Chapter 3
Expressions and Equations Part 1

Making connections from concrete (specific / numeric) thinking to algebraic (involving unknown quantities / variables) thinking is a challenging but essential step in the mathematical progression of every student. Chapter 3 focuses on facilitating this transition by making connections to patterns (mathematical properties) already experienced for numbers and through repeated problems involving basic real-life examples. After developing understanding and procedural fluency with arithmetic properties of rational numbers, students learn how to manipulate equations to find solutions. Physical representations, including algebra tiles and area models, aid in understanding these operations for integer values and then extending them to include rational numbers. Using the distributive property "in reverse" helps students begin to master the important skill of factoring.

Once students have understood and achieved fluency with the algebraic processes, they then take real world situations, model them with algebraic equations, and use properties of arithmetic to solve them. The chapter concludes by having students model and solve percent increase and percent decrease problems that involve a little more algebraic thinking than the set of problems at the end of Chapter 1.

The objective of Chapter 3 is to facilitate students' transition from concrete representations and manipulations of arithmetic and algebraic thinking to abstract representations. Each section supports this transition by asking students to model problem situations, construct arguments, look for and make sense of structure, and reason abstractly as they explore various representations of situations. Throughout this chapter students work with fairly simple expressions and equations to build a strong intuitive understanding of structure. For example, students should understand the difference between $2x$ and x^2 or why $3(2x - 1)$ is equivalent (equal in value) to $6x - 3$ and $6x + (-3)$. Students will continue to practice skills manipulating algebraic expressions and equations throughout Chapters 4 and 5. In Chapter 6, students will revisit ideas in this chapter to extend to more complicated contexts and manipulate with less reliance on concrete models.

Another major theme throughout this chapter is the identification and use in argument of the arithmetic properties. The goal is for students to understand that they have used the commutative, associative, additive and multiplicative inverse, and distributive properties informally throughout their education. They are merely naming and more formally defining them now for use in justification of mathematical (quantitative) arguments.

In essence, Chapter 3 is an extension of skills learned for operations with whole numbers, integers and rational numbers to algebraic expressions in a variety of ways. For example, in elementary school students modeled 4×5 as four "jumps" of five on a number line. They should connect this thinking to the meaning of $4x$ or $4(x + 1)$. Students also modeled multiplication of whole numbers using arrays in earlier grades. In this chapter they will use that logic to multiply using unknowns. Additionally, in previous grades, students explored and solidified the idea that when adding/subtracting one must have *like units*. Thus, when adding $123 + 14$, we add the "ones" with the "ones," the "tens" with the "tens" and the "hundreds" with the "hundreds." Similarly, we cannot add $1/2$ and $2/3$ without a common denominator because the unit of $1/2$ is not the same as a unit of $1/3$. Students should extend this idea to adding variables. In other words, $2x + 3x$ is $5x$ because the unit is x, but $3x + 2y$ cannot be simplified further because the units are not the same (three of x and two of y). This should be viewed as a situation similar to that posed by: $2/3 + 4/5$, which cannot be simplified because the units are not the same (but here we have fraction

equivalence to replace both fractions by equivalent fractions with the same denominator.

In 6th grade, students solved one-step equations. Students will use those skills to solve equations with multiple steps in this chapter. Earlier in this course, students developed skills with rational number operations. In this chapter, students will be using those skills to solve equations that include rational numbers.

Section 3.1 reviews and builds on students' skills with arithmetic from previous courses, as previously noted, to write basic numeric and algebraic expressions in various ways. In this section students work on understanding the difference between an expression and an equation. Further, they should understand how to represent an unknown in either an expression or equation. Students will connect manipulations with numeric expressions to manipulations with algebraic expressions. In connecting the way arithmetic works with integers to working with algebraic expressions, students name and formalize the properties of arithmetic. By the end of this section students should be proficient at simplifying expressions and justifying their work with properties of arithmetic.

Section 3.2 uses the skills developed in the previous section to solve equations. Students will need to distribute and combine like terms to solve equations. In Grade 7, students only solve linear equations in the form of $ax + b = c$ or $a(x+b) = c$, where a, b, and c are rational numbers. This section will rely heavily on the use of models to solve equations, but students are encouraged to move to abstract representation when they are ready and fluent with the concrete models.

We close the chapter with section 3.3 which is about converting contextual (story) problems into algebraic equations and solving them. Contexts involve simple equations with rational numbers so the focus will be on concept formation and abstraction (solving algebraic formulations). Percent increase and decrease is revisited here. Time will be spent understanding the meaning of each part of equations and how the equation is related to the problem context. Note that the use of models is to develop an intuitive understanding and to transition students to abstract representations of thinking.

As students move on in this course, they will continue to use their skills in working with expressions and equations in more complicated situations. The idea of inverse operations will be extended in later grades to inverse functions of various types. A strong foundation in simplifying expressions and solving equations is fundamental to later grades. Students will also need to be proficient at translating contexts to algebraic expressions and equations and at looking at expressions and equations and making sense of them relative to contexts.

