Chapter 6
Expressions and Equations

The relevant standard for this Chapter is

Apply and extend previous understandings of arithmetic to algebraic expressions. 6.EE.

The progression in the school curriculum, from arithmetic to algebra, is spread out over the grades 6-9. At first (and this begins in the primary grades) students are asked to perform arithmetic operations (such as $5 + 6$, and then 5×6) and then combining operations (as in $3 \times 2 + 4 \times 5$). Students learn that $3 \times 2 + 3 \times 5 = 3 \times 7$, and ultimately learn to identify the properties of arithmetic that permit these combinations. All of this can be put into the category of manipulation of *arithmetic* or *numeric* expressions.

Along the way, students begin to solve problems, not just of the sort "how much is 3+5?" but also "what has to be added to 3 in order to get 5?" This question is written as "3+? = 5" where the question mark may be replaced by a space or a box. This formulation is a prelude to algebraic notation, and in grade 5, it is suggested to use a symbol in place of the question mark, space or box, writing $3 + A = 5$, where "A" stands for "answer."

In grade 6 we begin to use letters systematically to represent unknowns, and introduce students to the rules of arithmetic; for example, those that allow us to transform $3 + A = 5$ into $A = 5 - 3$, and then do the computation to get the solution, 2. Thus, in a natural progression, calculations lead to problems, leading to the introduction of *unknowns* represented by letters, to the rules for *solving for unknowns.*

Until grade 6, students have been doing this work intuitively. The goal of this chapter is to develop understanding, in a structural way, what they have been doing, in order to achieve greater flexibility in thinking and application of processes. In this chapter, we formally describe the *process* in solving *one step* algebraic equations (specifically of the form $x + p = q$ and $px = q$.

We get there by detailing the transition from numeric expressions (such as $3(2 + 7)$) to algebraic expressions (such as $3(x + 7)$), as a prelude to understanding the mathematics of grades 7 and 8. The development of this material in the workbook is a little different from the sequence taken here in that the material is developed in a way that moves gradually from what students already know how to do to understanding what it is they are doing. Here, in the foundations we present the material as the objective to which the pedagogy is heading.

The major transition is that of using symbols as *unknowns* to using them as *variables*. This is a change from interpreting "x" as *a number to be determined* to *a placeholder into which any number may be put* in order to understand the behavior of the expression as that number is varied.

Students have seen variables in equations, particularly in Chapter 5, where certain *formulas* are introduced: $A = LW$, $V = LWH$ and so forth, but the text has only hinted at the possibility of their variation. Here, for example, L and W vary so as to produce a specific value of A. There are many rectangles of area A, and now we shift interest to the relationship of L and W as they vary.

This chapter should be seen as a broad overview of the way algebra is going to be used in subsequent grades. The idea is to acquaint students to what they are doing, and to broaden their understanding of these processes, something that will *not* be completed in grade 6. The idea is to expand their minds; that expansion of knowledge will accrue slowly over the subsequent years.

In section 1 we review numeric expressions from a structural perspective. Numeric (or arithmetic) expressions are viewed as a code for an computation, i.e., $3 + (5 - 1)$ encodes these two steps:

1) Subtract 1 from 5; 2) Add 3 to the result of step 1.

Two numeric expressions are said to be *equivalent* when the results of the computation they describe produce the same number. Another definition of equivalence is that we can move from one expression to the other by applying the rules of arithmetic. For a summary and discussion of these rules, see Chapter 0: Fluency. Indeed, we should view Chapters 0 and 6 as the bookends for the mathematics of grade 6. These two definitions of equivalence are to be viewed as interchangeable, using whichever is most convenient. For example, the expression $3 + (5 - 1)$ is equivalent to the expression $4 + (2 + 1)$ since they both produce the result 7. We also can get from $3 + (5 - 1)$ to $4 + (2 + 1)$ by calculating $5 - 1 = 4$, and then replacing 3 by 2+1, and finally, using the commutative property.

An *algebraic expression* is, in structure, the same as a numeric expression, except that now some numbers are represented by letters. These are thought of *unknowns*, in the sense that placing conditions on the expression allows us to solve for the value of the unknown. For example, $4 + (x + 1)$ is an algebraic expression; if we require it to produce the value 7, then we *solve* using the rules of arithmetic, to find that x must be 2.

But also, we can look at x as a variable, and we want to understand how $4 + (x + 1)$ varies as x varies. A simple observation is that $4 + (x + 1)$ will increase as x increases. A more subtle observation (to be made in grade 8) is that a change from x to $x + c$ produces a change in $4 + (x + 1)$ by the same amount.

Two algebraic expressions are *equivalent* if, *every* replacement of the unknowns by numbers produces equivalent numeric expressions. We can show that equivalence fails by finding just one such substitution which gives different answers. But to demonstrate equivalence, we have to show that for *every* substitution, we get the same answer. Since that is impossible in finite time, we employ the alternative definition: two expressions are equivalent if there is a sequence of rules of arithmetic that take one to the other. For example, $4 + (x + 1)$ is equivalent to $3 + (x + 2)$, which we can see by removing parentheses and adding $4 + 1 = 5$ in the first, and $3 + 2 = 5$ in the second, so that both expressions are equivalent to $5 + x$. Testing as many substitutions for x as we can still won't show the equivalence, because there are an infinite number of choices for x.

It is a fact (that will not be explored until grade 8) that to show equivalence of two linear algebraic expressions, we need only show that the give the same value for just two different substitutions of numbers for the unknown. There is a good chance that, while doing problems on equivalence, students may discover this fact.

In section 2, we introduce and discuss the concept of *simplification*. We have already seen one operation, that of combining terms (as in $3 + x + 2 = 5 + x$). Here, the word *combine* refers to addition. In an expression such as $5 \times (x \times 2)$, we combine by replacing this by $10 \times x$; here *combine* refers to multiplication. To emphasize the distinction between addition and multiplication, we begin by distinguishing the words *terms* (addends) and *factors* (multipliers). To further drive this home, algebraic expressions are represented by algebra tiles, where the word "like" is made evident.

Nest we discuss the distributive property as the second most important mechanism for simplification. "Order of operations" is a driving force here, without actually mentioning it. That is, to calculate $2(x + 6) + 2(x - 5)$ we first use the distributive property (getting $2x + 12 + 2x - 5$) and then combine like terms (getting $4x + 17$).

Section 2 ends with evaluating expressions and writing algebraic expressions to model real world problems.

The instruction "simplify" has to be understood in terms of the desired end result. For example, if we are discussing a rectangle whose length is twice as long as its width, we'll write down the formula for area as $A = w(2w)$, where w is the width. This formula is as simple as it can be if we are intent on showing the relation of the width and length of the rectangle. But if we want to find area in terms of width, we *simplify* to $A = 2w^2$. Here we are introducing the idea of *function* without mentioning it. The language used here: "in terms of" will become, in grade 8, "as a function of."

In section 3, we discuss this question: given two algebraic expressions, if they are not equivalent, for what substitutions of the variables (if any) do they give the same result? This set of values is called the *solution set* for the problem. This definition of *equation* (meaning, set two expressions equal to each other) becomes increasingly significant throughout the subsequent three grades. In grade 6 we consider only the simplest of these possibilities: those where one expression is a number, and the other is of the form $x + p$ or px. What is important here is the meaning of *equation* and the technique of its solution. In grade 6, the technique of solution is one-step: in one case we subtract, and in the other we divide. All of the rest of algebra (that is, linear algebra) is based on combinations of these two operations. In grade 6, we want to emphasize the essential nature: to solve $x + p = q$, subtract p from both sides; to solve $px = q$, divide by p on both sides (where $p \neq 0$).

Finally, we turn to inequalities, for which the preceding paragraph applies; but the answer will be expressed as an interval, and not just a single number.

The fact that the solution to an equation is a number, but the solution to an inequality is an interval (in general), can be an issue for students.

This relation, that the solution to an equation is a single number, while the solution to an inequality is an interval of numbers, is easy for a formalist: the same set of rules apply except that the form of the answer is different. For future work, it is important that students' understanding goes deeper into the difference of objective, despite the similarity of methods.

Section 1. The Structure of Numeric and Algebraic Expressions

Write, read, and evaluate expressions in which letters stand for numbers. 6.EE.2.

This section transitions students from numeric expressions (with which they have worked previously) to algebraic expressions. To facilitate this change we start with a review of numeric expressions. In particular, we want to emphasize that algebraic expressions are the same as numeric, except that some numbers are "unknown."

Numeric Expressions

The important building block in algebra is the idea of an expression. The term *expression* means a "phrase that makes sense" made up of numbers, letters, and operations. A numeric expression is made up of numbers and arithmetic operations (such as addition, subtraction, multiplication, division, etc.).

Examples of numeric expressions are the first three below:

a) $5 \cdot 4 + 3$, b) $(27/9)/2$, c) $7^2 - 6 \cdot 3 \cdot 2 + 9$ d) $5 + \div 4 - 3$ e) $7 \ \ 2 +$

Numeric expressions are to be interpreted as a set of instruction to perform a multi-step arithmetic computation. This implies that there is a syntax to be understood. It is not a good idea to write down and exercise the rules of this syntax (just as in grade 2 one should not perseverate on the rules of grammar), but it is important to direct students to an intuitive understanding. For example, a) above tells us that multiplication precedes addition, so $5 \cdot 4 + 3$ directs us to multiply 5 by 4 and then add 3; and not to add 4 and 3 and then multiply by 5. The latter instruction involves parentheses: $5 \cdot (4 + 3)$ tells us to first add 4 and 3, and then multiply by 5. As this illustrates, a numerical expression is not to be read from left to right as a set of instructions, but instead the parentheses indicate the way to be read. So, b) is read as "divide 27 by 9; then divide the result by 2." On the other hand, the expression $27/(9/2)$, instructs us to divide 9 by 2, and then divide 27 by that result.

