Chapter 5
Geometric Figures and Scale Drawings

Here we connect concepts developed about ratio and proportion in the previous chapters to concepts in geometry. In the first section we start by exploring conditions necessary, in both angle measure and side length, to construct unique triangles with ruler and protractor. The concept of 'uniqueness' is discussed as an introduction to the idea of equivalence under a rigid motion. Students will distinguish with more precision than in previous years that two figures can be exactly the same size and shape, or can be the same shape, but different size, or can be of different shape. The focus of the second section is on figures that are the same shape but different size. Students construct scaled drawings of polygons first, and then other figures. Through explorations we discover that polygons that are the "same shape but different size" (more precisely, are scaled images of each other) have angle measures that are the same and side lengths that are proportional. Essential here is the notion of *scale*. Students will connect the concept of scale to ideas associated with ratio and proportion in order to reproduce images. We note that side lengths change by the same factor but area changes by the square of that factor. In the third section, we turn to circles and observe that all circles are scaled drawings of each other; from which it follows that for any circle, the circumference (length of the perimeter) is proportional to the length of the radius, and the area is proportional to the square of the radius. Students will discover the remarkable fact that the constants of proportionality are related. In fact, we have $A = \pi r^2$ and $C = 2\pi r$, where π is both the circumference of a circle of diameter 1 unit and the area (in square units) of a circle of radius 1 unit. The chapter ends with students examining angle relations as a means to solve problems, a theme to be further explored in the next chapter.

Students will also observe that there are many triangles with given angle measures at the vertices, and that they are all scale drawings of one another. This is a significant characteristic of similarity that is further explored in grades 8 and 10; in grade 7 we simply observe that it is true for the triangles that we construct with ruler and protractor. In section 4, we gather together, through exploration, other statements that appear to be true: for example that the sum of the angles of a triangle is a straight angle, and use that fact to solve problems involving angles.

Looking ahead, in Grade 8, using the concept of "dilation" along with the rigid motions. In grade 7 we use terms such as "same shape" and "same size." Additionally, Grade 8 students will extend their understanding of circles to surface area and volumes of 3-D figures with circular faces. Grade 9 students will formalize the triangle congruence theorems (SSS, SAS, AAS, ASA) and use them to prove facts about other polygons. Also, Grade 8 students will extend the idea of scaling to that of dilation of right triangles and then to the slopes of lines. Grade 10 students will formalize dilation with a given scale factor from a given point as a non-rigid transformation (this will be when the term "similarity" will be defined) and will solve problems with similar figures. The understanding of how the parts of triangles come together to form its shape will be deepened in Grade 8 when students learn the Pythagorean Theorem, through to Grade 11 and trigonometry (numerical geometry of the right triangle) and its generalization to all triangles through when they learn the Law of Sines and Law of Cosines.

Geometry is the study of shapes and forms with attention to defining properties and relationships among them. In the elementary grades, student have learned much about these forms and their properties, in terms of lengths, angles and area. In this chapter, and again in the geometry chapters of 8th grade, we undertake a review of this knowledge, and start to give it some logical structure that finally will be fully studied in secondary mathematics. Here we will rely on constructions and diagrams to illustrate and explore concepts. While emphasizing that all geometric knowledge comes out of understanding these constructions, we must caution that a good picture is just an example, and each picture will have features that are not characteristic of the situation prescribed by the context. Nonetheless, working with diagrams is an essential component of geometric thinking.

Geometry of the plane was well understood in antiquity. When Alexander the Great, toward the end of the 4th century BCE, founded the library at Alexandria the Greek philosophers and mathematicians moved there to set up their schools. They set as a primary goal the creation of an exposition of plane geometry in the strict logical style advocated by Aristotle. This was the "Elements" of Euclid, which remained the standard exposition up to today. At the beginning of the 20th century CE, David Hilbert wrote what was to become the definitive Euclidean geometry in this logical format. Around the same time, the mathematician Felix Klein suggested a new way of looking at geometry – as the study of properties of objects in a set that are unchanged by a particular collection of transformations of the set. For example, the *length* of a line segment remains the same, no matter where we put it on the plane. According to Klein, the basis of study of planar geometry lies not the axioms and theorems, but in the rigid motions: rotations, shifts and reflections. Two objects are considered *congruent*, of the same shape and dimension, if there is a rigid motion taking one onto the other. Similarly, the fundamental objects in spherical geometry are the rotations of the sphere, and so forth. This perception of geometry is most useful in its applications, and, in particular, provides the mathematics for online applications for geometry (Geogebra, Geometer's Sketchpad, etc.). For that reason, as well as the closer correlation to intuition than the the axiomatic approach, transformational geometry has been adopted by the Common Core, and the Utah Core Standards for the exposition of geometry starting in seventh grade and going through secondary mathematics.

Section 5.1: Constructing Triangles from Given Conditions

In this section students discover the conditions that must be met to construct a triangle, first using only straightedge and compass, and then introducing measure through ruler and protractor. It is important to keep in mind the difference between the "thing" and the "measure of the thing." A line segment has a certain measure, its length, and an angle has its measure (degrees, and much later, radians). This numerical quantification of geometric concepts is relatively new in human history, relative to the understanding of the basic facts relating lines and angles. For a carpenter, a plank is of a certain length, width and thickness, but also of a certain cost and a certain material. These measures of a plank are its characteristics, and distinct from the object. If the carpenter says that "here we will use 137 linear feet of plank," that gives us some information, but not the information about material, the strength of the material, and its cost. Making this point here helps immeasurably later.

