ square yards. The small rectangle has $\frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8}$ square yards.

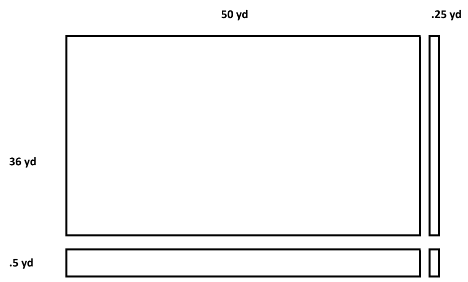


Figure 20

So, our easy attempt, multiplying estimates of the linear measures, lost 25 + 9 = 34 square yards, which is significant, and an additional $\frac{1}{8}$ square yard, which is not significant. We see that, in estimating the area of a rectangle, if we just multiply estimates of the side dimensions, we could be off by quite a bit. When making estimates of this kind, it is desirable to draw a diagram like that to the right in order to really include all the relevant measurements. The following few examples illustrate this principle, and leave it to the reader to make the appropriate diagrams.

Example 16. In order to estimate the cost of materials on a job, the painter measures the dimensions of the walls to be painted in quarter feet. Our painter has measured a particular wall to be 8.25 feet in height and 32.5 feet in length. Given that a quart of paint covers about 100 square feet, how many quarts of paint are needed?

SOLUTION. Use a similar figure as in the preceding example. Rounding these dimensions off by feet, we get $32 \times 8 = 256$ ft^2. But we need to add the areas of the missing strips, as in Figure 20. These give an additional $0.5 \times 8 + 32 \times 0.25 = 4 + 8 = 12$. So in total,we have a little more than 268 ft^2. Thus we need three quarts.

In this particular example, the simple calculation (32×8) would have sufficed, because the purpose was to get within 1 quart of paint (or 100 square feet of surface) and so we need not be so fussy. In Example 15, the farmer has to make a decision based on cost, and cost per square yard of preparation, seeding, maintaining and harvesting could be quite significant.

Example 17. The Head of a dancing school in New York City has engaged an architect to build a new school. The main exercise room has to be about 800 square feet. The architect sketched the new structure with a rectangular exercise room of dimensions 40 feet by 20 feet. The Head said that he would prefer a room that is more square. In his next attempt the architect gave a proposal of four different sets of dimensions:

a) 25 feet by 32 feet b) 28 feet by 28 feet, c) 27 feet by 30 feet, d) 29 feet by 29 feet .

Given that the cost of construction is \$670 per square foot, which of the four options is the best?

SOLUTION. This is a difficult question to answer for the word "best" is not defined. It the question were "cheapest," that would define it, but the questioner probably means to include aesthetics.That's fine: difficult questions generate rich discussions. This discussion will center about the importance of various measures: "squareness", total area, cost.

Squareness: b) and d) are square; a) is the least square; c) slightly better.

Total area: a) 800 sq. ft. b) 784 sq. ft. c) 810 sq. ft. d) 841 sq. ft.

Cost: a) \$536,000 b) \$525,280 c) \$542,700 d) \$563, 470 .

It looks like option b) is winning: it is square, and it is the least expensive.

Section 3. Volume of Three-dimensional shapes

Find the volume of a right rectangular prism with fractional edge lengths by packing it with unit cubes of the appropriate unit fraction edge lengths, and show that the volume is the same as would be found by multiplying the edge lengths of the prism. Apply the formulas $V = lwh$ and $V = bh$ to find volumes of right rectangular prisms with fractional edge lengths in the context of solving real world and mathematical problems. 6.G.2

Now we move to three dimensional geometry to understand the concept of *volume* and how to measure it. The basic 3D figure is the *prism*, or - more accurately - the right rectangular prism see Figure 21), or the rectangular parallelepiped.