It is through practice that students learn how to read them as a list of instructions. It might be useful at this time for students to use calculators to practice transforming numeric expressions into calculations. In this activity, the role of parentheses is significant, and time should be taken to understand how parentheses help with the order of operations. For example, consider these three expressions as three instructions for computation:

$$(3 + 5) \cdot 6 \qquad 3 + (5 \cdot 6) \qquad 3 + 5 \cdot 6 .$$

In order of operations, those within parentheses are to be done first. So, the first expression instructs: "add 3 and 5, then multiply by 6. The second expression instructs: "multiply 5 and 6, then add 3 to the result." The third expression is ambiguous: do we perform the addition first, or the multiplication? This ambiguity is resolved by the rule that, once parentheses have been observed, multiplications take precedence over addition. So, the third expression should be interpreted as the second. Nevertheless, parentheses always clarify ambiguities, so the second expression is preferable to the third. In grade 6 it is important to start to understand the conversion of an expression into a set of computational instructions, exploring the ambiguities that might arise.

As another illustration, b) above could have been written as $27/9/2$ or as $27 \div 9 \div 2$, without parentheses, and the understanding is this: first divide 27 by 9 and then by 2. However, the student might read the instructions from right to left, and interpret $27 \div 9 \div 2$ as $27 \div (9 \div 2)$ Although this is not likely, keep in mind that that interpretation is possible., and judicious use of parentheses can avoid confusion.

The last two expressions in the display above are not numeric expressions because they do not make sense. The reason these do not make sense is that most arithmetic operators require two inputs. Statement d) has too many operations between the 5 and 4. while e) does have two numbers and an operation but the operation is not between the two numbers.

Let's take another look at c) $7^2 - 6 \cdot 3 \cdot 2 + 9$ once again. This is not to be confused with

$$(7^2 - 6) \cdot 3 \cdot 2 + 9 \quad \text{or} \quad 7^2 - 6 \cdot 3 \cdot (2 + 9).$$

In an expression with parentheses, the calculations inside the parentheses should be completed first. If there are no parentheses, then the expression is to be viewed as a sum of terms, and the multiplications indicated in the individual terms are to be completed first. So, in $7^2 - 6 \cdot 3 \cdot 2 + 9$, we first compute $7^2 = 49$, $6 \cdot 3 \cdot 2 = 36$ and replace the expression by $49 - 36 + 9$, and then perform the indicated subtraction and addition to get 22.

Numeric expressions arise in many common situations. If a person has a job that pays \$8 per hour and he/she works for 6 hours in one day, we can create a numeric expression to model the amount he/she will be paid: $8 + 8 + 8 + 8 + 8 + 8$. If someone wants to find the perimeter of a rectangle of width 5 feet and length 4 feet, it can be done by adding the lengths going around the figure: $5 + 4 + 5 + 4$ feet. For many numeric expressions, we can say the same thing just written in a different way. For example, using the fact that repeated addition is multiplication, we could rewrite $8 + 8 + 8 + 8 + 8 + 8$ as $6 \cdot 8$. Here we have used "·" to represent multiplication (rather than the symbol "×" to which students may be accustomed) because it will become the predominant way of indicating the operation of multiplication. Since $6 \cdot 8 = 48$ represents the amount paid for the day's work, we would express this with the proper units of \$48. For the perimeter problem, we could have recognized that the perimeter of a rectangle will be the sum of twice the width and twice the length: $2 \cdot 5 + 2 \cdot 4$, which can be evaluated as 18, meaning 18 feet. That way, the person will know how much material to purchase in order to complete the project (perhaps fence for a dog run). Examples 8 and 9 further illustrate this conversion of a context to an expression.

Example 1. Evaluate these numeric expressions.

$$a)\ 5 + 3 \qquad b)\ 5 \cdot 3 \qquad c)\ 2 \cdot (5 + 3) \qquad d)\ \frac{3}{4} + (\frac{3}{5})^2;$$

that is, perform the arithmetic operations as indicated.

Solution. Here we want to stress the order of operations. For a) and b) there is no issue: there is only one operation to perform, and so we get the answers:

$$a)\ 8 \qquad b)\ 15 .$$

In c) there are two operations: a multiplication and an addition. We can't do the multiplication first, for to do that, we have to know the result of 5 + 3. So, we add first: 5 + 3 = 8, and then we multiply, getting the answer

$$c)\ 2 \cdot (5 + 3) = 2 \cdot 8 = 16\ .$$

Of course, we could have used the distributive property of multiplication over addition first (see appendix), leading to this calculation:

$$c)\ 2 \cdot (5 + 3) = 2 \cdot 5 + 2 \cdot 3 = 10 + 6 = 16.$$

For d) we have these operations: an addition, two divisions and a squaring. If using a calculator, this is easy; but let's look at what the calculator does. First, it clears the parentheses, so squares 3/5 to get 9/25. Then it does the division of 3 by 4, replacing the results by decimals, and finally the addition:

$$\frac{3}{4} + \frac{9}{25} = 0.75 + 0.36 = 1.11 = \frac{111}{100}.$$

The question arises, "How do we know when two expressions, A and B, are equivalent?" One answer uses the understanding of an expression as a computation, a sequence of operations to be performed. We'll say that A and B are *equivalent* numeric expressions if they compute the same number.

Example 2. Of the following expressions, which are equivalent?

$$a)\ [2 \times (3 + (5 \times 6)] \div 6 \quad b)\ (40 + 15) \div 5 \quad c)\ 25 + 25 + 25 \quad d)\ 20 + 30 + 3 \times 10 - 5$$

Solution. a) computes to 11, b) computes to 11, c) computes to 75 and d) computes to 75. Therefore, a) and b) are equivalent expressions as are c) and d).

A *rule of arithmetic* is a way of transforming one expression into another which is equivalent to the first. For example, the expression $5 + 3 \cdot (2 + 1)$ is equivalent to $5 + 3 \cdot 3$, because $2 + 1 = 3$, and this is equivalent to $5 + 9$ since $3 \cdot 3 = 9$, and so forth.

The rules of arithmetic are already understood by the students, but not yet codified. In the Fluency chapter, we provided a discussion of the rules of arithmetic. The important fact is this: two numerical expressions are equivalent if one can be transformed into the other by a sequence of rules of arithmetic.

For example, because of the commutative property of addition, the expression 5 + 4 + 5 + 4 is equivalent to 5 + 5 + 4 + 4 (note that the two middle addends "switched places"). By the definition of multiplication, 5 + 5 is the same as 2 groups of 5, namely $2 \cdot 5$ and similar reasoning transforms 4 + 4 to $2 \cdot 4$. Therefore, in a series of mathematically valid steps, we can move from 5 + 4 + 5 + 4 to $2 \cdot 5 + 2 \cdot 4$ so these numeric expressions are equivalent. We could also go a step further, using the distributive property: $2 \cdot 5 + 2 \cdot 4$ is equivalent to $2 \cdot (5 + 4)$. Of course, we could also do the calculations and observe all three are equivalent for they all compute to 18.

Example 3: Kay purchases and downloads four songs for $1.50 each and three movies for $5.00 each. Write two different numeric expressions for how much Kay spent.

Solution. There are many correct answers, including the following: $1.5 + 1.5 + 1.5 + 1.5 + 5 + 5 + 5$, $4 \cdot 1.5 + 5 + 5 + 5$, $4(1.5) + 3(5)$, $6 + 15$. Note that each of these expressions calculates to 21, meaning $21.

Algebraic Expressions and Equivalence

Write expressions that record operations with numbers and with letters standing for numbers. For example, express the calculation "Subtract y from 5" as $5 - y$. *6.EE.2a.*

An *algebraic expression* is the same as an arithmetic expression, except that some of the entries are letters representing numbers. One way of viewing this is that an algebraic expression is a numeric expression for which some of the numbers are unknown.

As students gain fluency with numeric expressions, they transition to the more abstract algebraic expressions. In addition to using numbers and operations that make sense (as in a numeric expression), *algebraic expressions* also involve other symbols that each represent an unknown quantity or a measure that can vary (take on many possible values) called a *variable*. Common symbols include letters such as x, y, s, or n. Some examples of valid algebraic expressions include $3+x$, $2y-1$, s^2, πr^2, and $n(n+1)/2$. While algebra is usually thought of as secondary mathematics, examples of algebraic thinking underlies many of the tasks of elementary mathematics. When a first grade student solves the problem $3 + \square = 9$, he or she is actually using algebraic concepts by letting a symbol stand for an unknown value. Here the symbol was $\square$ but it could have easily been named x.

Evaluate expressions at specific values of their variables. 6.EE.2.
Include expressions that arise from formulas used in real world problems. Perform arithmetic operations, including those involving whole number exponents, in the conventional order when there are no parentheses to specify a particular order (Order of Operations). For example, use the formulas $V = s^3$ and $A = 6s^2$ to find the volume and surface area of a cube with sides of length $s = \frac{1}{2}$. 6.EE.2c

Example 4. Here are some algebraic expressions involving the *unknowns* x and y:

$$i)\ 2 + 3x \qquad ii)\ 6 - (x + 1) \qquad iii)\ x + 2y \qquad iv)\ xy.$$

Evaluate these expressions for each set of values of the variables x and y:

x	0	2	2.5	3	5
y	2	1	0	3	4

Of course, the y values are not relevant for expressions i) and ii).