By constructing triangles students will note that the sum of the two shorter lengths of a triangle must always be greater than the longest side of the triangle and that the sum of the angles of a triangle is always a straight angle($180°$. They then explore the conditions for creating a unique triangle: three side lengths, two sides lengths and the included angle, and two angles and a side length, whether or not the side is included. This approach of explore-hypothesize-substantiate, and *then* seek the logical structure of those conclusions is integral to the new core. It is also the way science is done. In grades 9 and 10, on the basis of this exploration in middle school, students turn to the logical structure of geometry. Throughout this chapter, students and teachers use geometric terms with which they have become familiar: point, line, line segment, circle, etc. Though in 7th grade these terms will not be rigorously defined, it is important that they are used correctly and misconceptions are not developed, thus we take time here to provide a frame for using terms.

The most fundamental objects in the geometry of the plane are points, lines and circles. It is important to distinguish between the physical drawing of a point and the mathematical conception of a point. in that geometric points are ideal and have no size while the drawings we make do. In the same way, a drawn line segment will have thickness, but the ideal concept does not.

A *line segment* is determined by two points, called its *endpoints*. The line segment between two points is drawn with a straight edge aligned against the two points, and its length is measured by a ruler.

A *circle* s defined as all points of equal distance from a *c*enter point. A circle is drawn with a compass: the needlepoint is situated at the *center* of the circle, and the pencil point traces out a curve as it is rotated around the fixed center. When we speak of "the area of a circle," we are referring to the area of the region enclosed by the circle. Any line segment from the center to a point on the circle is a *radius*; all radii have the same measure, denoted by r. The curve that bounds the circle is called its *circumference*.

Once unit lengths have been chosen, distance on the plane is measured using a ruler whose markings are based on the chosen unit. Thus, we might have a yard ruler or a meter stick or an electron microscope; in any case it is important to understand that it is the distance between two points (or the length of the line segment) that is being measured, and (as pointed out in chapter 4), any two ways of measuring distance are proportional. When it comes to curved lines, like circles, there is no easy, ruler-like way to measure their length. We will discuss this further for the circle in the third section.

A *ray* is a piece of a line that extends from one point (called the *vertex*) on and on in only one direction. We name rays by listing the initial point or vertex first, so ray AB has vertex A and extends on in one direction through the point B.

Angles

An *angle* consists of two rays which share the same vertex. The rays are called the *sides* of the angle. The angle with rays AB and AC is shown in Figure 1. We refer to this angle using the symbol $\angle$, as $\angle CAB$. Note that when we name an angle, the vertex is listed in the middle, and the other outside letters designate points on the defining rays. The symbols $\angle CAB$ and $\angle BAC$ denote the same angle; in other words, we do not distinguish the way the angle is traversed (clockwise or counterclockwise). The distinction will become important in 8th grade when we discuss orientation,

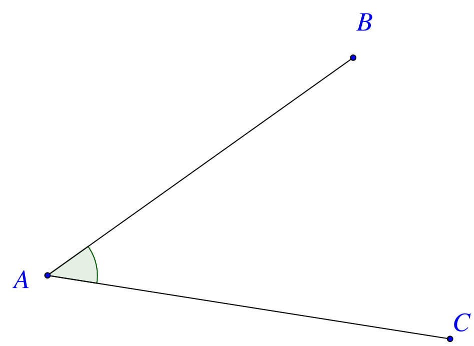


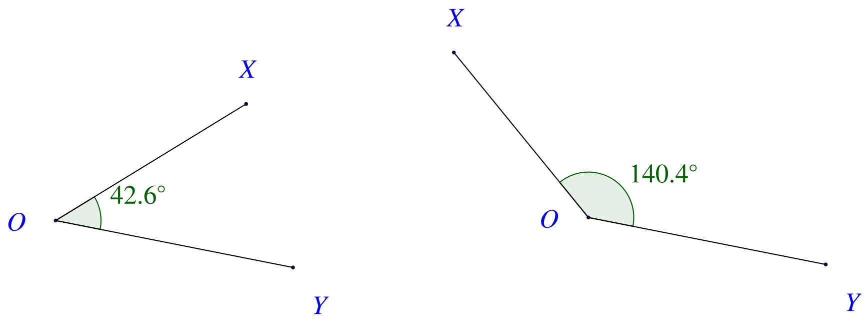
Figure 1

We measure angles in degrees using a protractor. A full circle rotation around a point is assigned the measure of 360°. The reason for this is historical and dates back to the times of the ancient Babylonians, who counted in a sexigesimal (base 60) system, for the simple reason that 60 has so many factors.

If the rays of an angle lie on the same line, but point in opposite directions, the angle is half the full rotation, and so has 180° and is called a *straight angle*. If one ray of the angle bisects the straight angle formed at the vertex by the line containing the other ray, the angle has a measure of 90° and is called a *right angle*. Note that in this case, *all* the angles at the vertex are right angles.

We classify angles in reference to these designations. An *acute angle* measures less than 90°. An *obtuse angle* measures greater than 90° and less than 180°.