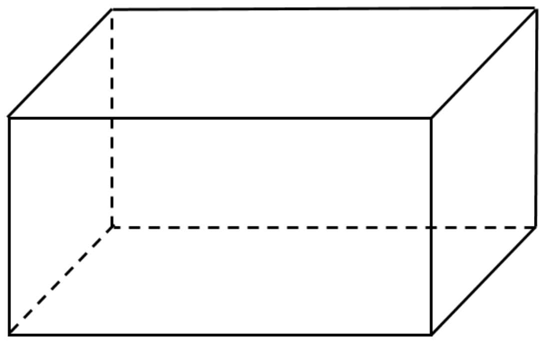


Figure 21. Rectangular Prism

A rectangular prism is a figure in space bounded by three pairs of congruent, parallel rectangles such that each pair meets each other pair in a right angle. (A figure bounded by three pairs of parallel parallelograms is called a parallelepiped.) The bounding rectangles are called the *faces* of the prism, the line segments in which the faces meet are the *edges*, and the points at which the edges meet are the *vertices*.

We identify the measure of a prism by specifying the measure of the three edges meeting at (any) one vertex, called the *length, width, height*. These designations are arbitrary, unless specified by the context.

A prism for which the length, width and height all have the same measure, is called a *cube*. A cube of side length 1 cm (called the *unit cube*) is said to have a volume of 1 *cubic centimeter*, denoted as 1 cm^3. A prism of length L cm, width W cm and height H cm (where L, W, H are positive integers) can be filled with 1 cm^3 cubes as follows. Along the side of measure L, we can place L unit cubes, with faces parallel to the faces of the prism.

Repeat this array W times in the direction of the width. This gives us LW unit cubes. This is demonstrated in Figure 22, where $L = 3$, $W = 5$, and $H = 6$. Now repeat this square array of cubes H times in the direction of the height. We have now filled the prism with LWH unit cubes, so it has volume LWH cm^3. The rectangular prism in Figure 22 thus has volume $3 \times 5 \times 6 = 90$ cm^3.

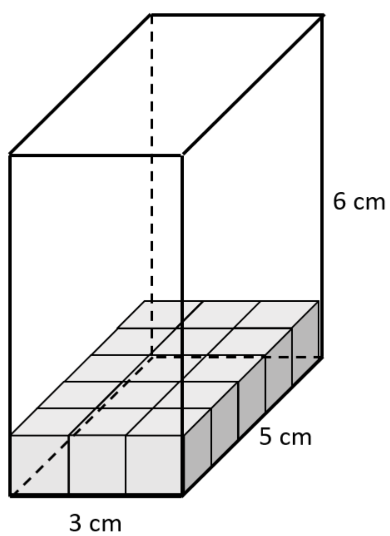


Figure 22. 3x5x6 Rectangular Prism

Example 18. Grigor wants to ship a violin to his cousin Ivan. The outside dimensions of the violin are: length: 23 in, width: 13.5 in, height: 5.5 in. He has decided, in order to safely send the violin, to get a box that is bigger than the violin in each dimension by half an inch, and fill the remaining space with padding. What are the dimensions of the box and its volume?

Solution. If we increment each of the dimensions by 0.5 in, we get a box of these dimensions: length: 23.5 in; width: 14 in; height: 6 in. Using the formula, the volume is $L \times W \times H = 23.5 \times 14 \times 6 = 1974$ in^3.

Note that the length is not an integer, so - if we return to the concept of area as counting the number of unit cubes we can fit inside the figure - we realize that at the last step, we cannot find a layer of unit cubes. But still the idea holds through: if instead we fill the region with half cubes, it works.

Caution: A cubic inch is of dimensions 1 inch by 1 inch by 1 inch. When we speak of "half a cube," we mean that one of those dimensions is halved (not all). That is, half a cube has half the volume of the original cube.

Note how the above arguments have gone: first we pick two dimensions, which determined a face of the prism, and find the area (by filling it up with unit squares) of that face. Say that area is B square units. Now we build B unit cubes on top of that face, and then repeat that all along the third dimension of length H, and conclude that the volume is BH cubic units. This remains true whatever the 2D figure selected at the beginning. Such a figure is still called a *prism, based* on the starting 2D figure. Let us illustrate this in the following example.

Example 19. A *wedge* is a 3D figure formed by taking a right triangle on a plane, and sliding it out in space in a direction perpendicular to that plane. In Figure 23 we start with a triangle of base length 6 cm and height 8 cm, and drew it out to a wedge whose width is 3.5 cm. What is the volume of this wedge?