SOLUTION.

i)	x	Value	ii)	x	Value	iii)	x	y	Value	iv)	x	y	Value
	0	2		0	5		0	2	4		0	2	0
	2	8		2	3		2	1	4		2	1	2
	2.5	9.5		2.5	2.5		2.5	0	2.5		2.5	0	0
	3	11		3	2		3	3	9		3	3	9
	5	17		5	0		5	4	13		5	4	20

Example 5. Here are some algebraic formulas from geometry (Chapter 5) involving the *unknowns* a, b and c:

$$i)\ A = ab \qquad ii)\ V = ab^2 \qquad iii)\ W = \frac{1}{2}(a + b)c.$$

Evaluate these expressions for each set of values of the variables a, b, c:

a	5	1	3	3.5	4
b	8	6	1.5	1.5	4.5
c	2	2.5	3	4	5

Once again, the c values are not relevant for parts i) and ii).

SOLUTION.

a	b	c	A	V	W
5	8	2	40	320	13
1	6	2.5	6	36	8.75
3	1.5	3	4.55	6.75	6.75
3.5	1.5	4	5.25	7.875	10
4	4.5	5	18	81	21.25

Identify when two expressions are equivalent (i.e., when the two expressions name the same number regardless of which value is substituted into them). For example, the expressions y + y + y and 3y are equivalent because they name the same number regardless of which number y stands for. 6.EE.4

As with numeric expressions, algebraic expressions are equivalent when they mean the same thing. Recall that two numeric expressions are equivalent if they both prescribe a sequence of operations that leads to the same number. Since an algebraic expression becomes a numeric expression when we replace all unknowns by numbers, surely we want equivalent algebraic expressions to give, upon *any* such substitution, equivalent numeric expressions. Otherwise said, two algebraic expressions are equivalent if, for every substitution of numbers for the unknowns, the computations so defined produce the same numbers. For a moment, let's look ahead to grade 8, where the concept of *function* is introduced. In that context, this definition of equivalence comes down to: the two expressions defining the same function.

It is impossible to verify the condition "for every substitution of numbers for unknowns . . . ," for there are infinitely many such substitutions. Fortunately, there is a workable alternative: we can make the test of equivalence of algebraic expressions by the rules of arithmetic, as we did for numeric expressions. A rule of arithmetic transforms an expression, numeric or algebraic, into an equivalent expression.

More precisely, two algebraic expressions, A and B, are *equivalent* if expression B may be arrived at by applying rules of arithmetic to expression A. To illustrate: the algebraic expression $3 + x$ is equivalent to the expression $x + 3$ because of the commutative property of addition. In brief, these are the rules of arithmetic.

Algebraic Properties of Arithmetic

Commutative Property

Additive: Changing the order of terms in an addition problem does not change the sum.

$$A + B = B + A.$$

Multiplicative: Changing the order of the factors in an multiplication problem does not change the product.

$$A \times B = B \times A.$$

Associative Property

Additive: You can group addends in different ways to add:

$$A + (B + C) = (A + B) + C.$$

Multiplicative: You can group factors in different ways to multiply:

$$A \times (B \times C) = (A \times B) \times C.$$

Distributive Property

When you multiply a sum (or difference) by a number, you multiply each term in the sum or difference by that number.

$$A \times (B + C) = A \times B + A \times C.$$

The point of making explicit these properties (or rules) of arithmetic is that they are to be used in deciding upon the equivalence of expressions. The rule is this: if we change an expression to another via one of these properties, we arrive at an equivalent expression. Of course, if we have two expressions and want to decide whether or not they are equivalent, if we fail to find a sequences of applications of the properties, we have not shown the inequivalence. But, the definition of equivalence is easier to apply: we need to just find one number whose substitution for the unknown in each expression produces different answers.

Example 6. Of these pairs of expressions, explain the equivalence (or inequivalence) of the pair.

a) $2x + 1$ and $2(x + 1)$, b) $3(x - 2)$ and $3x - 6$, c) $2x + 1$ and $2x + 3$,

d) $y(y - 1)$ and $y(1 - y)$, e) $2x - 3 - 2(x - 3)$ and 0.

Solution. a) The use of parenthesis is incorrect. In the first case we add 1 to $2x$, and in the second case, we add one to x and then multiply by 2. These are not the same operation, so we presume that the expressions are inequivalent. Try $x = 0$: the first gives 1 and the second, 2. In fact there are *no* numbers to be substituted to produce the same result.

b) We can use the distributive property to free the first expression of parentheses: $3(x - 2) = 3 \cdot x - 3 \cdot 2 = 3x - 6$. The expressions are equivalent.

c) The expressions are clearly inequivalent, since addition of 1 to an expression (in this case $2x$) is not the same as adding 3. Of course, we need only have shown that the two expressions give different results when 0 is substituted for x. Indeed substitution of any number for x gives inequivalent numeric expressions.

Suppose instead that we ask if the expressions $2x + 1$ and $x + 3$ are equivalent. A respondent may mistake $2x$ for $2 + x$ and say,"yes, they are equivalent because 2+1=3." Pointing out the mistake doesn't show that the expressions are inequivalent: we still have to show that there are numerical substitutions that give different answers. Our respondent, by chance, substitutes 2 for x and says, "Aha, $2 \times 2 + 1 = 2 + 3$! In that case, the student should try another number, and as it turns out, for any choice of number a, that is *different* from 2, $2a + 1 \neq a + 3$.

d) The expressions look different, so start with the assumption that they are inequivalent. In fact one is the opposite of the other. The only number which is its own opposite is 0, so the only numbers for which this is a correct statement are $y = 0$ and $y = 1$. We needn't make that analysis, setting $y = 2$: $2(2 - 1) = 1$ and $2(1 - 2) = -1$, so we get different results. We included this problem, because an error (misuse of the distributive property) leads to a different conclusion.

e) Using the distributive property, and then collecting terms we get

$$2x - 3 - 2(x - 3) = 2x - 3 - 2x + 6 = 3.$$

So the first expression is equivalent to 3. Since 3 is not 0, the expressions are inequivalent: in fact, no matter what substitution for x is made, the first always produces 3 and the second is always 0.

For linear expressions (like those in Example 6), there is a simple test for equivalence, based on grade 8 mathematics. However, it is so useful that it makes sense to let sixth graders know it. Basically it is this: if two linear expressions give the same results for two different substitutions, then they are equivalent. Here is the test:

> Two expressions of the form $ax + b$ and $cx + d$
> a) are equivalent if $a = c$ and $b = d$. Otherwise they are inequivalent.
> b) If $a = c$ and $b \neq d$ then there is no substitution for x for which the subsequent computation comes to the same result.
> c) If $a \neq c$, there is precisely one substitution which produces the same result in both expressions.
> d) If $a \neq c$ but $b = d$, the only substitution to make the expressions the same is 0.

Example 7. Of these pairs of linear expressions, explain the equivalence (or inequivalence) of the pair.

a) $2x + 1$ and $x + 3$, b) $2x + 1 + 3x - 1$ and $5x$, c) $2x - 3$ and $2(x - 3)$, d) $3(x - 2)$ and $3x - 6$.

SOLUTION. a) Substitute $x = 0$ to get 1 and 3, so they are not equivalent. However, if we substitute 2 for x, both expressions produce 5. Later on we will refer to this situation as "2 is the *solution* of the equation $2x + 1 = x + 3$." For now it suffices to note that finding one numeric substitution resulting in computations of the same number does not suffice to show equivalence.

b) Collect like terms in the first expression: we get the equivalent expression $5x$, which is the second expression, so they are equivalent.

c) Apply the distributive property to the second expression to get $2x - 6$, which is *always* 3 less than $2x - 3$. The expressions are inequivalent.

d) Here the distributive property on the first expression produces $3x - 6$, which is the second: the expressions are equivalent.

Example 8. If Manuel earns \$8 per hour mowing lawns, write two different, equivalent algebraic expressions for how much money he will earn by working for h hours.

SOLUTION. As with numeric expressions, there are many equivalent algebraic expressions. For each hour Manuel mows a lawn, he earns \$8. Therefore, he will earn $8 + 8 + \cdots + 8$ (where there are h addends of 8) dollars. Since this expression represents h groups of 8, we may also write this as $h \cdot 8$, which is equivalent to $8 \cdot h$ by the commutative property of multiplication. A common convention is to omit the multiplication symbol when dealing with an algebraic expression that is the product of a number and a symbol and write the number (called a *coefficient*) before the variable, meaning $8 \cdot h$ and $h \cdot 8$ would both be written $8h$.

An algebraic expression may involve more than one unknown/variable. This is illustrated in the next example.

Example 9: In Example 3, Kay purchased 4 songs at \$1.50 each and 3 movies at \$5.00 each. Write an expression for how much Kay will spend by downloading s songs and m movies.

SOLUTION. There are many equivalent expressions. The songs will cost $1.5 + 1.5 + \cdots + 1.5$ (s addends) which is the same as s groups of 1.5 or $s \cdot 1.5$. The movies will cost $5 + 5 + \cdots + 5$ (m addends) which is the same as m groups of 5, or $m \cdot 5$. Using our convention of writing the number first in a product, the total cost will be $1.5s + 5m$. This expression is helpful because it will let us know the cost to download any choices for number of songs or number of movies.

There are two more Properties of Arithmetic that can be brought up in grade 6, as they describe the special properties of the specific numbers 0 and 1. The reason to introduce them here is to initiate student understanding of the special roles played by these numbers. Their algebraic roles are central (pun intended) to the understanding of additive and multiplicative inverses.