One of the fundamental uses of mathematics is to model real-world problems and find solutions using valid steps. Very often, this involves determining the value for an unknown quantity, or unknown. In Grade 8 we extend this concept to include that of a variable, a representation for a quantity that can take on multiple values. In this chapter we want to work on making several important distinctions. First is that between expressions and equations. The analogy is with language: the analog of "sentence" is equation and that of "phrase" is expression. An equation is a specific kind of sentence: it expresses the equality between two expressions. Similarly, an inequality makes the statement that one expression is greater than (or less than) another; a topic that will be further developed in Chapter 6. We note that statements can be true or false (or meaningless). In fact, the problem to be dealt with in this chapter is to discover under what conditions an equation is true; that is, what is meant by solving the equation?

These equations involve certain specific numbers and letters. We refer to the letters as unknowns, that is they represent actual numbers, but they are not yet made specific; our task is to do so. If an equation is true for all possible numerical values of the unknowns (such as $x + x = 2x$), then the equation is said to be an *equivalence*. Arithmetic operations transform expressions into equivalent expressions. Simplifying an expression or equation is a similar process of applying arithmetic properties to an expression or equation, but, in the case of an equation, may not go so far as to find the solution. Why are we interested in simplifying? When we simplify we make things "simpler" or reduce the number of symbols used while retaining equality, that is, to make the expression as short and compact as possible. Ultimately we are interested in finding that (or those, if any) numbers which when substituted for the unknown make the equation true. These are called the solutions.

Section 3.1: Communicate Numeric Ideas and Contexts Using Mathematical Expressions and Equations

This section contains a brief review of numeric expressions, with recognition that a variety of expressions can represent the same situation. Models are utilized to help students connect properties of arithmetic in working with numeric expressions to working with algebraic expressions. These models, particularly algebra tiles, aid students in the transition to abstract thinking and representation. Students extend knowledge of mathematical properties (commutative property, associative property, etc.) from purely numeric problems to expressions and equations. The distributive property is emphasized and factoring, "backwards distribution," is introduced. Work on naming and formally defining properties appears at the beginning of the section so that students can attend to precision as they verbalize their thinking when working with expressions. Throughout the section, students are encouraged to explain their logic and critique the logic of others.

Algebraic thinking doesn't just begin in Grade 7; it was started in the very early grades, in the sense that mathematics is about solving problems, and algebra provides the necessary algorithms, as in problems like "$3 + \underline{\quad} = 5$."

Early algebraic problems such as this are first solved using concrete manipulatives. However, as students develop their mathematical reasoning, they begin to use more abstract representational processes. Patterns in solving similar problems lead to both an understanding of, and familiarity with, arithmetic properties.

Apply properties of operations as strategies to add, subtract, factor, and expand linear expressions with rational coefficients. 7.EE.1

Students begin this section with a review of numeric examples and then extend that understanding to the use of unknowns. The following examples model the progression of thinking.

EXAMPLE 1.

For the following problems, a soda costs $1.25 and a bag of chips costs $1.75.

    a. Mary bought a soda and a bag of chips. How much did she spend?

    b. Viviana bought 3 sodas and 2 bags of chips. How much did she spend?

    c. Martin bought 2 sodas and some bags of chips, and spent a total of $7.75. How many bags of chips did Martin buy?

    d. Paul bought s sodas and 4 bags of chips. Write an expression for how much Paul spent.

    e. Domingo bought s sodas and 3 bags of chips and spent $10.25. How many sodas did he buy?

SOLUTION.

    a. $1.25 + $1.75 = $3.00.

    b. Viviana bought 3 sodas at $1.25 each, so spent 3($1.25) = $3.75 on soda. She bought 2 bags of chips at $1.75 a bag, so spent $3.50 on chips. All together she spent $7.25.

    c. Martin spent $2.50 on his sodas, and all the rest on chips. So, he spent $5.25 on chips. Since the cost of chips is $1.75 a bag, he bought $5.25/$1.75 = 3 bags of chips.

    d. At $1.25 each, the cost of s sodas is $1.25($s$) ; at $ 1.75 each, the cost of 4 bags of chips is $7.00. So, Paul spent 1.25s$ + $7.00.

e. At $1.25 each, the cost of s sodas is $1.25(s)$; at $ 1.75 each, the cost of 3 bags of chips is $5.25. Now, in total, Domingo spent $10.25, so we can write this equation:

$$1.25s + 5.25 = 10.25 \,,$$

from which we conclude that $1.25s = 5.00$, so $s = 4$.

One of the primary goals of this section is to help students recognize that properties of arithmetic, the rules they've informally observed throughout their formal and informal education that govern whole numbers and fractions, extend to all quantities (integers, rational numbers, and unknown quantities represented by letters or symbols). As the chapter unfolds, students will begin to name these properties, but first students must understand that addition and multiplication with unknown values "work" the same as they do with known values. The transition, in the above example, from **c.** through **e.** illustrates this, and the explicit use of these properties in solving problems. In **e.** we *simplify* the equation $1.25s + 5.25 = 10.25$ to $s = 4$ in order to solve it: First we subtract 5/.25 from both sides, and then we divide both sides by 1.25. Since these operations do not change the meaning of the assertion (of equality of the two expressions), they are allowable. The sentences, $1.25s + 5.25 = 10.25$ and $s = 4$, both tell us the same thing about the unknown s; in the second equation it has become known.