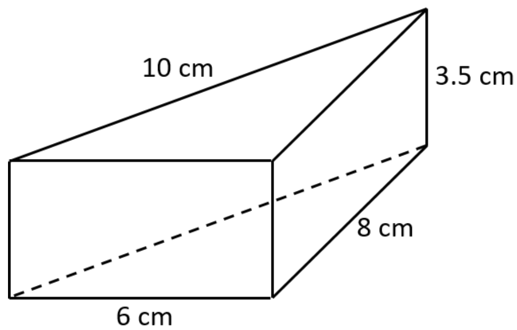


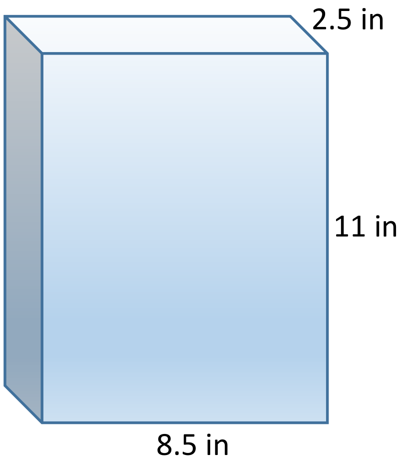
Figure 23. Wedge

Solution. There are five faces of this wedge, but for only one, the 6×8 triangle, can we say that the figure is drawn out in a perpendicular direction. So, we will use this face as the "base" and the vertical dimension (of 3.5 cm) the distance it is drawn out. The area of the base is $\frac{1}{2}(6 \times 8) = 24$ cm^2. Therefore, the volume is $bh = 24 \cdot 3.5 = 84\ cm^3$.

Example 20. José wants to wrap his favorite book as a Christmas present for his best friend. The book is 2.5 inches thick, and the pages (and covers) are usual page size: 8.5×11. What is the volume of the package?

SOLUTION. To the right find an image of the proposed package. The package will be a rectangular prism whose sides are of dimensions 2.5, 8.5, and 11 inches. The volume then is $2.5 \times 8.5 \times 11 = 233.75$ cubic inches.

In real life, even if the objects are polyhedra, the configurations are not such simple figures.However, in construction, for example, they are always composites of simple figures with lengths measured in the horizontal or vertical directions. The following (construction) examples illustrate the problems builders have to solve.



Example 21. Scythia is creating bookends for her shelves; these bookends are to be shaped as in Figure 24a below. The relevant dimensions (in inches) are shown. What is the volume of material of which each bookcase is to be made?

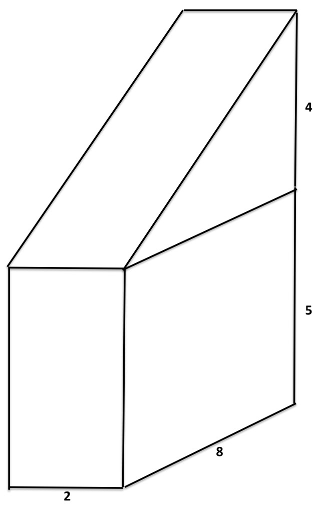


Figure 24a

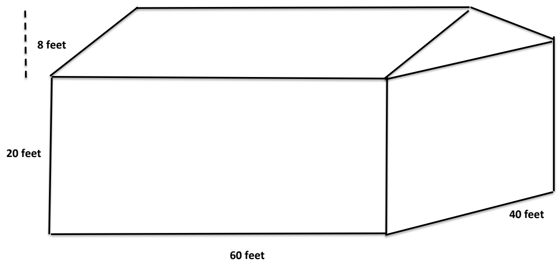


Figure 24b

SOLUTION. The solid consists of a wedge lying above a rectangular prism. The wedge is formed by dragging a right triangle of side lengths 4" and 6" in a perpendicular direction for 2". The prism has dimensions $2 \times 8 \times 5$, in inches. So the Volume of the entire object is

$$\text{Volume} = \text{Volume of wedge} + \text{Volume of prism} = A \times H + L \times W \times H =$$

$$= (\frac{1}{2}(4 \times 8)) \times 2 + 8 \times 2 \times 5 = 32 + 80 = 104 \text{ cu. in.}$$

Example 22. Ricardo is planning to air condition his house (see Figure 24b above). The power requirement of the air conditioner is decided by the volume of air to be conditioned. The image to the right is that of his house, together with the relevant measurements. What is the volume of the house?