Identity Properties of 0 and 1

Additive Identity Property of 0. When you add zero to a number/expression, the number/expression does not change.

$$A + 0 = A.$$

Multiplicative Identity Property of 1. When you multiply a number/expression by one the number/expression does not change.

$$A \times 1 = A.$$

Students will understand that if you add nothing to A, the result is still A; and if you take one copy of A, then all you get is A. But the point here is to move to the (formal) algebraic understanding of what is going on. For example, in grade 7 the idea of changing units is discussed. For example, 3 feet equals 1 yard. So, to change 87 feet to yards, we write

$$87 \text{ feet } = 87 \text{ feet } \times \frac{1 \text{ yard}}{3 \text{ feet}} = \frac{87}{3} \text{ yards} = 29 \text{ yards}.$$

What we have done here is to change from feet to yards by multiplying by 1, in the form

$$1 = \frac{1 \text{ yard}}{3 \text{ feet}}.$$

Section 2. Writing, Simplifying and Evaluating Algebraic Expressions

Identify parts of an expression using mathematical terms (sum, term, product, factor, quotient, coefficient); view one or more parts of an expression as a single entity. For example, describe the expression 2 (8 + 7) as a product of two factors; view (8 + 7) as both a single entity and a sum of two terms. 6.EE.2b.

Evaluate expressions at specific values of their variables. Include expressions that arise from formulas used in real-world problems. Perform arithmetic operations . . . in the conventional order when there are no parentheses to specify a particular order (Order of Operations). 6.EE.2c.

Terms and Factors

In the previous section, students began working with expressions. There are several mathematical words associated with expressions. First of all, an expression consists of numbers and letters connected by arithmetic operations. The numbers are called *constants* and the letters are called *unknowns* or *variables*. The operations are addition, subtraction, multiplication and division. It is customary to reduce this to the operations of addition and multiplication, since subtraction is the same as "adding the opposite," and division is the same as "multiplying by the inverse. Expressions may also include parentheses to indicate the intended order of operations. For example, $3 + 2 \cdot 6$ may be ambiguous and can be interpreted as either $(3 + 2) \cdot 6$ or $3 + (2 \cdot 6)$. The parentheses tell us which operations to perform first: in the case of $(3 + 2) \cdot 6$, first add 3 and 2; and in the case of $3 + (2 \cdot 6)$, first multiply 2 and 6. So, an expression of the form $x - 3$ is to be read as "$x + (-3)$ (add the opposite of 3 to x)" and $5 \div 3$ as $5 \cdot \frac{1}{3}$ (multiply 5 by the inverse of 3).

A *term* is a piece of an expression that is being added (also called an *addend*). This can be either a single symbol, a product of symbols or another expression in parentheses. So, in the expression $7 + 3x$, the terms are 7 and $3x$. in the expression $8 - 2$, the terms are 8 and -2. In the expression $3(y + 2) + 6$, $3(y + 2)$ is a (compound) term, as is 6. In the expression $1.5s + 5m$ from Example 9 in section 1, $1.5s$ and $5m$ are each terms, in fact, compound terms. In the expression $7 - (3 - 1)$, the terms are 7 and $-(3 - 1)$.

A *factor* of an expression is a piece of the expression that is multiplied into the expression. Thus 1.5 and s are factors of the term $1.5s$, 3 and $x + 1$ are factors of $3(x + 1)$, and x and y are factors of xy.

It is important to be careful using this language, for it is too easy to confuse terms and factors. For example, the expression $2x + xy$ consists of two terms, each is a compound term, and the first has the factors 2 and x and the second has the factors x and y. Later in this section we will discuss the distributive property and use the fact that x is a factor of both terms, to show that the expression has an equivalent representation that makes explicit that x is a factor of the expression (the equivalent representation being $x(2 + y)$). As this discussion can be difficult to follow, at this point it is essential to become fluent about terms and factors.

Example 10. For the following expressions, identify the terms and the factors of the terms.

a) $x + (2 + y)$ b) $3xy^2$ c) $7x - 2 + (3x + 2)$.

SOLUTION. a) The terms are x and $2 + y$.

b) There is just one term, $3xy^2$. Its factors are $3, x, y, y$.

c) There are three terms: $7x, -2, 3x+2$. The factors of the first term are 7 and x; of the second, $-1, 2$. The third term itself consists of two terms, $3x$ and 2. The factors of $3x$ are 3 and x. Why are the terms $7x, -2, 3x+2$, and not

Terms that involve the same variable raised to the same exponent are called *like terms*. In the expression $3x+5y+7+4y+8$, the terms are $3x, 5y, 7, 4y$, and 8. Here, $5y$ and $4y$ are like terms because they involve the same variable, y. 7 and 8 are both constants and are also like terms. However, $3x$ has no like terms in this expression.

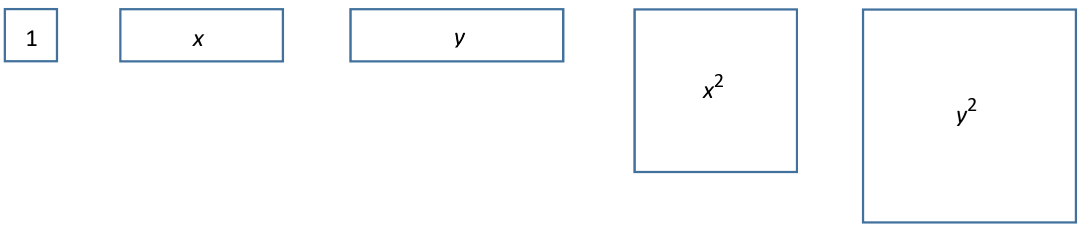
The importance of distinguishing like terms is that they can be combined to make the expression easier to read and compute, a process called *simplifying*. Students have been doing this for years, combing units with units, tens with tens, hundreds with hundreds, etc. The process of simplification may be illustrated through an analogy. If you have 5 apples and you add 4 apples, you have 9 apples. If you have 7 oranges and you add 8 oranges, you end up with 15 oranges. However, 3 bananas added to 9 apples do not make 12 bananas nor 12 apples because they are different kinds of fruits (unlike terms). In a similar way, we combine like terms (the same kinds of fruit) making $5y+4y$ worth $9y$ and $7+8$ can be replaced by 15; but $3x$ has no other like terms in this expression so it must remain as is. Therefore, using properties of arithmetic (commutative and associative), the expression $3x+5y+7+4y+8$ is equivalent to $3x+9y+15$. Since this involves the fewest possible terms (all terms are unlike), we say that it is in simplified (or simplest) form.

Example 11. Collect like terms in these expressions.

a) $x+3+4+y+2$ b) $3(x+1)+5x+1$ c) $3x+1+5(x+1)$ d) $2x-3y+2(x+y+2)$.

SOLUTION. a) $x+y+9$ b) $8x+4$ c) $8x+6$ d) $4x-y+4$.

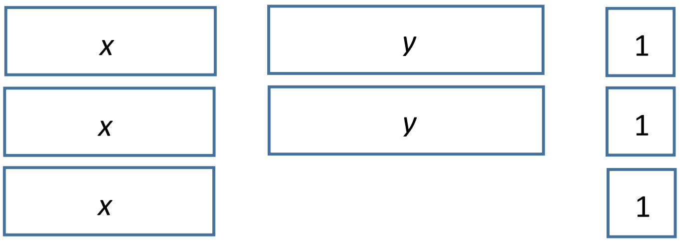
Algebra tiles are an aid in understanding simplification. They are readily available from online images to print paper versions. There are also many online applets students can use to model with algebra tiles. The basic tiles $(1, x, y, \ldots)$ are all the same width, 1, but differ in length based on the variable: units are squares, x-tiles have one length, and y-tiles have a different length. We can also create tiles for x^2 by making a square where each side has length x. Similarly, y^2 is a square where each side has length y. Because of this geometric nature, x^2 is commonly read as "x-squared." Some sample tiles are shown below.



Example 12. Model and simplify the expression $x+y+1+1+y+x+1+x$ using algebra tiles.

SOLUTION. Step 1: Model the expression with the tiles.

Step 2: Rearrange tiles so that tiles of the exact same shape (like terms) are together. This is essentially the step of "combining like terms."



Step 3: Convert to an algebraic expression. In the first column of step 2, we have three x-tiles, or $3x$. Next we see two y-tiles, or $2y$. Finally, we have three unit tiles, or 3. Therefore the expression $x + y + 1 + 1 + y + x + 1 + x$ in equivalent, simplified form is $3x + 2y + 3$.

Simplification of expressions is not easy, primarily because the objective of *simplify* is often dependent upon the context. For example, we might simplify $5(3+4x)$ to $15+20x$ if we wanted the simplest computational procedure. But if the question is to find out the value of x that gives this expression the value of 100, then the factored form is simpler, as it tells us the first step in solution: divide by 5. (The remaining steps of the solution will be discussed later in the chapter.) The idea of the following problem is to emphasize that the word "simplify" may need to be interpreted in terms of the context.

Example 13. In the table, on the next page, the instruction is "simplify the expression in the second row." The third row line gives an answer to the instruction. For each expression explain why the answer given is correct or incorrect. If incorrect, give the correct answer.

Problem	a	b	c	d	e
Expression	$x-x-x+y-y-y$	$x+x+x+y+y+y$	$4 + (2 + x)$	$12x$	$12x + 18y$
Simplification	$x + y - 2x - 2y$	$3xy$	Simplified	$3 \cdot 4 \cdot x$	$6(2x + 3y)$

Solution.

a). This is not in simplest form, for the terms x and $-2x$ (and y and $-2y$) can still be combined. The student who gives this answer vaguely understand "add together like terms" but is probably having difficulty understanding that the term $-2x$ means "add the opposite of $2x$."

b). This answer is incorrect because of a confusion of addition with multiplication. The student has seen 3 x's and 3 y's, so sees 3 xy's. But nowhere in the given expression are x and y connected by a multiplication sign. The correct simplification is $3x + 3y$; or $3(x + y)$, using the distributive property.

c). Here the student has decided that the expression has two unlike terms, 4 and $2+x$, and so is in simplified form. But, suppose that we remove the parentheses (a legitimate move) to get $4+2+x$. Then we have three terms, the first of which are like terms, and so we can combine them to get 6. The student has a point, and that is that the parentheses are there to set the $2+x$ apart from the 4. However, given no such reason, we should always rearrange parentheses (legitimally) to our advantage, and the preferred result is $6 + x$.

d). This is definitely a toss up. For purposes of computation, $12x$ is as simple as it can get. But, suppose (as did this student) that one was interested in exhibiting the factorization of the product. Then, the answer is almost the one sought for, which would be $3 \cdot 2 \cdot 2 \cdot x$.

e). The expression as it stands is simplified (for computational purposes), but if our interest is in the relationship between x and y, then the second line shows that better. This interest will be central in grade 7 mathematics, so here we should accept both forms as correct.