Before turning to equations, we first discuss simplification of expressions. For example, when simplifying $(3x + 2) + (4x + 6)$, students have learned that they can't simply add up all the digits ($3 + 2 + 4 + 6 = 15$). Instead, the place value involved in the notation tells students that there are three groups of x and two units in the first set and four groups of x and six units in the second set. From previous experience (with numeric place value), they know that to add you need to join the three x's and the four x's together because they are "alike" (based on the same size pieces). Also, the two units and six units are put together because they are units of the same value. Therefore, the simplified result is seven x's and eight units, written $7x + 8$.

To help make the desired connections between concrete numbers and abstract algebraic expressions, manipulatives similar to base-ten blocks, called algebra tiles, are introduced. These algebra tiles aid in understanding the processes of addition, subtraction, multiplication, and combining like terms because constants (positive and negative) and variables (positive and negative) each have a distinct shape. This commonality of shape encourages students to group the appropriate terms together, helping to avoid common mistakes. Although algebra tiles are for expressions with integer coefficients, the process extends to rational numbers. Early examples involve only one variable but later examples involve multiple variables, as shown below.

Algebra Tile Key:
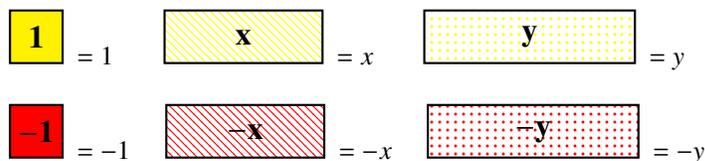
EXAMPLE 2.

Model the expression $3 - x - 1 + 2x$ and then simplify, combining like terms.

SOLUTION. First model each component of the expression with the corresponding algebra tile(s).

As with the base-ten blocks used in earlier grades, next put like terms together

Since $1 + -1 = 0$ and $x + -x = 0$, we have left the expression $2 + x$ (or, equivalently, $x + 2$).

EXAMPLE 3.

Model the expression $x + 3y - 2x + -y + 3$ and then simplify.

SOLUTION. First lay out the corresponding algebra tiles.
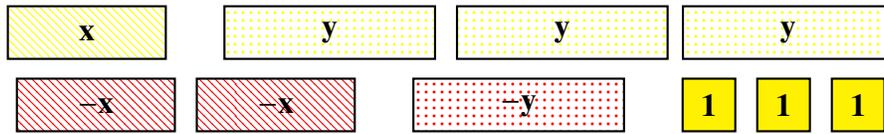
Next, combine like terms by grouping tiles of the same shape.
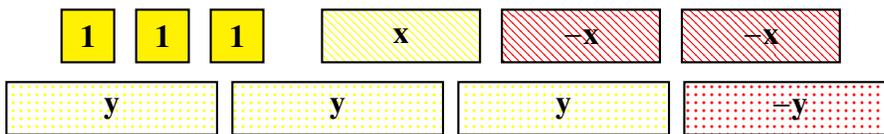
Since $1 + (-1) = 0$, $x + (-x) = 0$, and $y + (-y) = 0$, we eliminate some tiles, giving $-x + 2y + 3$.

The representation by tiles exposes various subtleties in the operations. Notice how easily tiles are reorganized so as to facilitate the cancellation of positive and negative tiles; after all, an expression like $x + -x$ can be eliminated, but first we have to recognize its existence.

After simplifying expressions, students then move to iterating groups and the distributive property as other ways to view the same expression. The understanding for this is built off of the area model for multiplication learned in earlier grades. Students have already learned that $3 \cdot 2$ can be represented as three rows of two, yielding a product (inside area of six).
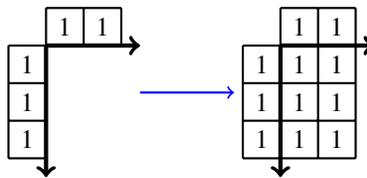
Similarly, we can use base-ten blocks to understand the meaning of $3 \cdot 21$ as three groups of the quantity two tens and one unit. In previous grades, students have written this out as $3 \cdot (20 + 1)$.
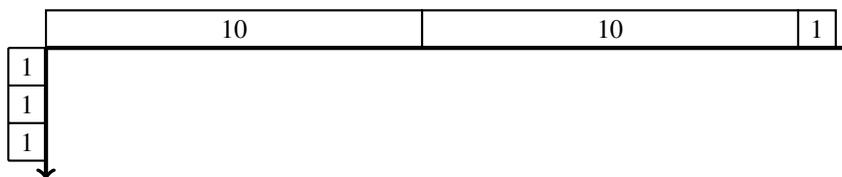
Once modeled, the students see that the inside area is six tens and three units or 63.

This naturally leads students to extend the knowledge to related algebraic expressions, for example $3 \cdot (2x + 1)$. Again the algebra tiles help make the desired connections, as in the following problem.

10	10	1
10	10	1
10	10	1

EXAMPLE 4.

Model the expression $3 \cdot (2x + 1)$ and then simplify.