It is important that students acquire facility in using tools and technology in mathematics, especially for the ability to draw geometric figures, to illustrate concepts and to solve problems. The classical tools of plane geometry are the straightedge and compass; the tools for measurement are ruler (distances) and protractor (angles). It is

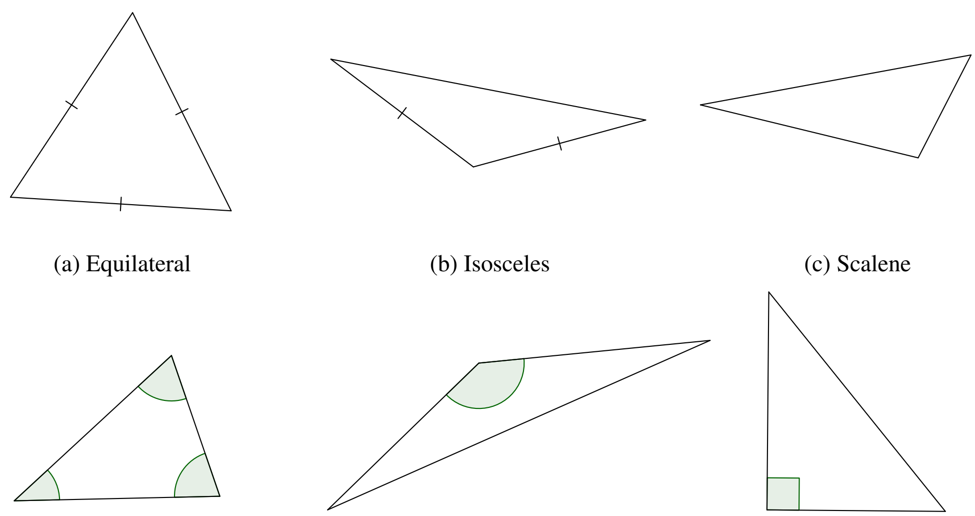


(a) (b)

Figure 2: (a) an acute angle, (b) an obtuse angle

(The example angles shown here were constructed and measured using Geogebra.)

important to learn how to use these tools, even though these tasks are greatly simplified through modern technology. For that reason it is important to become acquainted to the many online programs for drawing and analyzing geometric constructions; to name a few: Excel, Geogebra, Geometer's Sketchpad, Maple and Mathematica. Excel is noteworthy in the sense that the software is at a basic level, and so a lot of the work of creation of a good image is left to the student. Geogebra and Geometer's Sketchpad are very sophisticated instruments, allowing for dynamic manipulation of drawings; as such they can provide real insight into the concepts and procedures of geometry. Maple and Mathematica are research-level tools, incorporating all kinds of graphing capability, but also great facility in numeric and symbolic computation. In an appendix, we have provided a basic introduction to the use of hand-held tools as well as Geogebra.



(a) Equilateral (b) Isosceles (c) Scalene

(d) Acute (e) Obtuse (f) Right

Figure 3: Triangles

Triangles

A *triangle* is a region in the plane enclosed by three line segments. Figure 3 (on the preceding page) illustrates several types of triangles. When two or more sides have hash marks, those line segments with the same number of hash marks are of equal length.

Triangle (a) is equilateral, (b) is isosceles, (c) is neither (scalene), (d) is acute (all angles are acute), (e) is obtuse (one angle is obtuse), and (f) is a right triangle (the angle marked is the right angle).

Draw (freehand, with ruler and protractor, and with technology) geometric shapes with given conditions. Focus on constructing triangles from three measures of angles or sides, noticing when the conditions determine a unique triangle, more than one triangle, or no triangle.7.G.2.

Let us pause to introduce (or remember) certain vocabulary which will make it easier to talk about triangles. A *vertex* of a triangle is a point where two sides meet. A triangle has three vertices and three sides. Typically, the vertices of a triangle are labeled with capital letters, such as A, B, C, and the *opposite side* by the corresponding lower case letter (a, b, c). Given two vertices, the *included side* is the side joining the two vertices, which is also the side opposite the third vertex. Given a side, the *adjacent vertices* are the vertices at the ends of the side. Finally, we use the symbol Δ to designate a triangle; so ΔABC means the triangle with vertices A, B, C.

The question that students will now explore is this: given three positive numbers, a, b, c, is there a triangle with sides of these lengths? First lets look at the case where the lengths are the same.

Example 1.

Given a length a, how many triangles are there with all sides of length a?

An important question here is: what do we mean by "how many?" For example, triangle (a) above is a triangle all of whose sides are of the same length. If we move triangle (a) horizontally, do we get a different triangle? If we move triangle (a) vertically, or in any direction, should we call that a different triangle? We'd rather not: we want to say that these are the same triangles, only in different positions. Similarly, if we rotate the triangle around some point, once again we get the same triangle, but in a different position. So, let's rephrase our question:

Let a be a positive number. On a piece of graph paper, let A be the point on the horizontal axis of distance a from the origin O. How many triangles are there with one side OA, and all sides of the same length?

Solution. Draw, with a compass, or with appropriate technology, the circles of radius a centered at O and A. These circles will intersect at two points; one above the horizontal axis, and one below. Call these points B^+ and B^-. These are the only possibilities for the third vertex of the triangle (see Figure 4).

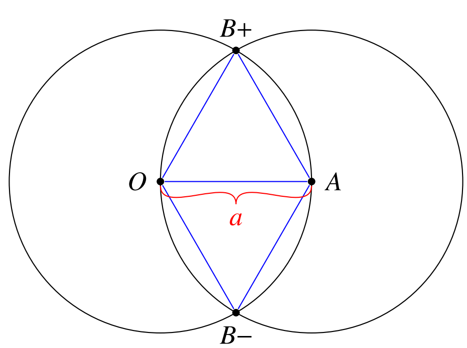


Figure 4

Are these triangles "different?" Not really, because one is the reflection of the other in the horizontal axis . So, we can conclude:

> Given a length $a > 0$, we can construct a triangle of side length a with one side on the horizontal axis with the origin as one endpoint, so that every triangle of all side lengths equal to a can be moved by rotations and slides to this one, or its reflection in the horizontal axis.