SOLUTION. The solid consists of a wedge lying above a rectangular prism. The wedge is formed by dragging an isosceles triangle of base 40' and height 8' in a perpendicular direction for 20'. The prism has dimensions $60 \times 40 \times 20$, in feet. So the Volume of the entire object is

$$\text{Volume} = \text{Volume of wedge} + \text{Volume of prism} = A \times H + L \times W \times H =$$

$$= (\frac{1}{2}(40 \times 8)) \times 20 + 60 \times 40 \times 20 = 3200 + 48,000 = 51,200 \text{ cu. ft.}$$

Example 23.

Zybranka is putting together a support system for her canoe, which will keep it off the ground. She has decided on two T shaped brackets in the shape shown in Figure 25 using 4×4 inch posts. The canoe is of width 3 feet, 8 inches. She wants the bracket post to be 1.5 feet in the ground, and 2.5 feet above the ground. The side brackets are to rise 2 inches above the side of the canoe.

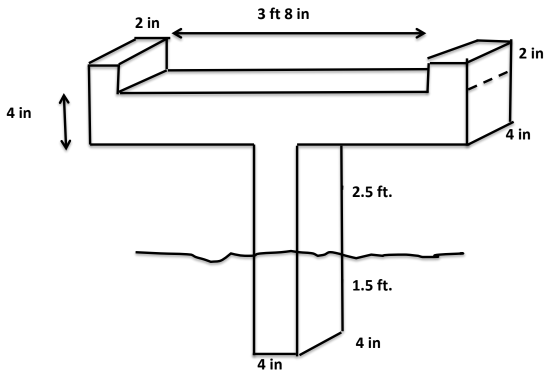


Figure 25

a) To make 2 such supports, how many linear feet does she need of 4×4 inch post material?

b) Her lumber yard sells 4×4 inch posts in 6 foot lengths. How many posts should she buy?

c) What is the total volume of the wood to be purchased?

Solution. This problem is more complex than the preceding, but still we can make the required analysis. Since all faces of the figure are either horizontal or vertical, we can decompose it into a set of prisms. One way to do so is shown in Figure 25a. The components are: to the left the support post; above that a horizontal beam and then the ends of the beam are topped by the two small prisms on the lower right.

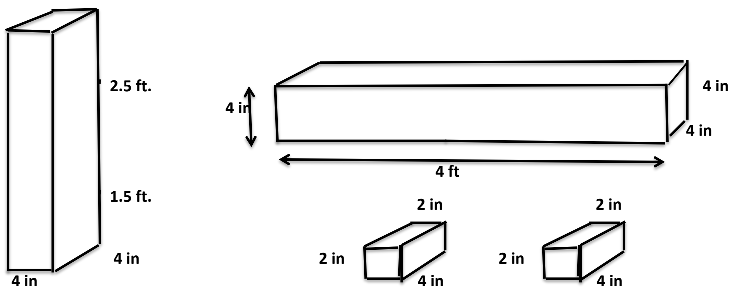


Figure 25a

a) Now the two larger posts have a 4×4 inch cross section, one of length 2.5 + 1.5 = 4 feet, and the other also of length 4 feet (3'8" +2"+2" = 4 feet). That comes to 8 feet. Now, the small 2" × 2" × 4" prisms that hold the canoe in place, contributing another 2" length of 4×4 post. One support needs 8'2" of 4×4; to make 2 she needs 16 feet, 4 inches.

b) Three 6-foot lengths of 4×4 post comes to 18 feet, so it would seem that all she needs is 3 6-foot posts. But she needs 4 4-foot lengths, and she can only get one out of each 6 footer. Thus she has to buy 4 6-foot lengths.