SOLUTION. Use the area model but include variable tiles.
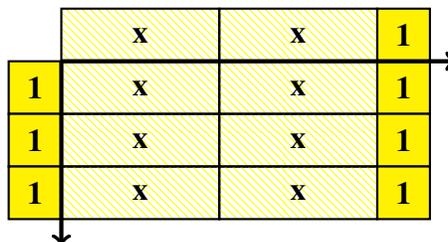
We see that the product (inside area) consists of 6 x-tiles and 3 1-tiles, so another way to write $3 \cdot (2x + 1)$ is $6x + 3$. This is perfectly analogous to the previous example except that we have used x-tiles in place of "base-ten" blocks.

Only one more step is needed when modeling a problem where an expression is multiplied by a negative integer. Emphasis needs to be placed on "−" meaning "the opposite": $2 + (−2) = 0$, $3 + (−3) = 0$, and in fact, for any number a, $a + (−a) = 0$. Subtraction can be viewed as adding the opposite: $a − b = a + (−b)$. All this may take time to unravel, say for an expression like $a − (−b)$. This should be read as $a + −(−b)$: in words, a plus the opposite of $−b$. Since the opposite of $−b$ is b, this becomes $a + b$. So, $5 − (3 − 4) = 5 + (−(3 − 4)) = 5 + (4 − 3)$ (since the opposite of $3 − 4$ is $4 − 3$, and so the result is $5 − (3 − 4) = 6$. This is not easy to understand without parsing the phrase $5 − (3 − 4)$ just as we parse a phrase in language.

Recall from the discussion in the preceding chapter, that, for whole numbers a and b, $a(−b)$ is considered as a groups of $−b$. Note that this is the same as the opposite of a groups of b: $a(−b) = −(a \cdot b)$. Finally, the extension of the laws of arithmetic leads to $−(a \cdot b) = (−a)b$.

EXAMPLE 5.

Model and simplify $−3 \cdot (2 − x)$.

SOLUTION. Realizing that $−3$ is the opposite of 3, first model $3 \cdot (2 − x)$ with algebra tiles. Recall that $2 − x$ is the same as $2 + −x$.
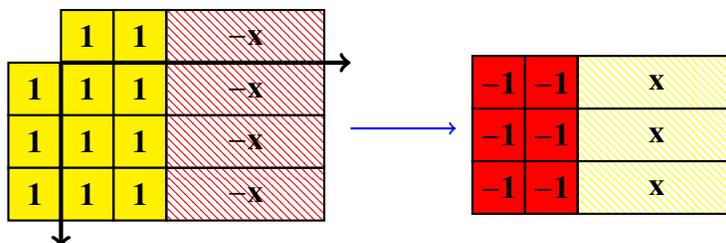
The inside area is $6 − 3x$. But this was the model for $3 \cdot (2 − x)$ and the original problem was $−3 \cdot (2 − x)$. Therefore, we need to take the opposite of all the inside tiles, getting $−6 + 3x$.

Students should recognize this as a model for the distributive law.

We continue to emphasize that the same patterns that held for numeric expressions also hold for algebraic expressions. At this point, students are introduced to the names of these properties and they practice using them as justification for each step in solving mathematical problems. It is also important to note that these same properties continue to hold as we extend to the real numbers in Grade 8 and complex numbers in Secondary II. In more advanced classes, systems that have the same properties will be given the name of field and will behave in similar ways. The table below summarizes these properties, making explicit that "iterating groups" is the distributive property.

Property Name	Concrete Example	General Equation
Identity Property of Addition	$7/10 + 0 = 7/10 = 0 + 7/10$	$a + 0 = a = 0 + a$
Identity Property of Multiplication	$9.04 \cdot 1 = 9.04 = 1 \cdot 9.04$	$a \cdot 1 = a = 1 \cdot a$
Multiplicative Property of Zero	$5/8 \cdot 0 = 0 = 0 \cdot 5/8$	$a \cdot 0 = 0 = 0 \cdot a$
Commutative Property of Addition	$5.8 + 2.4 = 2.4 + 5.8$	$a + b = b + a$
Commutative Property of Multiplication	$3.6 \cdot 7.1 = 7.1 \cdot 3.6$	$a \cdot b = b \cdot a$
Associative Property of Addition	$1/2+(3/5+1/4) = (1/2+3/5)+1/4$	$a+(b+c) = (a+b)+c$
Associative Property of Multiplication	$2 \cdot (3.1 \cdot 7.5) = (2 \cdot 3.1) \cdot 7.5$	$a \cdot (b \cdot c) = (a \cdot b) \cdot c$
Distributive Property of Addition over Multiplication	$2.8 \cdot (6.1+4.35) = 2.8 \cdot 6.1 + 2.8 \cdot 4.35$	$a \cdot (b+c) = a \cdot b + a \cdot c$
Additive Inverse Property	$7/16 + (-7)/16 = 0$	$a + (-a) = 0$
Multiplicative Inverse Property	$3/25 \cdot 25/3 = 1$	$a \cdot 1/a = 1$, for $a \neq 0$

Students practice applying these properties as they step through the simplification process logically. It is essential to concentrate on understanding, and not vocabulary. It does no good to be able to illustrate or define any one of these properties without understanding them.