In the following table we show the best response to "simplify" with the caveat that in c)-e) we can go with both answers, depending upon the purposes of the simplification.

Problem	a	b	c	d	e
Expression	$x-x-x+y-y-y$	$x+x+x+y+y+y$	$4+(2+x)$	$12x$	$12x+18y$
Simplification	$-x-y$	$3x+3y$ or $3(x+y)$	$6+x$	Simplified	$6(2x+3y)$

As a lead in to the next subsection, notice that the the answer to b) could be: there are 3 x's added to 3 y's, or, after rearranging, there are 3 $(x+y)$'s.

The Distributive Property and Factoring

In addition to the commutative and associative properties of arithmetic, another very important property is the distributive property of multiplication over addition:

$$A(B+C) = AB + AC.$$

We have seen this property, without naming it, at work in part d) of Example 11: collect like terms in the expression $2x-3y+2(x+y+2)$. As presented, there are three terms, $2x$, $3y$ and $2(x+y+2)$, where the second term is subtracted, and the last term is added, so there is nothing to do. However, using the distributive property on the last term $(2(x+y-2) = 2x+2y-4)$, and then making that replacement we get the equivalent expression $2x-3y+2x+2y+4$. Now we can combine like terms to get $4x-y+4$, as given in the solution. In this expression the terms are $4x, y, 4$ where the second term is subtracted and the last is added.

This discussion demonstrates that the question: "what are the terms of an expression?" is ambiguous, for application of the distributive property gives us another. For our purposes, we should always apply the distributive property to produce the simplest "like" terms. Nevertheless, we see again that the request "simplify" can be ambiguous without reference to context.

Example 14. Two boys and four girls have been running chores all afternoon for the neighborhood grocer. At the end of the day, she paid them \$3 each for their work. How much, in total, did the grocer pay out?

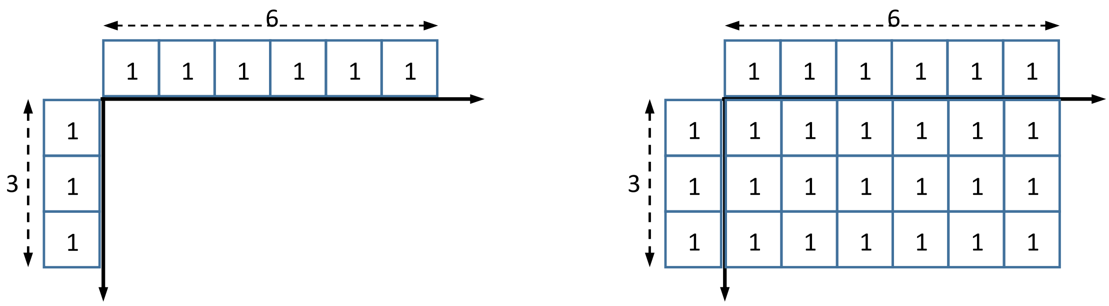
SOLUTION. The point of the distributive property is that there are several ways to make this computation. First, the grocer paid the boys \$3 each, so that comes to $3+3$, or \$6. As for the girls she put out $3+3+3+3$ or \$12. In total the grocer paid out \$18.

We could also reason this way: there are 2+4 workers, each of whom has been paid \$3, so the grocer has paid out $3(2+4)$, or $3 \cdot 6 = 18$ dollars.

Let's illustrate the computation using algebra tiles. Since $2+4=6$, this multiplication problem can be simplified to $3 \cdot 6$ and can be represented by an array of 3 rows of 6 columns, as shown to the right.

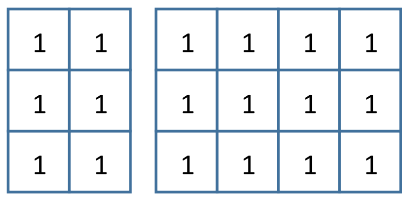
1	1	1	1	1	1
1	1	1	1	1	1
1	1	1	1	1	1

Some students benefit from making an "outline" or "frame" of the product $3 \cdot 6$ by first creating a column of 3 units and a row of 6 units, and then completing the product (rectangle) as shown below.

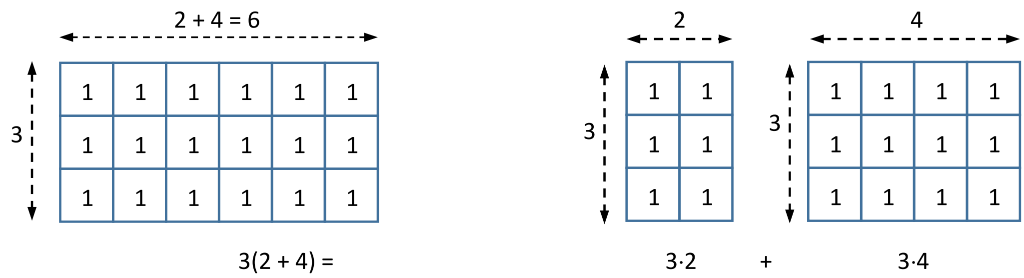


Using this method, we see that 3(2 + 4) has a value of 18.

If we look at the problem a little differently, we can separate the 3 by 6 rectangle into two pieces: two columns to represent the 2 and four columns to represent the 4 in the product. This is illustrated to the right.



The first block has three rows of two columns, representing the product $3 \cdot 2$. The second block has three rows of four columns, representing the product $3 \cdot 4$. Since we did not add or take away blocks (we only split the image into two pieces), the value must still be the same: $3 \cdot (2 + 4) = 3 \cdot 6 = 18$. Therefore, $3(2 + 4) = 3 \cdot 2 + 3 \cdot 4$ is a numeric example of the distributive property $a(b + c) = a \cdot b + a \cdot c$, where $a = 3, b = 2$, and $c = 4$. This is summarized in the figure below.



Example 15. Larry, Moe and Curly work Saturdays at the hamburg joint on Toledo Avenue. They each get paid \$12/hour for 6 hours, and then they all have dinner there at the joint, at \$7 per person. When they get home, they all deposit their "take-home" earnings in the same piggy bank. How much money is deposited in the piggy bank each Saturday?

Solution. Each of Larry, Moe and Curly worked 6 hours at a pay of \$12/hour, so each paycheck was $6 \cdot 12$ dollars. Then each spent \$7 for dinner. So the "take-home" for each is $6 \cdot 12 - 7$ dollars. Since this is the same for all of them, the aggregate "take-home" (that is, the deposit in the piggy bank is $3 \cdot (6 \cdot 12 - 7) = 195$ dollars.

We could instead first calculate the aggregate pay for work, which is $3 \cdot (6 \cdot 12)$, and then subtract the cost of dinner, which is $3 \cdot 7$ dollars.

Both computations are correct, telling us that

$$3 \cdot ((6 \cdot 12) - 7) = 3 \cdot (6 \cdot 12) - 3 \cdot 7 .$$

In this form it is easiest to compute the left hand side of this equation, to get $3 \cdot (72 - 7) = 3 \cdot 65 = 195$ dollars.

The following standard has been discussed in Chapter 0 (Fluency); this is a good place to bring that discussion in and further develop students' abilities in mental arithmetic.

Find the greatest common factor of two whole numbers less than or equal to 100 and the least common multiple of two whole numbers less than or equal to 12. Use the distributive property to express a sum of two whole numbers 1-100 with a common factor as a multiple of a sum of two whole numbers with no common factor. For example, express $36 + 8$ *as* $4(9 + 2)$. *6.NS.4.*

The distributive property is useful in doing mental arithmetic. Consider the numeric expression 18+30. We observe that each term is divisible by 3, giving us $3\times(6+10)$. This is $3\times 16 = 48$. If we had thought a little more, we could have made the computation even easier: both terms are divisible by 6, giving us $6\times(3+5) = 6\times 8 = 48$.

In the above case, we have looked for the largest number that divides both terms; this is called the *greatest common factor*, or GCF. Then we have reduced the problem to an addition and multiplication in its simplest form.

Example 16. Use the distributive property to mentally calculate the following sums.

a) $35 + 25$ b) $60 + 30 + 90$ c) $77 + 33$ d) $84 - 28$.

SOLUTION.

a) $35 + 25 = 5(7 + 5) = 5 \cdot 12 = 60$ b) $60 + 30 + 90 = 30(2 + 1 + 3) = 30 \cdot 6 = 180$

c) $77 + 33 = 11(7 + 3) = 110$ d) $84 - 28 = 4 \cdot (21 - 7) = 4 \cdot 7 \cdot (3 - 1) = 4 \cdot 7 \cdot 2 = 56$.

In general, when we have a numeric expression whose terms have no common factors (other than 1), the expression is in completely factored form.

These methods (of factoring and simplification) apply directly to algebraic expressions, since the letters in the algebraic expression are placeholders for numbers. The following example illustrates this.

Example 17. Write the expression $15 - 20x + 35y$ in completely factored form.

SOLUTION. Note that each coefficient is a multiple of 5: $15 = 5 \cdot 3, 20 = 5 \cdot 4$, and $35 = 5 \cdot 7$. Therefore, $15 - 20x + 35y$ is equivalent to $5(3 - 4x + 7y)$. Since there are no other common factors shared between the terms in the parentheses, the expression is in completely factored form.