c) Note that the question is not about the total volume of the support, but is about the total volume of her purchase. Zybranka is purchasing 4 pieces of lumber 6 feet long, and of cross section 4×4 square inches. Since 4 inches is 1/3 foot, we calculate the total volume as

$$4 \times 6 \times \frac{1}{3} \times \frac{1}{3} = \frac{4 \cdot 6}{3 \cdot 3} = 2\frac{2}{3} \text{ cubic feet.}$$

Section 4. Surface Area of Three-dimensional Shapes

Represent three-dimensional figures using nets made up of rectangles and triangles, and use the nets to find the surface area of these figures. Apply these techniques in the context of solving real world and mathematical problems. 6.G.4.

In several of the problems above, the issue is not really that of the volume, but of the area of the surface. In the example of a book as a Christmas present, what is relevant is how much paper we use to wrap it. In all of our examples, the figure are polyhedra - figures bounded by planar polygons. Since we know how to find the area of those polygons, we can find the area of the surface of a polyhedron: it is the sum of the areas of its faces.

Example 24. a) Referring to Example 20, what is the surface area of the of the book that José wants to wrap? b) If José wants to have at least a one inch extension on each of the edges of the wrap, what are the dimensions of the rectangle of paper he must cut out?

SOLUTION. a) Figure 26 shows all of the faces of the book, opened up and laid out as a polygon in the plane. Region 1 is the beginning of the book and region 3 is the end of the book. Regions 2 and 4 represent the binding and its opposing side, and regions 5 and 6 represent the top and bottom of the book. The areas of the various regions are tabulated:

The surface area of the book is the sum of the areas

Region	1	2	3	4	5	6
Area	93.5	27.5	93.5	27.5	21.25	21.25

of these regions: 284.5 in^2.

b) The rectangle of wrapping paper that José has to be larger than the figure above by 1 inch on each side of the largest width and largest length of the figure above. The largest width is $2.5+11+2.5 = 16$, and the largest length is $2(8.5 + 2.5) = 22$. The area of a rectangle of dimensions 16 and 22 is $16 \times 22 = 352$ square inches. So to have a 1 inch extension on each side, we need a rectangle of 18 by 24 inches, and $18 \times 24 = 432$ in^2.

This figure is called a *net*: a planar depiction of the surface area of a three dimensional polyhedron. By (figuratively) cutting and splaying out, the surface of any polyhedron can be visualized by a net. We conclude with a few more examples.

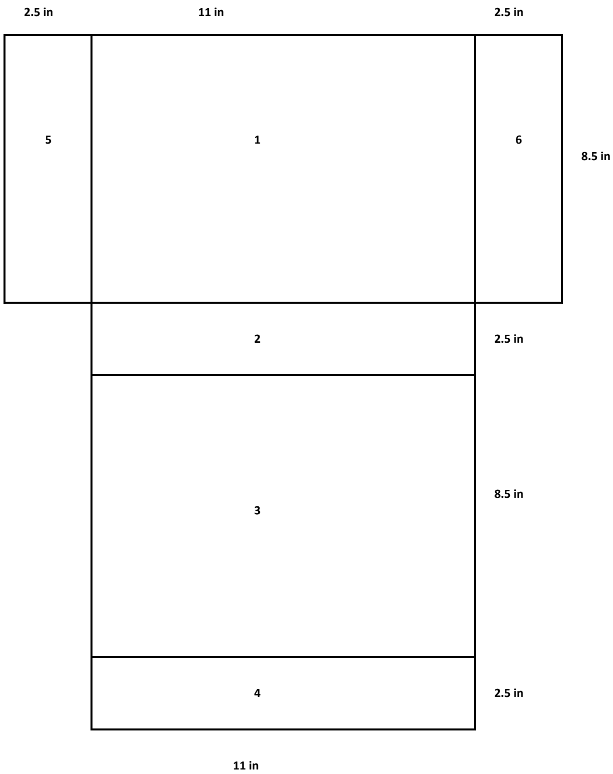


Figure 26. Net for book package area

Example 25. a) What is the surface area of a cube whose side length is 3 cm?
b) What is the surface area of a rectangular prism, given the measures: length = 3 cm, width = 5 cm and height = 6 cm?