The properties of arithmetic allow us to transform expressions so that they become more clear to us, *without changing*. But what does that mean? It can only mean that, when we replace unknowns by specific number, we end up with the same result. For example $2(x + 3)$ and $2x + 6$ are *equivalent* expressions because a replacement of x by *any* number gives us the same answer in both expressions. $2(51 + 3)$ and $2(51) + 6$ are the same number, and this is true if 51 is replaced by any number.

Checking the equivalence of expressions can become a tiresome project: we want to check that we get the same numerical result for every substitution of a number for an unknown, and that could take forever. In fact, it will. But we know that if we apply algebraic operations (i.e, combining like terms, eliminating parentheses, etc) we make changes in representation that do not make changes in meaning (as in replacing $2(x + 3)$ by $2x + 6$. A test for whether or not an algebraic operation creates an equivalent expression is that it can be undone. This applies to all the properties listed above, *e*xcept multiplication by zero which clearly cannot be undone. Actually, multiplying an expression by any number other than 1 changes its meaning, so is not allowable. To summarize:

I. To show that two expressions are *e*quivalent it suffices to show that we can get from one to the other by a sequences of applications of the laws of arithmetic.

II. To show that two expressions are *n*ot equivalent it suffices show that, for some number, its substitution for the

unknown in both expressions produces different results.

If expression B is derived from expression A, then the substitution of any number into the two expressions gives the same result.

EXAMPLE 6.

Consider the expression $2(3x + 1) + -6x + 3$. In the first column of the following table are various expressions. Some are equivalent to the given expression, and others are not. In the second column, if the expression is equivalent to the given expression, give the arithmetic rules that demonstrate the equivalence,. In the third column, justify the instruction by the appropriate property. If the expression is not equivalent to the given expression, just write *No* in the second column.

	Expression	Steps	Justification
a.	$2(3x + 1) + -6x + 3$	No change	Given expression
b.	$6x - 3$	No	
c.	0	NO	Multiplication by 0
d.	$6x + 2 + -6x + 3$		
e.	$6x + 2 + -6x - 3$		
f.	$6x + -6x + 2 + 3$		
g.	$6x + 1 - 6x + 3$		
h.	$0 + 5$		
i.	5		

SOLUTION.

	Expression	Steps	Justification
a.	$2(3x + 1) + -6x + 3$	No change	Given expression
b.	$6x - 3$	No	
c.	0	No	
d.	$6x + 2 + -6x + 3$	Multiplied both $3x$ and 1 by 2	Distributive Property
e.	$6x + 2 + -6x - 3$	No	
f.	$6x + -6x + 2 + 3$	Change the order of the terms	Commutative Property of Addition
g.	$6x + 1 - 6x + 3$	No	
h.	$0 + 5$	From **f**, $6x + (-6x) = 0$	Additive Inverse
i.	5	Delete 0+ from **h**	Additive identity

EXAMPLE 7.

1. In the above problem, in each case of an entry of "No" in the second column, explain how you arrived at that conclusion.

2. In each case, can we reverse the process: return to the expression $2(3x + 1) + -6x + 3$ from the expression in column 1?

SOLUTION.

Part 1.

b. If $6x - 3$ were equivalent to $2(3x + 1) + -6x + 3$, then they should give the same result under any substitution of a number. But, for $x = 1$, we get 3 for **b** and 5 for the original. Indeed, almost any number substitition gives different results.

c. Of course we can move from any expression to 0 by multiplication of zero. But there is no way to get back.

e. By the *value* test, since the substitution of 0 for x produces -1 in **e**, it cannot be derived from the original (which gives 5).

g. The same test works: substitute 0 for x in **g**, and get 4, not 5.

Part 2.

a. Since no change got us from the original to **a**, then "no change" gets us back.

d This results from applying the distributive property to the first term of the expression. This action is reversed by factoring the first two terms by 2.

f. This results from **d** by reordering terms according to type. This ca be undone.

h. This results from combining like terms in **f**. Since $6x + -6x$ is the same as 0, we can add it to **h** to get **f**.

i. We can add 0 to 5 to get $0 + 5$ without changing meaning.

Extension

Recalling the area model for multiplication, students understand why $x \cdot x$ is called "x-squared" and expand the algebra tile key.
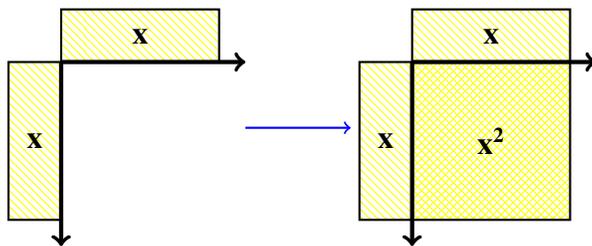
Key:
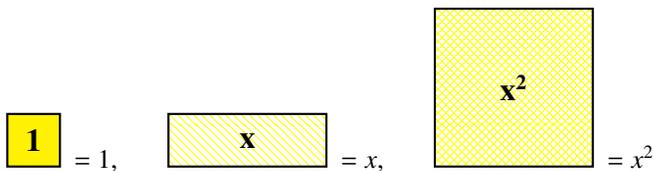
Combined with previous knowledge, students can now see the connection between arithmetic properties and expressions involving variables, especially multiplication and the distributive law. This is an important step as the student progresses from concrete to abstract.