Repeated Multiplication: Exponents

Write and evaluate numerical expressions involving whole-number exponents. 6.EE.1

When students learned multiplication, they saw that it was a quicker way to do *repeated addition*. For example, 2 + 2 + 2 + 2 + 2 is five groups of two, or $5 \cdot 2$. Using variables, $a + a + a + \cdots + a$ (where there are b addends of a) is b groups of a, or $b \cdot a$.

Now, we consider the process of repeated multiplication, such as $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$, which is the product of five twos. We use exponents to describe this repeated multiplication: 2^5 represents this product of five 2s, and if we do the calculation, we get 32.

$$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^5 = 32 .$$

For any number a and any positive integer b we express the product

$$a \cdot a \cdot a \cdot \dots \cdot a \quad \text{where there are } b \text{ factors of } a, \text{ as } a^b \text{ .}$$

The superscripted number (or variable) is the *exponent* and tells us how many times to multiply the *base* (the non-superscripted number or variable) by itself. In this example, the exponent of 5 and base of 2 tell us to multiply 2 by itself 5 times.

Exponents come up in many context, and eventually lead to a method of transforming multiplication into addition (*logarithms*; a topic in grade 10 mathematics). Here we will discuss in more detail two such contexts: factoring and geometric measurement.

Example 18. Factor these numbers as a product of the smallest numbers you can find as factors.

a) 4 b) 12 c) 20 d) 32 e) 40 f) 48 g)120 h) 1000.

Solution.

a) $4 = 2 \cdot 2 = 2^2$.

b) $12 = 4 \cdot 3 = 2 \cdot 2 \cdot 3 = 2^2 \cdot 3$.

c) $20 = 4 \cdot 5 = 2 \cdot 2 \cdot 5 = 2^2 \cdot 5$.

d) $32 = 2 \cdot 16 = 2 \cdot 4 \cdot 4 = 2 \cdot (2 \cdot 2)(\cdot 2 \cdot 2) = 2^5$.

e) $40 = 5 \cdot 8 = 5 \cdot (2 \cdot 2 \cdot 2) = 2^3 \cdot 5$.

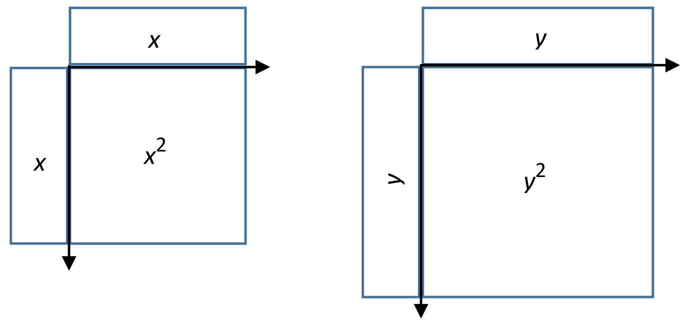
f) $48 = 4 \cdot 12$. Now, using a) and b): $48 = 2^2 \cdot 2^2 \cdot 3 = 2^4 \cdot 3$.

g) $120 = 10 \cdot 12 = (5 \cdot 2) \cdot (2^2 \cdot 3) = 2^3 \cdot 3 \cdot 5$.

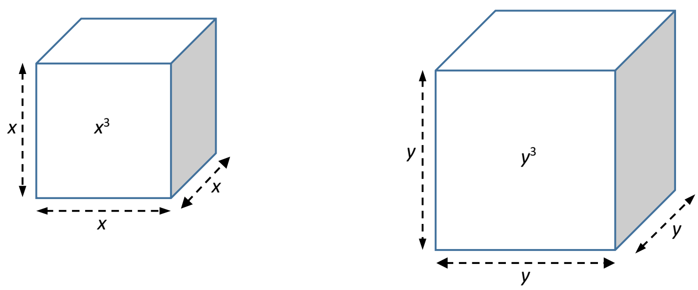
h) $1000 = 10 \cdot 10 \cdot 10 = 10^3$. But now, $10 = 2 \cdot 5$. So 1000 represents (2 multiplied together 3 times) times (5 multiplied together 3 times), which is represented by $2^3 \cdot 5^3$.

The phrase "the smallest numbers you can find as factors" may require some discussion, and students may recognize that those smallest numbers have the property that they can no longer be written as products of whole numbers greater than one, and thus are *primes*. Although such a discussion is desirable, it is not the point of this exercise, which is to gather together the same prime factors using exponents. Another point is that each of these problems has many routes to a final answer, and the final answer is always the same. For example, the solution to b) could be $12 = 6 \cdot 2 = (3 \cdot 2) \cdot 2 = 2^2 \cdot 3$. The recognition that there are many routes to the answer, and they all lead to the same place is a significant piece of mathematical understanding of the rules of arithmetic, and the fundamental fact that no matter how you got there, there is only one final answer.

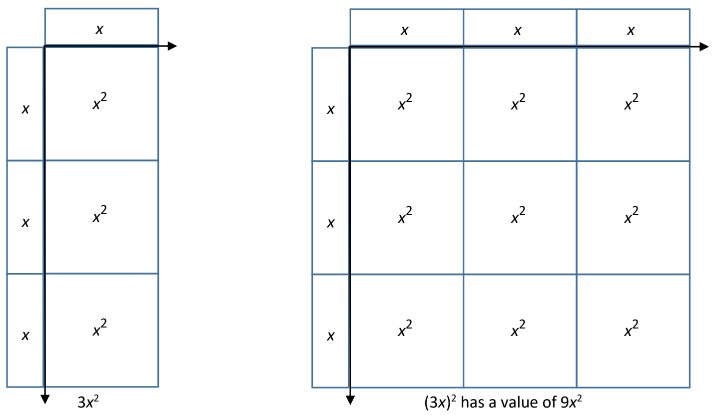
Using algebra tiles, we can see how expressions can involve variables raised to exponents. With the same technique as before, we interpret x^2 as $x \cdot x$, which can be viewed as a square of side length x, and y^2 as a square of side length y. This can be seen in the frames below. Note that since x and y are different variables (as represented by different sized algebra tiles), the squares for x^2 and y^2 are of different sizes.



The geometric interpretation carries over to variables raised to the third power: x^3 corresponds to a cube where each side is of length x and the volume of the cube is $x \cdot x \cdot x$. Similar reasoning yields a cube of side length y has a volume of y^3 cubic units. These are shown in the figure below.



When learning about algebraic expressions involving exponents, sometimes confusion arises. For example, there is an important difference between $3x^2$ and $(3x)^2$. The first expression, $3x^2$, means three groups of x^2. Note that $3x^2$ can be written $3 \cdot x^2$ to help emphasize that it is 3 times (copies of) x^2. Using the associative property, $3x^2$ can also be written as $(3x)x$ or equivalently, $(x + x + x)x$. The second expression, $(3x)^2$, means a square where each side length is $3x$, and gives a value of $(3 \cdot x) \cdot (3 \cdot x) = 3 \cdot 3 \cdot x \cdot x = 9x^2$. Both of these expressions are demonstrated in the figure below to show the difference between them.



Evaluate expressions ... that arise from formulas used in real world problems. Perform arithmetic operations, including those involving whole number exponents, in the conventional order when there are no parentheses to specify a particular order (Order of Operations). For example, use the formulas $V = s^3$ and $A = 6s^2$ to find the volume and surface area of a cube with sides of length $s = \frac{1}{2}$. 6.EE.2c

In the preceding chapter, in area and volume calculations, students became aware of the geometrically specific exponents 2 and 3: a^2 is the area of a square of side length a; a^3 is the volume of a cube of side length a. For these reasons, the expression a^2 is read as "a squared," and a^3 is read as "a cubed." In this language a square of side length 4 units is $4^2 = 16$ square units, and a cube of side length 7 units has volume $7^3 = 343$ cubic units.

Example 19.
a) What is the volume of a cube of edge length 6 cm?
b) What is the volume of a rectangular prism, the length of whose edges are 4 cm, 6 cm and 9 cm?
c) What is the volume of a rectangular prism of if one face is a square of area 27 square centimeters, and the length of the remaining side is 8 cm?

Solution. a) We use the formula $V = s^3$ for the volume of a cube of side length s, by substituting 6 for $s : V = 6^3 = 216$ cu. cm.

b) We use the formula $V = LWH$ for the volume of a prism of side lengths L, W, H, by substituting 4 for L , 6 for W and 9 for $H : V = 4 \cdot 6 \cdot 9 = 216$ cu. cm.

c) We use the formula $V = AH$ for the volume of a prism with a face a rectangle of area A and the remaining side length H, by substituting 36 for A and 6 for $h : V = 27 \cdot 8 = 216$ cu. cm.

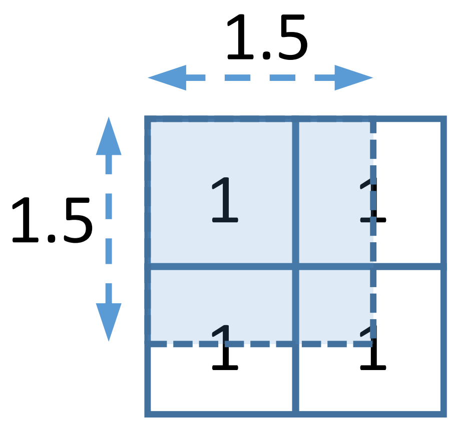
Notice that all these prism have volume 216 cu. cm, even though they have different shapes. It is good to emphasize this: if a rectangular prism has an area whose measure happens to be the cube of a number (in this case 6^3), it is not necessarily a cube.

Notice also that even though area can be computed as LWH or AH, we cannot conclude that these algebraic expressions are equivalent: they are equivalent only in the given context of the volume of a prism, and in particular need the additional statement that $A = LW$.

Of course, the formulas work for fractional numbers as well, as the following examples show.

Example 20. What is the area of a napkin that has a side length of 1.5 ft?