SOLUTION. a) A cube has 6 faces, all of them congruent squares. If the length of a side of the cube is s cm, then the area of a face is s^2 cm. Since the cube has 6 congruent faces, the surface area of a cube of side length 3 cm is $6 \times 3^2 = 54$ square cm.

b) Figure 27 depicts the surface area of the prism with the given dimensions. Here we tabulate the areas of the component rectangles: and the sum is

Region	1	2	3	4	5	6
Area	30	18	30	18	15	15

Surface Area = 2(30+18+15) = 60+36+30 = 126 square inches.

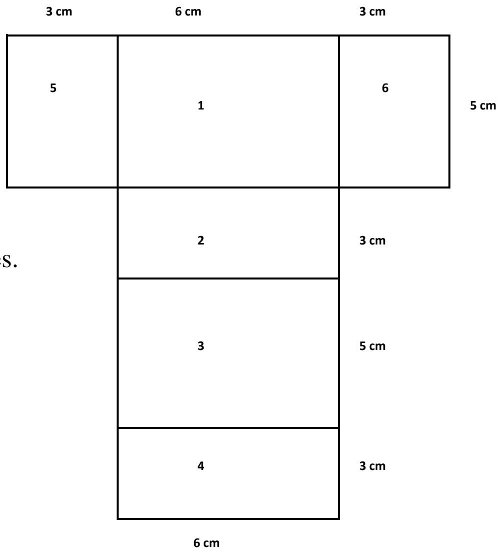


Figure 27

Example 26. a) What is the surface area of the box of Example 18?

b) What is the surface area of the wedge of Example 19?

Solution. a) This is the same type of problem as that of the preceding example, but with different side lengths. So, the net diagram to the right will serve for this problem if we refer to the side lengths of Example 18.

$$\text{Surface Area} = 2(23.5 \times 14 + 23.5 \times 6 + 14 \times 6) = 1108 \text{ square inches.}$$

b) Figure 28 depicts a net drawing of the surface of the wedge of Example 19. The drawing comprises two

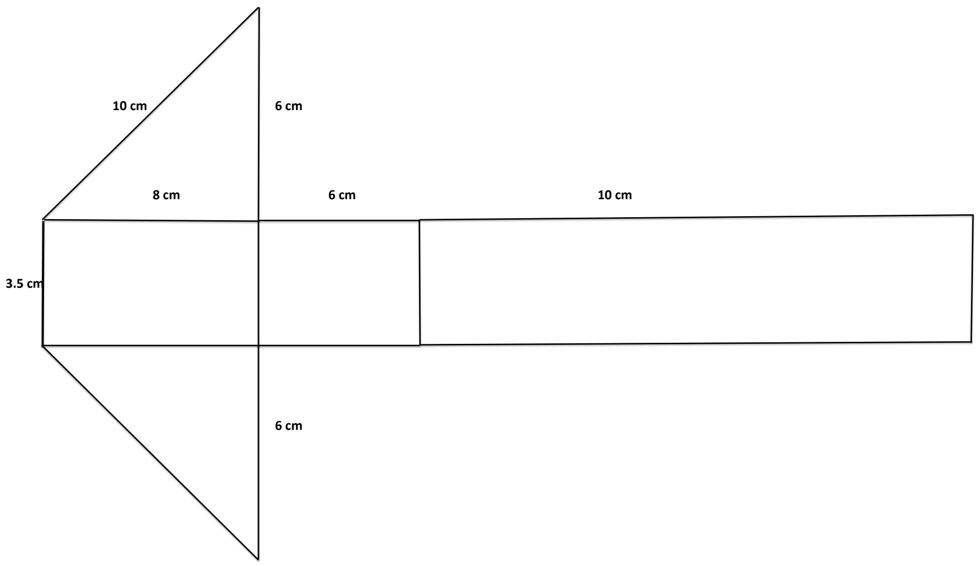


Figure 28. Surface Area Net of Wedge

congruent right triangles of leg lengths 6 cm and 8 cm, and three rectangles of width 3.5 cm and lengths of 8, 6, 10 cm each. This gives us an area of

$$\text{Surface Area} = 2(\frac{1}{2}(6 \times 8)) + 3.5(6 + 8 + 10) = 48 + 84 = 132 \text{ square cm.}$$

Let's now gather together, in formula form, the instructions for calculating volume and surface area of three dimensional polyhedra.