EXAMPLE 8.

Use algebra tiles to find $2x \cdot (x + 3)$. Then write an equation of how this models the distributive property.

SOLUTION. Use the factors of $2x$ and $x + 3$ and the area model.
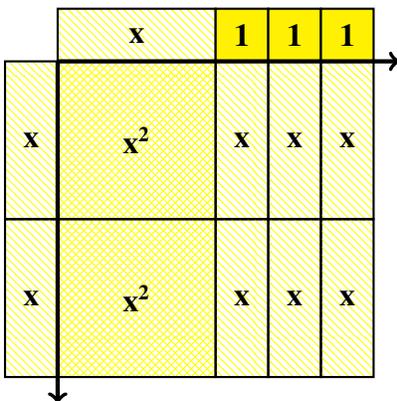
There are 2 x^2 pieces and 6 x pieces for an area of $2x^2 + 6x$. Therefore,

$$2x \cdot (x + 3) = 2x \cdot x + 2x \cdot 3 = 2x^2 + 6x \,.$$

Reversing the above process allows students to use the distributive property backwards, a process called factoring that will be used heavily in later courses. This is seen in the following example.

EXAMPLE 9.

Use the area model to factor the expression $x^2 + 3x$.

SOLUTION. Start with the given information, namely one tile for x^2 and three x pieces.
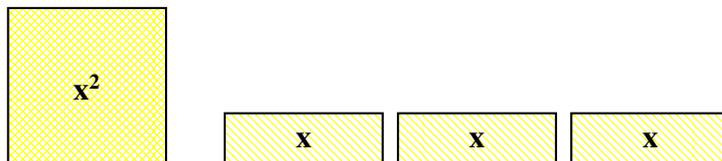
Since the area model for multiplication makes a rectangle, combine the pieces in such a way as to make one large rectangle. One method is shown below.
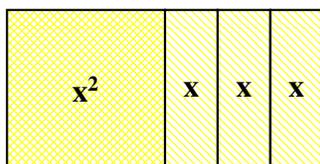
Looking at this as a multiplication in reverse, we see that this would be x down the left and x plus three 1s across the top. So the factors of $x^2 + 3x$ are x and $x + 3$. The equation showing the factorization (and connecting to the distributive property) would be $x^2 + 3x = x \cdot (x + 3)$.
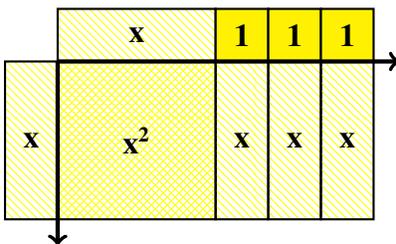
End Extension

Section 3.2: Solving Algebraic Equations (AX + B = C)

Having achieved proficiency at using arithmetic properties to manipulate expressions, students are now ready to take the next step and apply their knowledge to solving equations. Recall that an equation is an assertion that two expressions are equal. In Grade 7 an algebraic equation will assert the equality between two expressions involving an unknown represented by a letter, usually x, but often, in context, with a letter that suggests the specific unknown to be determined. For example, if one is given information to use to discover Darren's age, we might use the symbol d to represent Darren's age. If the given information is that two years ago Darren was 11 years old, we can write this as the algebraic equation $d \times 2 = 11$. The object of this section is to have students understand and use the laws of arithmetic in order to solve these equations for the unknown.

Solve multi-step real-life and mathematical problems posed with positive and negative rational numbers in any form (whole numbers, fractions, and decimals), using tools strategically. 7.EE.3

Students begin the section by modeling and solving equations with integer coefficients. Patterns in solving similar problems with models (algebra tiles) lead to solution methods that do not require diagrams. This then allows the process to extend to problems with rational numbers. Also, students will reinforce the properties learned in the previous section as they justify each step in the solution process.

EXAMPLE 10.

Solve $2n - 5 = 3$.

SOLUTION. (using algebra tiles). Model with tiles:

Add the same amount, 5, to both sides. This is legitimate since the values of n that make the above equation true are precisely the same values of n making the next statement true.
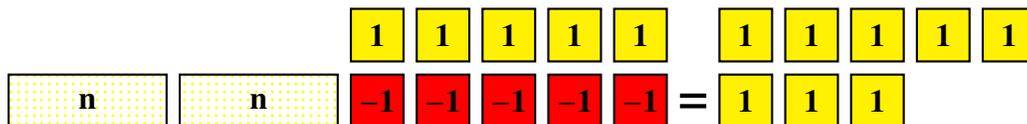
Using $1 + -1 = 0$, the additive inverse property, combine like terms. Rearrange tiles and the solution is 4.
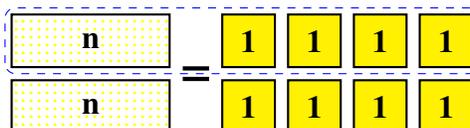
SOLUTION. (without algebra tiles).

Given $2n - 5 = 3$, add 5 to each side (addition property of equality) of the equation: $2n - 5 + 5 = 3 + 5$.

By the additive inverse property, $-5 + 5 = 0$ so $2n + 0 = 8$.

Using the additive identity property, $2n = 8$.

Dividing both sides of the equation by 2 gives $n = 4$.