Solution.
In the upper-left piece of the image to the right, we see a 2×2 square with a smaller square of side length one and a half units shaded. We can calculate the area of the shaded piece in two ways:

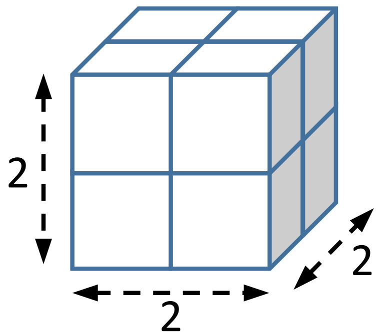


1) The shaded area consists of 1 unit square and 2 halves of a square unit (the shaded portions of the upper right unit square) and the lower left unit square, and one-fourth of the lower right unit square, telling us that the area of the napkin is $1 + 2(\frac{1}{2}) + \frac{1}{4} = 2.25$ square feet.

2) We can subtract the unshaded portions from the full 2×2 square, telling us that the area of the napkin is the area of the large square less the area of the unshaded pieces:

$$= 4 - 2 \cdot \frac{1}{2} - \frac{3}{4} = 4 - 1 - 0.75 = 2.25 \text{ sq. ft.}$$

When we move to 3 dimensions and the concept of volume, students have learned that the volume of a cube is the three times repeated product of the side length. That is: the volume of a cube of side length a is a^3 (as is shown in the image to the right, where $a = 2$).



Note that there are two layers of squares of side length 2, explaining for us why

$$2^3 = 2 \cdot 2^2.$$

Example 21. a) What is the volume of a cube of side length $\frac{3}{5}$ feet? What is the surface area of the same cube?

SOLUTION. We know (recall section 4 of Chapter 5) that a cube of side length L feet has volume L^3 cubic feet and surface area $6L^2$ square feet. So, just substitute $\frac{3}{5}$ for L:

a) $$\text{Volume } = (\frac{3}{5})^3 = \frac{3 \cdot 3 \cdot 3}{5 \cdot 5 \cdot 5} = \frac{27}{125} \text{ cu. ft.}$$

Note that if we had converted to decimals ($\frac{3}{5} = 0.6$), the computation would be Volume = $(0.6) \cdot (0.6) \cdot (0.6) = 0.216$ cu. ft.

b) $$\text{Surface Area } = 6 \cdot (\frac{3}{5})^2 = 6 \cdot \frac{9}{25} = \frac{54}{25} = 2\frac{4}{25} \text{ sq. ft.}$$

In decimals, this computes to 2.16 sq. ft.

In the above problem, it was just a coincidence that the surface area is 10 times the volume. This is specific to these numbers, because $\frac{3}{5} = \frac{6}{10} = 0.6$. It might be useful to demonstrate this in detail, especially if the issue comes up.

Example 22.

Another concept that students easily deduce from problems with whole numbers is: "the square of a number is bigger than the number, and the cube is *much* bigger." Then, when they work with figures of side length less than 1, they are surprised at how small the volume is compared to the side length. The intent of the next example is to make clear the difference of exponentiation of numbers less than one from those greater than one.

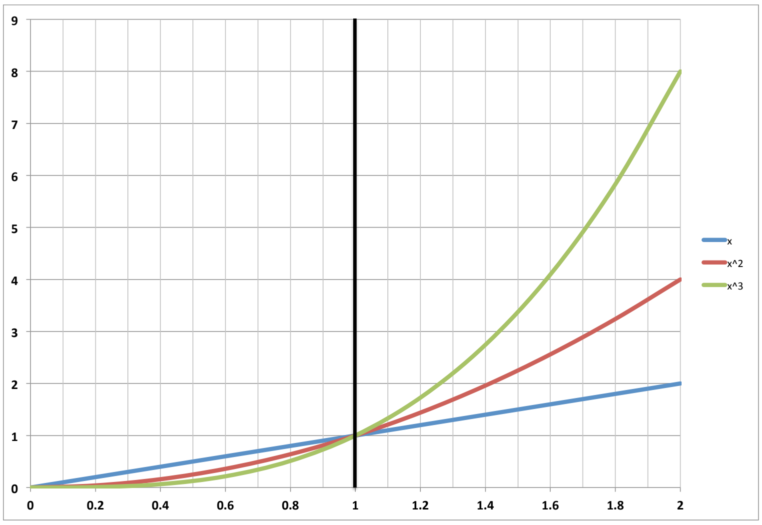
a) Fill in the blanks in the table to the right.

b) Graph the values.

x	x^2	x^3
0		
0.2		
0.4		
0.6		
0.8		
1		
1.2		
1.4		
1.6		
1.8		
2		

SOLUTION. The table of values of x run from 0 to 2, at a spacing of 0.2. We are asked to calculate the values of the squares and cubes of these numbers. The result is in the table below.

x	x^2	x^3
0	0	0
0.2	0.04	0.008
0.4	0.16	0.064
0.6	0.36	0.216
0.8	0.64	0.512
1	1	1
1.2	1.44	1.728
1.4	1.96	2.744
1.6	2.56	4.096
1.8	3.24	5.832
2	4	8



Section 3. Equations and Inequalities in One Variable

The kind of problems that led to the development of algebra are those to determine the set of values of certain quantities that satisfy certain conditions. For example:

- What are the dimensions of a rectangle of area 1800 sq. ft. whose length is twice its width?
- This 50 foot by 160 foot lot is to include a house whose rectangular footprint is 2800 sq. ft. and has a frontage of at most 100 ft. What are the allowable dimensions?

The problems above are high school algebra problems, so we shouldn't expect 6th graders to solve them, but they should be adequately prepared so that when the time comes, they will be able to perform the tasks.

Reason about and solve one-variable equations and inequalities. 6.EE.5-9

This objective is the beginning of algebra, and will remain the over-arching goal driving the mathematics of the next four grades. For grade 6, students learn what is meant by these terms, and how to solve one step linear equations. These are the equations that ask:

i) for what values of x does the expression $x + p$ produce the specific result q?

ii) for what values of x does the expression px produce the specific result q?

We might say that the first is an "additive" question, while the second is "multiplicative." Students have been solving this kind of problem for some time now. To solve the first, subtract p from both sides, and to solve the second, divide both sides by p (it is assumed that $p \neq 0$). The point here is to have students articulate, in the algebraic context, the processes they have been using.

A more complex situation (to be taken up in grades 7 and 8) is this: for what values of x do the expressions $ax + b$ and $cx + d$ give the same result (i.e., "solve $ax + b = cx + d$")? Historically, these equations were considered different from the equations studied in grade 6. Indeed, they are: in grade 6 we ask: "for what value of the unknown do we get the result q", and in grade 7 we ask" "for what values of x do the expressions $ax+b$ and $cx+d$ give the same result?" In grade 8, students are introduced to the concepts of *function* and the *graph of a function*, and this question becomes: For what values of x and y do the graphs of $y = ax + b$ and $y = cx + d$ intersect.

Understanding Equations

Understand solving an equation … as a process of answering a question: which values from a specified set, if any, make the equation … true? Use substitution to determine whether a given number in a specified set makes an equation … true. 6.EE.5.

An *equation* is a question asking of two expressions: for what values of the unknowns do the expressions give the same result. Those values of the unknown which produce the same answer are called the *solutions* of the equation.

Example 23. Find the solutions of these equations:

a) $3 = x + 1$ b) $3x = 15$ c) $2x + 3 = x + 2 + x + 1$ d) $x + 1 = 3 + x$ e) $|x| = 2$.

SOLUTION.

a) asks for the whole number just preceding 3 and that is 2.

b) asks for the number which, when multiplied by 3 produces 15, and that is 5.

c) is more complicated, but if we simplify the right hand side, we get the left hand side. Thus the two expressions are equivalent, so *every* number is a solution of the equation.

d) This can be written as $x+1 = x+3$, and since $3 = 1+2$, for any number x, the right hand side is greater than the left by 2. So, there are *no* solutions.

e) There are two solutions, both +2 and −2 have absolute value 2.

As with expressions, it is not going to be possible to try all possible numbers in order to find out for which numbers the two expressions produce the same result. What we do is to manipulate the equations, using properties of arithmetic, so as to eventually end up with an equation for which the solution set is obvious, as:

$$a)\ x = 5 \qquad b)\ x = x \qquad c)\ x = x + 1.$$

In the first case the solution set is 5, in the second, all numbers are solutions, and in the last, no numbers solve the equation.

We say that two equations are *equivalent* if they have the same solution set. Since we can't check the two equations for all numbers, we resort to rules of arithmetic: these are rules that change an equation into an equivalent equation. They include all the rules of equivalence of expressions (applied to either side of the equation) and three more specific to equivalence of equations.

> A. If we replace any expression in the equation by an equivalent expression, we obtain an equivalent equation.
> B. If we add or subtract the same expression to both sides of the equation, we obtain an equivalent equation.
> C. If we multiply or divide both sides of the equation by the same nonzero number, we obtain an equivalent equation.

Of course, two equations are *not* equivalent if there is a number that is a solution of one, but not of the other.

Solving Equations

Solve real-world and mathematical problems by writing and solving equations of the form $x + p = q$ and$px = q$ for cases in which p, q and x are all nonnegative rational numbers. 6.EE.7.

Because of its importance, we repeat what was said in the preceding subsection: to solve an equation, apply the properties of arithmetic and the rules of equivalence A, B, C above until we arrive at an equation whose solution set is obvious. The equations with which students in Grade 6 will work are of the form $ax + b = q$, with either $a = 1$ or $b = 0$. In both of these cases, solving the equation requires just one step. Let's illustrate:

Example 24. Solve these equations:

$$\text{a) } x + 5 = 7 \qquad \text{b) } x + \frac{3}{5} = \frac{7}{10} \qquad \text{c) } 8x = 100 \qquad \text{d) } \frac{4}{5}x = 9.$$

Solution.

a). Rule B allows us to subtract 5 from both sides of the equation, to get the solution: $x = 2$.

b). Putting everything over the common denominator 10, we get:

$$\frac{10x + 6}{10} = \frac{7}{10}; \quad \text{now multiply by 10 to get} \quad 10x + 6 = 7.$$

Subtract 6 from both sides to get $10x = 1$. Now divide by 10 to get the solution $x = \frac{1}{10}$.