- Cube whose edge is of length L:

$$\text{Volume} = L^3, \qquad \text{Surface Area} = 6L^2;$$

- Rectangular Prism of length L, height H and width W:

$$\text{Volume} = LHW \qquad \text{Surface Area} = 2(LW + LH + WH).$$

- Wedge of height H based on a right triangle of side length A, B, C, where C is the longest:

$$\text{Volume} = \frac{1}{2}ABH \qquad \text{Surface Area} = 2AB + (A + B + C)H.$$

Reliance on formulas is not a good strategy to solve real-world problems; as we have seen in the preceding examples, in the real world we rarely have simple polyhedra with which to work. So, the major task for the student is to learn effective techniques to decompose polyhedra into simple figures, for each of which we can apply a formula. To illustrate this, let us return to Example 23 and calculate the volume and surface area of the polyhedron in Figure 25a.

Example 27. What are the volume and surface area of the polyhedron (one of the canoe supports) of Example 23?

SOLUTION. Let's copy Figure 25a, showing the decomposition of the canoe support into rectangular prisms.

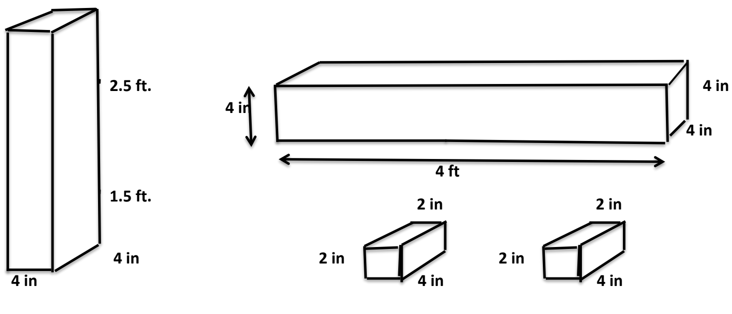


Figure 25a

As for volume, convert inches to feet and then apply the formula $V = LWH$ for each prism:

$$\text{Volume} = (2.5 + 1.5) \cdot (\frac{1}{3} \cdot \frac{1}{3}) + 4 \cdot \frac{1}{3} \cdot \frac{1}{3} + 2 \cdot (\frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{3}) = \frac{4}{9} + \frac{4}{9} + \frac{1}{54} = \frac{49}{54} \text{ cubic feet.}$$

Now, to surface area: we can calculate the surface area of each of the four prisms, and then subtract the area of those surfaces that that abut other surfaces in the support. For example, the top piece of the leftmost prism abuts the middle of the bottom surface of the top right prism. The area of overlap of these two pieces is 16 sq. in., so we have to subtract that twice from the sum of the areas of the pieces. Let's do the calculation by converting all lengths to inches:

$$\text{Surface area of upright post} = 2 \times (4 \times 4 + 4 \times 48 + 4 \times 48) = 2(16 + 192 + 192) = 800 \text{ sq. in.}$$

The top right (horizontal) post is of the same dimensions as the upright, so it too has a surface area of 800 sq. in. Finally, the two small pieces each have surface area

$$2(2 \times 2 + 2 \times 4 + 2 \times 4) = 40 \text{ sq. in.}$$

This gives a total of $800 + 800 + 2 \cdot 40 = 1680$ sq. in. Now we have to subtract the abutting faces. The two large posts abut a 16 sq. in rectangle, and each small retaining prism abuts the large horizontal post in 8 sq. in. rectangles. So, we have to subtract from the above calculation, $2 \cdot (16 + 2 \cdot 8) = 64$ sq. in. The surface area of one support is $1680 - 64 = 1616$ sq. in, or $11\frac{2}{9}$ sq. ft.