When arriving at a result, it is always good practice to verify that it is the desired value. Substituting $n = 4$ into the original equation, we see that $2(4) - 5 = 3$. A little arithmetic shows that the equation is true when n = 4 so we have checked that the result is in fact the solution to the equation.

Use variables to represent quantities in a real-world or mathematical problem, and construct simple equations to solve problems by reasoning about the quantities. 7.EE.4.

After achieving procedural fluency, students then model real world situations by equations and find corresponding solutions as illustrated in the example below.

EXAMPLE 11.

Bill has twice as much money as I do. Together we have $18.45. How much money do we each have?

SOLUTION. (with a model).

Since we do not know how much money I have, let's assign it an unknown, x. This can be represented by an x algebra tile. Bill has twice as much money as I have which would be modeled by two x algebra tiles, as is illustrated below.

My money   Bill's money (twice my money)
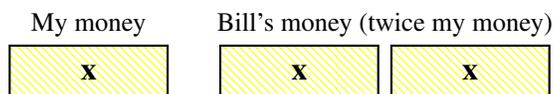
Since the problem tells us how much money we have together, we add our algebra tiles (join our resources). This yields $x + (x + x)$ or $3x$.
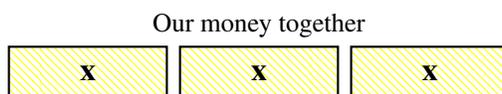
Our money together

Since our money together is represented by $3x$ and that is equal to the amount $18.45, we have the following model.

x	$1	$1	$1	$1	$1	$1	$1	$1
x =	$1	$1	$1	$1	$1	$1	$1	$1
x	$1	$1	25¢	10¢	10¢			
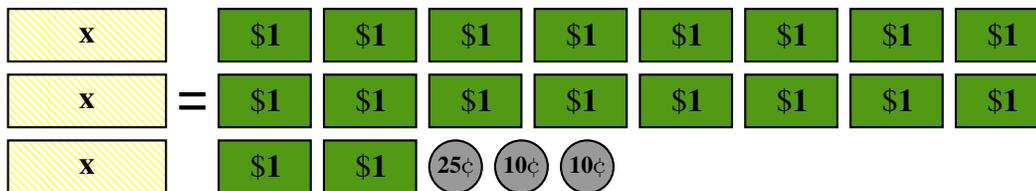
Trading the quarter for a dime and three nickels and then rearranging into equal groups next to each x-tile, we see that $1x = x = \$6.15$.

x	$1	$1	$1	$1	$1	$1	10¢	5¢
x =	$1	$1	$1	$1	$1	$1	10¢	5¢
x	$1	$1	$1	$1	$1	$1	10¢	5¢

Therefore, I have $6.15 and Bill has double that amount, $12.30. Adding these amounts together gives $18.45 so we have verified that this is the correct answer.

SOLUTION. (without a model). Let x be the amount of money I have. Then, since Bill has twice as much money, $2x$ is the amount of money he has.

Putting the amounts together leads to the equation $x + 2x = \$18.45$.

Combining like terms gives $3x = \$18.45$.

Dividing both sides by 3 yields the equation $x = \$6.15$.

Therefore, I have $6.15 and Bill has twice that amount $2 \cdot \$6.15 = \$12.30$.

Again, a quick check shows that $6.15 + $12.30 equals $18.45 so we have verified that our result is the solution to the equation.

Section 3.3: Solve Multi-Step Real-World Problems Involving Equations and Percentages

This chapter concludes with the modeling of, and finding the solution to, common real-world percent problems. We begin with a quick review of the content of Chapter 1, Section 3. Percent increase problems arise in everyday scenarios such as price plus sales tax, tip on a meal, and pay raises. Percent decrease problems occur when dealing with discounts or similar reduction scenarios. Students use a diagram to model the situation, convert the model into an expression or equation, and then simplify or solve using the skills recently developed. Models are emphasized in this section to help students understand the structure of the problems and they will rely on more algebraic reasoning later in the course. Note that if the problem involves an x percent increase of a quantity q then the result is $q + (x/100)q$ which can also be written as $q(1 + x/100)$. Similarly, for an x percent decrease problem, the expression will be $q - (x/100)q$ or $q(1 - x/100)$.

Understand that rewriting an expression in different forms in a problem context can shed light on the problem and how the quantities in it are related. For example, $a + 0.05a = 1.05a$ means that "increase by 5% " is the same as "multiply by 1.05" 7.EE.2

EXAMPLE 12.

Lucia takes some friends out to lunch and the bill comes to $b. She wants to leave a 20% tip for the waitress. How much money does she need for the food and tip together?

    a. Draw a model to represent the situation.

    b. Write at least two equivalent expressions to represent the total amount Lucia pays.

    c. If the cost of the food was $28.50, how much will Lucia need to pay?

SOLUTION.

    a. Since b is the amount of the lunch bill, draw a rectangle to represent it.

b
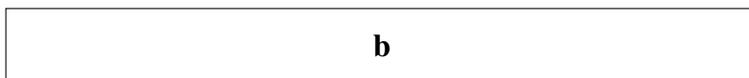
Because 20% is equal to one-fifth, divide the original rectangle into five equal parts.