We now turn to consider general triangles, exploring what conditions suffice to construct a triangle, and in what sense it is unique (the only solution possible, ignoring its position on the plane). First, we ask if there are conditions on a set of three positive numbers for them to be the lengths of the sides of a triangle. Try the lengths 3, 6, and 10 units, and then lengths 3, 9, 10 units. We see inFigure 5, that we cannot find a triangle with sides given by the first set of numbers, but we can for the second.

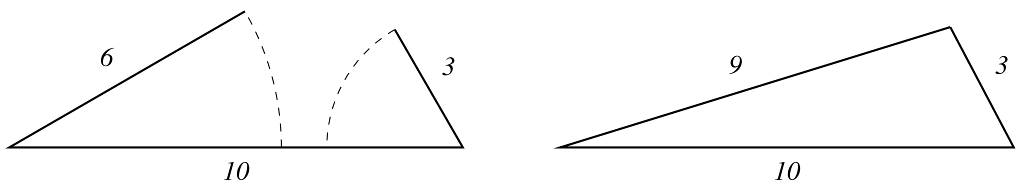


Figure 5

Example 2.

What are the possible values for the third side of a triangle if the other two sides are 2 and 12?

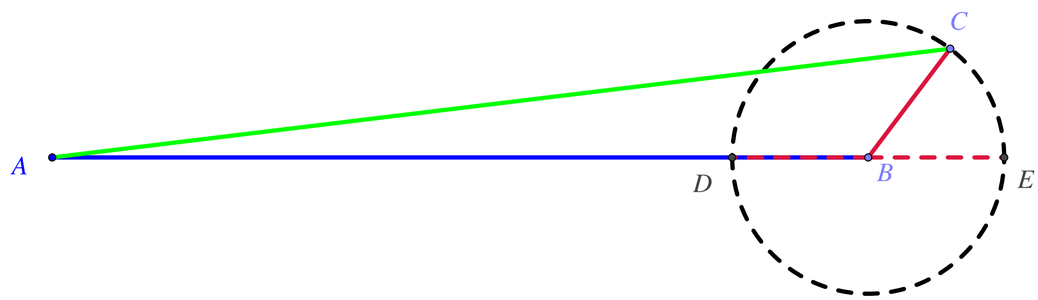


Figure 6: Building a triangle with side lengths 12 and 2.

In Figure 6, AB is the side of length 12, and BC the side of length 2. Imagine swinging the segment BC around point B, then point C will always be somewhere on the circle shown. No matter what angle we choose between the two segments, the third side of the triangle must connect point C to point A. For all points C on the circle (except D and E) there is a triangle with side lengts 2 and 12. The triangle (except for the possibility of flipping in the line $ADBE$) is unique. Now, the shortest line segment between A and a point on this circle is AD of length 10 units, and the longest such line segment is AE of length 14 units. Since the third vertex of our triangle cannot be either D or E (for in those cases all sides of the triangle lie on the same line), we can conclude that the third length must be strictly between $10 = 12 - 2$ and $14 = 12 + 2$.

There was nothing special about the numbers 12 and 2 in this argument, we can replace them with any two positive numbers a and b with $a \geq b$, and assert if c is the length of the third side of a triangle with sides of length a and b, we must have $c > a - b$ and $c < a + b$. The general statement is: the longest side length of a triangle is less than the sum of the lengths of the other two sides. The best way to state this is:

> **Triangle Inequality**. For any triangle, the sum of the lengths of two sides is greater than the length of the third.

Extension. Explain why the triangle inequality is true. Show that this extends to arbitrary polygonal paths: the total length of a path (the sume of the lengths of the line segments that form it) is no less than the length of the straight line between its endpoints. Consequently the length of any polygon side is always less than the sum of the other polygon side lengths. **End Extension**.

Now, through exploration, students will make this important observation:

> If a, b, c are three positive numbers satisfying the triangle inequality, then there is a unique triangle (up to motions in the plane) with those numbers as side lengths.

To see this, pick three numbers a, b, c that satisfy the triangle inequality. On a coordinate plane, label the origin as A and label a point B on the positive horizontal axis so that the line segment AB has length a (Consult the above figure, but with 12 replaced by a and 2 replaced by b) . Now, draw a circle with center at B and of radius b. Because of the triangle inequality, c is between $a - b$ and $a + b$, so there is a point C on the circle above the horizontal axis that is of distance c from A. These three points are the vertices of a triangle of side lengths a, b, c. Now, suppose that we have another triangle with these side lengths. We can move (by a slide and rotation) that triangle so that the side of length a coincides with the segment AB. Then the side of length b has an endpoint at A or B. If it is at A, reflect the triangle in the horizontal line through the midpoint of AB. Now, the side of length b has B as an endpoint, and the side of length c has one endpoint at A and the other on the circle of radius b centered at B. But there is only one point on that circle whose distance from A is c, so the moved triangle coincides with the triangle we constructed.

Oops, not exactly - there is a point C' on the circle lying below the axis of distance c from A, that forms the triangle $\Delta ABC'$. But, this is the reflection of ΔABC in the horizontal axis, so is still the same triangle as constructed.