Alternatively, we can write everything as decimals: $x + 0.6 = 0.7$. Now subtract 0.6 from both sides to get $x = 0.1$.

c). Dividing both sides by 8 (or multiplying both sides by $\frac{1}{8}$) gives $x = 100/8 = 25/2 = 12.5$.

d). Multiplying both sides of the equation by 5/4 gives $x = 45/4 = 11.25$.

Understanding Inequalities

Understand solving … an inequality as a process of answering a question: which values from a specified set, if any, make the inequality true? Use substitution to determine whether a given number in a specified set makes an inequality true. 6.EE.5.

An *inequality*, just like an equation, asks a question about two algebraic expressions, specifically to find the numbers which, when make the inequality true (the solution set of the inequality).

Basically, inequalities are of two kinds: "for what values of the unknown is expression A less than expression B?" and "for what values of the unknown is expression A at most expression B?" Students may initially think the two are the same but the subtle difference is between "less than" and "at most" (sometimes said "less than or equal to") has to be respected. The statement "A is less than B" is written $A < B$, and the statement "A is at most B" is written $A \leq B$.

Since the statement "A is less than B" is the same as "B is more than A", and the statement "A is at most B" is the same as "B is at least A," we have two more possibilities of expressing an inequality. Finally, there is one more possible statement: A is within c of B, where c is some positive number. This statement involves absolute values, and is written $|A - B| < c$. Or, is it $|A - B| \le c$? The algebraic expressions resolve that ambiguity, and in any given context, the student will have to decide from the language, which is intended.

This table summarizes the discussion; we will follow it with specific examples.

Statement	Algebraic	Alternative	Algebraic
A is less than B	$A < B$	B is greater than A	$B > A$
A is at most B	$A \le B$	B is at least A	$B \ge A$
A is strictly between B and C	$B < A < C$ or $C < A < B$		$B > A > C$ or $C > A > B$
A is between B and C	$B \le A \le C$ or $C \le A \le B$		$B \ge A \ge C$ or $C \ge A \ge B$
A is strictly within c of B	$\|A - B\| < c$	A is strictly further than c from B	$\|A - B\| > c$
A is within c of B	$\|A - B\| \le c$	A is further than c from B	$\|A - B\| \ge c$

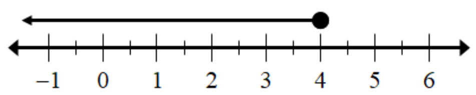
Example 25. For each of the following inequalities, determine whether or not the numbers $-1, 0, 6, 8$ satisfy the inequality

a) $x \le 4$ b) $x < 6$ c) $x \le 3/2$ d) $x > 3/2$ e) $|x - 3| < 1$

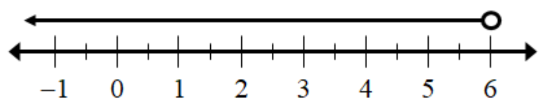
Solution.

Inequality	-1	0	6	8
a) $x \le 4$	T	T	F	F
b) $x < 6$	T	T	F	F
c) $x \le 3/2$	T	T	F	F
d) $x > 3/2$	F	F	T	T
e) $\|x - 3\| < 1$	F	F	F	F

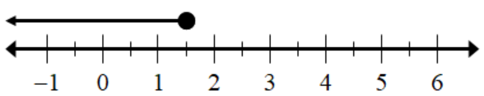
We can also show the solution set graphically. For all of these inequalities, the solution set is an interval, it could be an interval starting at some number, and preceding to the left or the right, or an interval between two numbers. In the following graphics we indicate the interval by an arrow. The numerical endpoints of the arrow can either be excluded ($<$ or $>$) from the solution set, or included ($\le$ or $\ge$). We illustrate inclusion by a a filled dot, and exclusion by an empty dot.



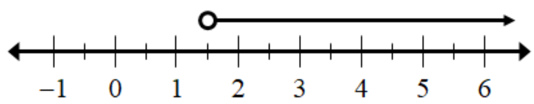
a) $x \le 4$



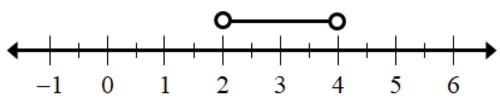
b) $x < 6$



c) $x \le 3/2$



d) $x > 3/2$



e) $|x - 3| < 1$

We should note that composite inequalities are also possible. For example, part e) could also be written as $2 < x < 4$. Similarly, if the circle on the right were filled, the inequality represented is $2 < x \leq 4$.

Solving Inequalities

Write an inequality of the form $x > c$ or $x < c$ to represent a constraint or condition in a real-world or mathematical problem. Recognize that inequalities of the form $x > c$ or $x < c$ have infinitely many solutions; represent solutions of such inequalities on number line diagrams. 6.EE.8.

To solve inequalities, just as with equations, we use the same rules of equivalence, with these changes:

A. If we replace any expression in the inequality by an equivalent expression, we obtain an equivalent inequality.
B. If we add or subtract the same expression to both sides of the inequality, we obtain an equivalent inequality.
C. If we multiply or divide both sides of the equation by the same **positive** number, we obtain an equivalent inequality.

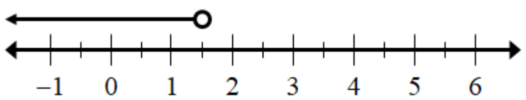
Example 26. Solve and graph the solution of these inequalities:

a) $2x < 3$, b) $x-1 \leq 5$ c) $2x > 3$, d) $x-3 < 1$, e) $|x-3| < 1$ f) $|x-3| > 1$.

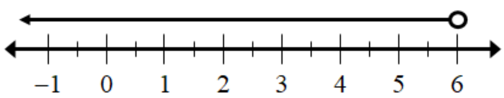
Jonathan: I made the change to leq in part b. Thanks for offering to make a new graphic. I have taken advantage to your offer; below I ask for 3 more graphics.

Solution.

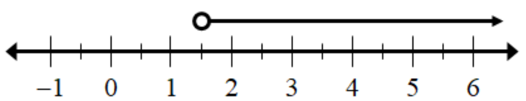
a) $x < 3/2$, b) $x < 6$ c) $x > 3/2$, d) $x < 4$, e) $2 < x < 4$ f) $x < 2$ or $x > 4$.



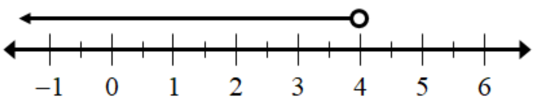
a) $2x < 3$



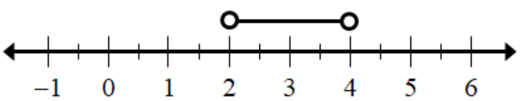
b) $x - 1 < 5$



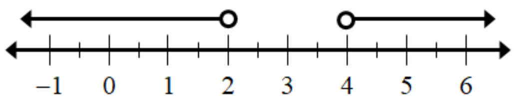
c) $2x > 3$



d) $x - 3 < 1$



e) $|x - 3| < 1$

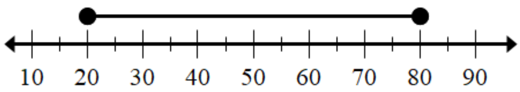


f) $|x - 3| > 1$

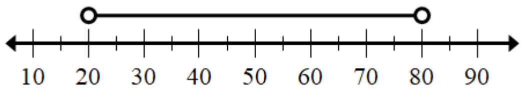
Example 27. A particular medication advises that the daily dosage is at least 20 mg and at most 80 mg. Using the letter D to represent dosage, express this advice algebraically.

Solution. The dosage has to between the values 20 and 80. The algebraic expression for these conditions is $20 \le D \le 80$, where D represents the dosage.

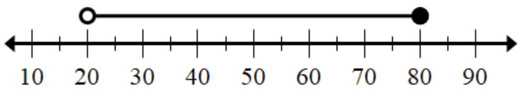
Note: if the instructions were "the dosage has to be greater than 20 and less than 80," then the algebraic expression would be $20 < D < 80$. Furthermore, if the instruction were "the dosage has to be greater than 20 and at most, 80," the algebraic expression would be $20 < D \le 80$. The following graphic illustrates the different possibilities, we a filled circle indicating the point is part of the solution set, and the empty circle indicating that the point is not in the solution set.



$20 \le D \le 80$



$20 < D < 80$



$20 < D \le 80$

Example 28. Your GPS informs you that the next stop sign is 35 miles away, and your destination is within 5 miles of that stop sign. Express this statement algebraically, both with and without absolute value.

Solution. Let D represent the distance to our destination. We are told that our destination is "within 5 miles of 35 miles," meaning that it is no less than 30 miles and no more than 40 miles. Algebraically: $D > 30$ and $D < 40$ or $30 < D < 40$. In terms of absolute value, we can write this as $|D - 35| < 5$.

As with the preceding example, it is not clear whether the term "within" includes or excludes the extrema. Our solution uses the exclusive definition, whereas the inclusive inequalities would be: $D \ge 30$ and $D \le 40$ and $30 \le D \le 40$ or $|D - 35| \le 5$.

In the case of Example 28, the ambiguity of the word "within" is not significant: the only meaning we can attach to the GPS report is that the next stop sign is about 35 miles way. However, in the case of Example 27, the difference can be significant (especially since the the pills probably come in units of 5 mg).