$\frac{1}{5}b$	$\frac{1}{5}b$	$\frac{1}{5}b$	$\frac{1}{5}b$	$\frac{1}{5}b$
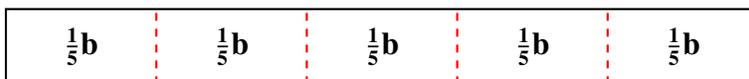
Since the tip is in addition to the bill, Lucia will have to pay an extra one of those five parts.

$\frac{1}{5}b$	$\frac{1}{5}b$	$\frac{1}{5}b$	$\frac{1}{5}b$	$\frac{1}{5}b$	$\frac{1}{5}b$

    b. This diagram leads to the expressions $b + (1/5)b$ and $b + 0.2b$. Another way to express this is that Lucia will pay for 6/5 or 1.2 times the original price, leading to the expressions $(6/5)b$ and $1.2b$.

    c. If the food cost $28.50, then the food and tip together will cost $1.2 \cdot \$28.50 = \$34.20$.

EXAMPLE 13.

Jordan wants to buy a soccer ball that costs s dollars. Today, all sports items are on sale for 25% off regular price.

    a. Draw a model to represent the situation.

    b. Write at least two equivalent expressions to represent the sale price of the ball.

    c. If the original price of the ball was $12.96, what is the sale price of the ball?

SOLUTION.

    a. Since s is the original price of the ball, draw a rectangle to represent original price.

s
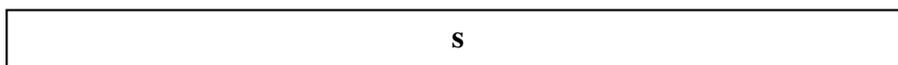
Since 25% is equal to one-fourth, divide the rectangle into four equal parts.

$\frac{1}{4}s$	$\frac{1}{4}s$	$\frac{1}{4}s$	$\frac{1}{4}s$

Since the ball is discounted 25%, Jordan will not have to pay for one of those 4 parts.

$\frac{1}{4}s$	$\frac{1}{4}s$	$\frac{1}{4}s$	$\frac{1}{4}s$

b. This diagram leads to the expression $1s - 0.25s$.

Another way to express this is that Jordan will pay for three-fourths of the original price, leading to the expression $3/4s$.

Note that $1s - 0.25s = 0.75s = 3/4s$.

c. Given that $s = \$12.96$, the sale price of the ball would be $0.75¢ \cdot \$12.96$ which has a value of $\$9.72$.

Solve word problems leading to equations of the form $px + q = r$ and $p(x + q) = r$, where p, q, and r are specific rational numbers. Solve equations of these forms fluently. Compare an algebraic solution to an arithmetic solution, identifying the sequence of operations used in each approach. For example, the perimeter of a rectangle is 54 cm. Its length is 6 cm. What is its width? 7.EE.4a

EXAMPLE 14.

Drake wants to buy a new skateboard with original price of s dollars. The skateboard is on sale for 20% off the regular price; the sale price is $109.60. Find the original price of the skateboard.

SOLUTION.

$$s - 0.2s = 109.60 \quad \text{or} \quad 0.8s = 109.60$$

from which we find that $s = 137$: the original price of the skateboard was $137.

The goal here is to understand structure of percent equations, that is, recognizing that the 20% off an original amount is the same as 80% of the original amount.

EXAMPLE 15.

Drake decided to buy the skate board for the sale price. When he got to the counter, he got an additional 10% off, but also had to pay 8.75% sales tax. What was the total amount Drake paid for the skateboard?

SOLUTION. The additional cut of 10% means he paid, 90% of $109.60, but then had 8.75% of that amount added in tax:

$$0.90(109.60) + 0.0875(0.90(109.60)) = 98.64 + 8.63 = 107.27 \, .$$

Solve multi-step real-life and mathematical problems posed with positive and negative rational numbers in any form (whole numbers, fractions, and decimals), using tools strategically. 7.EE.3

As students start to solve word problems algebraically, students also use more complex expressions, eliciting guidance in order to develop the strategy of working forwards, rather than backwards. In essence, working with numeric expressions prepares students for work with algebraic expressions.

EXAMPLE 16.

Bruno earns $43,250 a year. Of this amount, he pays 17.8% to taxes. Of the remainder, 1/3 is for living expenses, 2/5 for food and entertainment, and 1/4 for other insurance and car expenses. What percent of the $43,250 does Bruno have left over for miscellaneous expenses? How much money is left over?

SOLUTION. Bruno's taxes are $0.178 \times 43250 = 7698.5$ dollars. So, his after-tax income is $43250 - 7698.5 = 35,551.50$. Of that $1/3+2/5+1/4$ is allocated to recurring expenses. Now

$$\frac{1}{3} + \frac{2}{5} + \frac{1}{4} = \frac{20}{60} + \frac{24}{60} + \frac{15}{60} = \frac{59}{60} \ .$$

So what remains is 1/60 of his after-tax income, or $(1/60) \times 35551.50 = 592.53$. Since $592.53/43250 = 0.0137$, we can conclude that 1.37% of his total pay is available for misc. expenses, and consequently, all work and no play makes Bruno a dull boy.