Here is another set of conditions for which there is a unique triangle satisfying the conditions:

> Given an angle $\angle ABC$, and positive numbers a and c, then there is a unique triangle ΔABC with the given angle , and the sides adjacent to that angle of lengths a and c.

Measure off a distance a on the ray BA from the point B, and measure off a distance c on the ray BC from the point B. Draw the line segment joining the endpoints of those segments to get the desired triangle (see Figure 7). This is the unique triangle satisfying the given conditions, because if we have another such triangle we can move the angle to the angle ABC, and the side of length a is either on the ray BA or the ray BC. If it is on the first ray the two triangles coincide. But what if the side of length a is on BC, do we get a “different” triangle?

If we are given two side lengths and an an angle, in order that they describe a unique triangle it is important that the lengths be of the adjacent sides, as we see in Figure 8: there is one acute triangle and one obtuse triangle with given angle and side lengths.

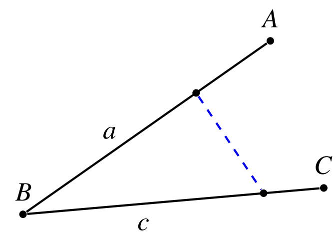


Figure 7

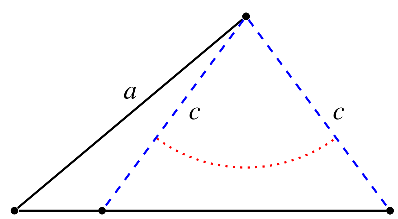


Figure 8

EXAMPLE 3.

Given two angles, and a positive number a, if there is a triangle with a side of length a whose adjacent angles are the given angles, then it is unique.

SOLUTION. Draw a line segment AB of length a, and copy the angles (as shown in Figure 9) at the endpoints of AB.

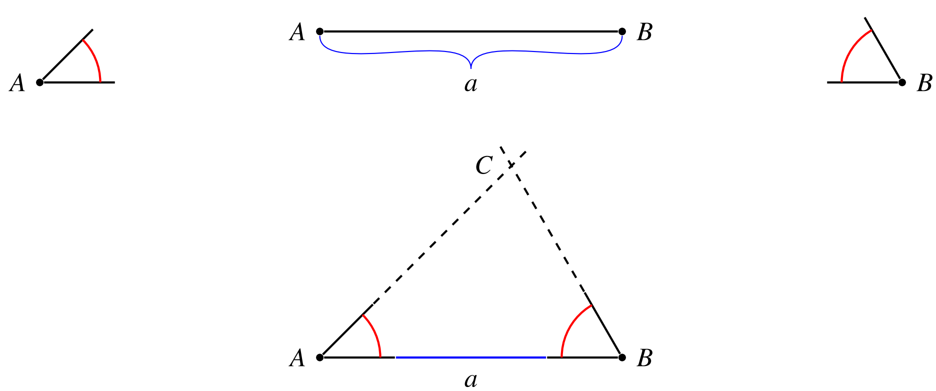


Figure 9

If the rays of the angles, other than the ray containing AB intersect, call the intersection point C, then ABC is a triangle.

This explanation is too easy, with some gaps in the logic. This might be a good time to explore the gaps in a preliminary way. First of all, Figure 9 is not the only possibility: if the two given angles are obtuse, then there will be a point of intersection on the side of AB opposite the one depicted. In this case, the given angles are *exterior* angles of the triangle. A student may observe that if the given angles are right angles, there will be no point of intersection C. This generalizes to: if the rays in question are parallel, there is no point of intersection and thus no triangle. It may come up that this statement is the same as saying that the sum of the whose measures of the given angles is 180°. The final realization that might happen is that the condition for there to be a point of intersection is that the sum of those measures is less than 180°, In any case, these two examples should help the students understand that if the measure of two angles of a triangle is known, then so is the third.

Students will observe that the the condition on the measures of the angles leads to an intersection point of the two outside rays of the angles. Denoting that point as C, the triangle ΔABC is the desired triangle.

At this point, the student may wonder: why did this turn out this way? That is a question that will be answered in secondary mathematics; for the time being, what is important is that students explore these ideas through constructions of their own. A partial answer to that question will come later in grade 7: that the sum of the measures of the angles of a triangle is 180°, so the sum of the measures of two angles has to be less than 180°. In fact, students will see that it follows from this fact, that if we know the length of one side and the measures of *any* two angles, then the triangle with these dimensions is unique.

The following animations at the website *Mathsonline* show how to construct triangles when certain measures are given. Before looking at the links, try the following using a ruler and compass. You may use a ruler, a protractor, and a compass.

- Construct a triangle with sides of length 10 cm, 12 cm, and 18 cm.

  Side-side-side animation here:
  `http://www.mathsonline.org/pages/animationPage.html?triangle3sides`
- Construct a triangle with 2 sides of length 15 cm and 9 cm and an included angle of 35°.

  Side-angle-side animation here:
  `http://www.mathsonline.org/pages/animationPage.html?triangle2sides`
- Construct a triangle with a 25° angle, a 10 cm included side, and a 100° angle.

  Angle-side-angle animation here:
  `http://www.mathsonline.org/pages/animationPage.html?triangle1side`

Here are some more facts about constructing figures that students might come across in their explorations:

- It is not true that there is a unique triangle with given angles, as Figure 10 shows. The question: "what do such triangles have in common?" will be taken up in the next section, and discussed in detail in 8th grade.

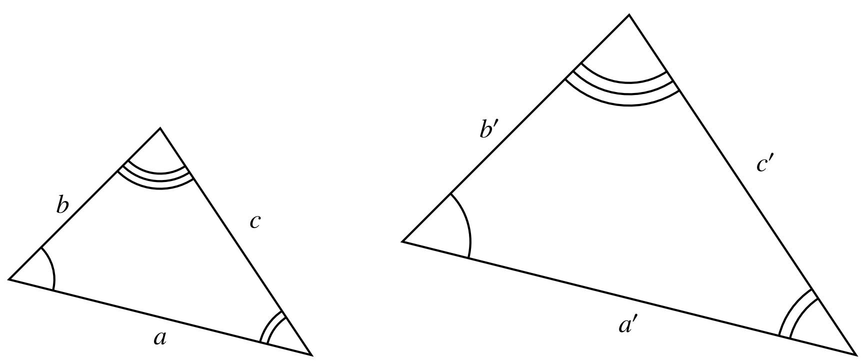


Figure 10

- There is a unique circle with given center and radius. Given two circles of the same radius, we can slide the center of one to coincide with the center of the other. Then the circles coincide as well.
- Given two positive numbers a, b we can construct a rectangle with side lengths a and b: just go from the origin along the horizontal axis a distance a, and along the vertical axis a distance b. These are two sides of the desired rectangle. Furthermore, given any other rectangle of side lengths a and b, we can move it by a slide and a rotation so that it coincides with the constructed rectangle.

Section 5.2: Scale Drawings (Objects that have the same shape)

The central idea of this section is scale and its relationship to ratio and proportion. Students will use ideas about ratio, proportion, and scale to a) change the size of an image and b) determine if two images are scaled versions of each other.

By the end of this section, student should understand that we can change the scale of an object to suit our needs. For example, we can make a map where 1 inch equals one mile; lay out a floor plan where 2 feet equals 1.4 cm from a diagram; or draw a large version of an ant where 3 centimeters equals 1 mm. In each of these situations, the "shape characteristics" of the object remain the same, what has changed is size. Objects can be scaled up or scale down. Through explorations of scaling exercises, students will see that all lengths of the given object are changed by the same factor in the scaled representation; that the factor is called the *scale factor*.

The term "similar" is not defined in Grade 7; here students continue to develop an intuitive understanding of "the same shape," so that the concept of similarity (introduced in Grade 8) will come naturally. Throughout this section students should clearly distinguish between two objects that are of the same shape and dimension and objects that are scaled versions of each other. In particular students will come to understand that two polygonal figures that are scaled versions of each other have equal angles and corresponding sides in a ratio of a:b where $a \neq b$. Students should also distinguish between saying the ratio of object A to object B is a:b and the scale factor from A to B is b/a. This idea links ratio and proportional thinking to scaling.

Students will learn to find the scale factor from one object to the other from diagrams, values and/or proportion information. Students should be able to fluidly go from a smaller object to a larger scaled version of the object or from a larger object to a smaller scaled version giving either or both the proportional constant and/or the scale factor.

Solve problems involving scale drawings of geometric figures, such as computing actual lengths and areas from a scale drawing and reproducing a scale drawing at a different scale. 7.G.1.

Scale drawings are diagrams of real measurements with a different unit of measurement, arranged so as to have the same shape as the original they represent. The *scale* describes the relation between the unit of measurement in the drawing and that of the original. Examples of scale models include photographs, doll houses, model trains, architectural designs, souvenirs, maps, and technical drawings for science and engineering. Today with computer image manipulation even in our word processing programs, scaling figures, text, and photos is a common activity. Dynamic visualization tools like Google Earth provide ample real life experience with scale maps and figures.

What exactly is the same and what is different about these scale models and their original counterparts? Linear dimensions on scale models are proportional to the corresponding length on the original: the ratio of any length in the drawing to the corresponding actual length of the original is the *scale* of the drawing, and is the same ratio for any measurement taken on the image. Distortions of a given shape do not count as a scale model. For example, a Barbie doll or cartoon character (like Wreck-it-Ralph) is not proportional to any real human.

Figure 11 (next page) shows a scale drawing of an ant. How long is this ant?

To find the real ant's length we measure the black scale line with a ruler, in order to discover the scale of the drawing. The actual length in your image will depend upon the platform on which you are working, so for this discussion, let us say that the length of that line in the figure is 3 cm, or 30 mm. Thus, the scale for this image is 30 to 1: every linear measurement on the image is 30 times the size of the corresponding measurement of the ant. To answer the question about the actual size of the ant, we have to judge the overall length of the ant's image: this can be a little tricky since we must decide where to place the ruler over the top of the image. Should one start measuring at the antennae and stop at the end of the tail? If one holds the ruler diagonally over the top from head to rear foot, a different measurement results. Nonetheless, it appears that the length of the ant image is about 10 cm, or 100 mm.

Since the scale of image is 30 to 1, every length on the image is 30 times the actual length. Or, the actual length

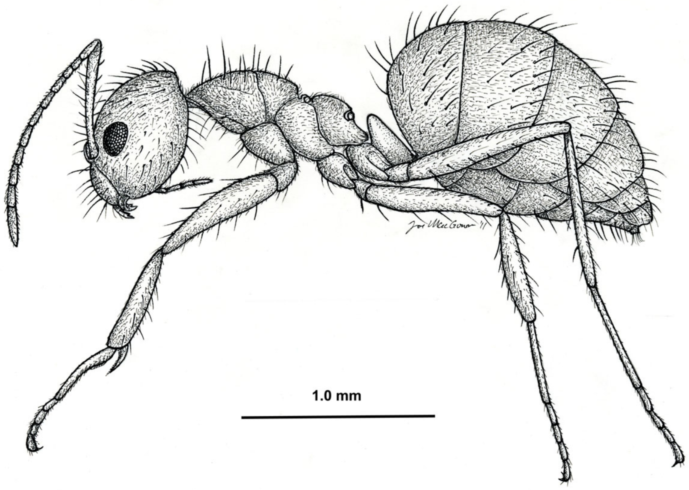


Figure 11

Nylanderia fulva, side view of a worker (drawing by Joe MacGown). Image comes from Mississippi Entomological Museum.

is one-thirtieth the length on the image, so that the actual length of the ant is 100/30 = 3 1/3 mm. To clarify the difference between "ratio" and "scale factor,:" the ratio is 30:1, however the scale factor is 1/30th. In other words, one multiplies each length of the original by 1/30 to get the original size of the ant. For example, if the ratio was original: half of 1:$\frac{1}{2}$, then the scale factor would be $\frac{1}{2}$.

Alternatively, we could set up a proportion. The ratio of 1 mm to 3 cm corresponds to the ratio of the ant's actual length in mm to the measurement on the drawing in cm. Hence,

$$\frac{1 \quad \text{mm}}{3 \quad \text{cm}} = \frac{x \quad \text{mm}}{10 \quad \text{cm}}.$$

After solving, we find that $x = \frac{10}{3}$, so the ant is $3\frac{1}{3}$ mm in length. That seems about right for one teeny little ant.

One might also wonder how the artist went about creating this picture. The artist probably looked at an ant specimen under a microscope and drew what he saw as accurately as possible. In real life, biologists often use tiny grids to help them determine the scale of small things they observe under a microscope.

Some important considerations when working with scale drawings:

- The scale of a drawing can be expressed with and without units. For the ant drawing above, the scale (from image to actual) is 3 cm to 1 mm, or 3:0.1 (or 30 to 1: the ant has been magnified 30 times). Maps are scale drawings of actual geography, and the scale may be indicated by a statement: "1 inch = 20 miles," or it might be expressed by labeling an actual line in the drawing with the actual length it represents (as in Figures 15 and 16 below). The scale can also be represented as a unit less ratio, such as 1:24,000. If this ratio appears on the map it is telling us that any length on the map represents an actual length that is 24,000 times as long, independent of whether the measurement is made in feet and miles, or in meters and kilometers. Notice (at the bottom left of Figures 15 and 16) that the scale is written with the dimension of the drawing first, and the dimension of the actual last.
- In this sense, scale factors are unitless constants that indicate the relationship between lengths in a scale drawing or model and its real life counterpart. So, a scale factor can be expressed as fraction or even a percentage. If a model is 1% of the real thing (or perhaps a model is 250% of the original), then the

percentage expresses the ratio of the length measures in the model to the length measures of the actual object. In this case, any unit may be used. In a model that is 1% of the original, one yard in the original will be 1/100th of a yard in the model. One centimeter in the model will represent 100 centimeters of the original.

- If one uses a scale factor bigger than one, the replica is larger than the original, while if the scale factor is less than one, the model is smaller than the original.
- It is important to keep track of units, if the units are made explicit. A 1:2 ratio or scale factor is different from a ratio of 1 inch: 2 yards.

  A specification of 1:2 ratio without explicit units tells us that the units are the same, so, for example, distances on the scale drawing are half the original distances, so 1 cm corresponds to 2 cm in real life, 1 inch corresponds to 2 inches, and 1 foot corresponds to 2 feet. The 1 inch: 2 yards ratio would be equivalent to a scale of 1:72 since there are 36 inches in a yard. A scale drawing where 1 cm corresponds to 1 km does not have a 1:1 scale factor, but rather 1:100,000 since a centimeter is 1/100000 of a kilometer.
- Angles in a scale drawing are the same as the corresponding angle measures in the original.

Some techniques for making scale drawings:

- Overlay a grid, then copy the figure from corresponding grid squares onto a grid of a different size.
- Use proportions to make corresponding side lengths to outline the figure, using the same angle measures from one segment to the next.
- Use computer programs to make scale drawings. (such as Geometer's Sketchpad or Geogebra). If you have a tablet or a smartphone, draw a figure and then expand or contract it in such a way as to have one image be a scale drawing of the other.

When comparing a scale model to the real thing, dimension is also important. As we have seen measures of length scale by the scale factor, but is this the same for area, or, in 3D modeling, volume? Consider this situation:

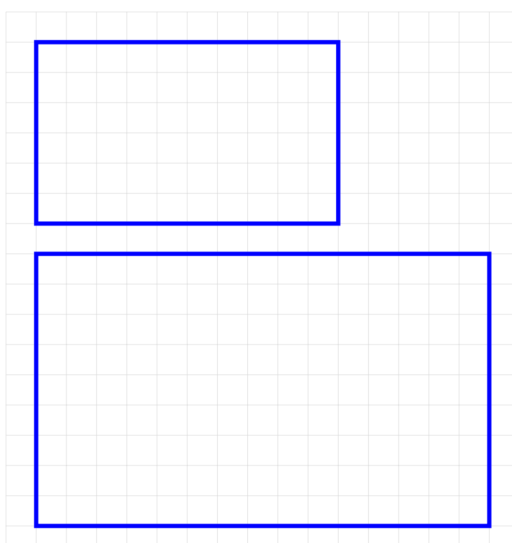


Figure 12

Although the ratio of the top image to the bottom (in Figure 13) is 2:3, the top rectangle contains 60 squares, while the right rectangle contains 135 squares; so in this case the area ratio is 4:9.

Example 4.

Draw a 2x6 rectangle on a piece of graph paper, and then another rectangle in the ratio 1:3. What is the ratio of the areas of these rectangles?

Solution. : The count gives the ratio 12:108, which simplifies to 1:9.

In Figure 13, the scale ratio across all squares is 1:2:3:4:5.. Note that the areas are in the ratio 1:4:9:16:25. These examples support the statement that the scale of area in a scale drawing is the square of the scale of length.

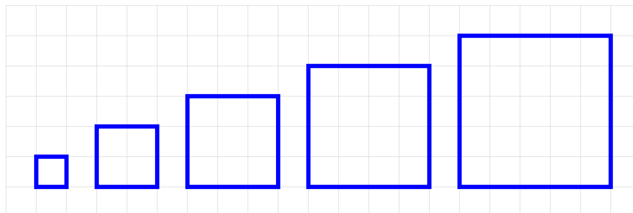
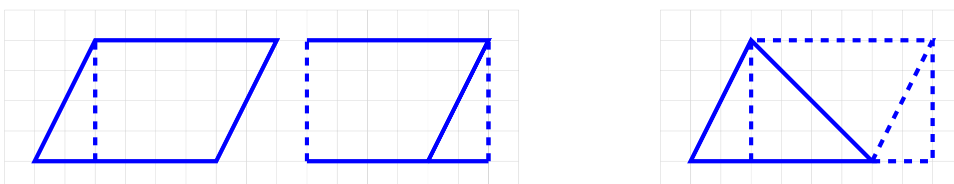


Figure 13

This is true not only for rectangles, but for all figures. For example, since we can move a triangle off a parallelogram to make a rectangle of the same area, it is true of parallelograms, and since a triangle s half a parallelogram it is also true for triangles. In the next section we will show this for the area inside a circle as well.



Area of Parallelogram = 30 Unit Squares

Area of triangle = 12.5 Unit Squares

Figure 14

A fact that comes out of the above discussion of rectangles is this: for two rectangles, if one is a scale drawing of the other, then the ratio of corresponding sides of the rectangles is the same. It is not hard to see that this is also true that two rectangles with the same ratio of sides are scale drawings of each other.

Example 5.

Dave was planning a camping trip to Salina, Utah. He used an online map to find the approximate area of the Butch Cassidy Campground, and to find the distance from the campground to his grandmother's house. His grandmother lives on the northeast corner of 100 East and 200 North in Salina. (See Figure 15). In fact, Dave's grandmother owns that entire block (between 100 and 200 East and 200 and 300 North). What is the area of her property?

Solution. The campground, he reasoned using his index finger tip to measure, looks like it is about 1000 feet by 500 feet, so that's 500,000 square feet in area. He wondered how many acres that would be and quickly looked up the information that 1 acre = 43,560 square feet. Okay, he thought, and punched in 500,000 divided by 43,560 into his calculator, so the campground is about 11.5 acres. Now, to get to Grandma's house, he thought and continued using his finger to measure. It looks like it is over 5000 feet to get to Main Street from the campground entrance, and then probably just over another 1000 feet after that. Recalling that about 5000 feet make a mile, he decides it will be over a mile, but less than a mile and a half, to grandma's house. As for the area of grandmother's property: her block seems to be about 800 feet on a side, so the area of that block s 64,000 square feet, or about 1.5 acres.

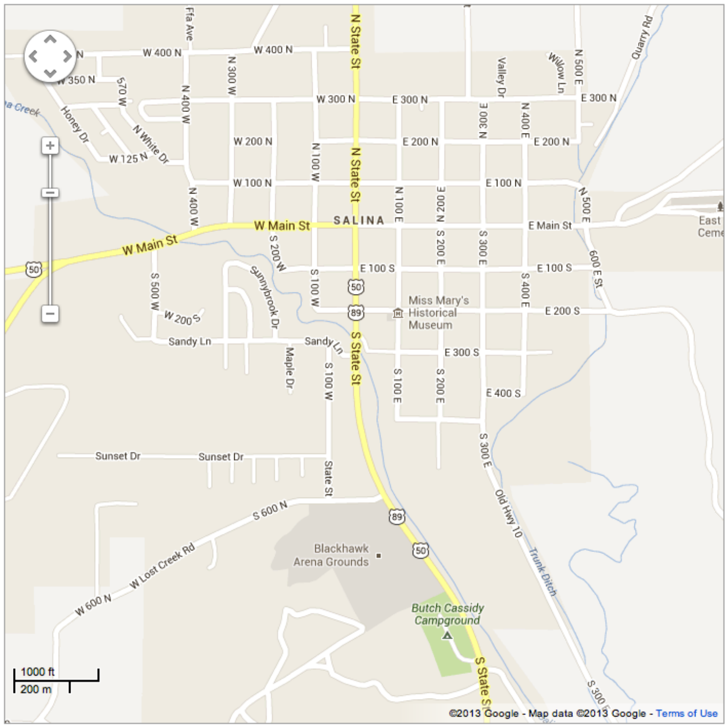


Figure 15: Google maps - Salina, Utah.

Dave then used a different scale map to approximate the distance he'd have to bike to get from the Butch Cassidy Campground to Palisade State Park. This time he decided to print the map and use his ruler. (Figure 16)

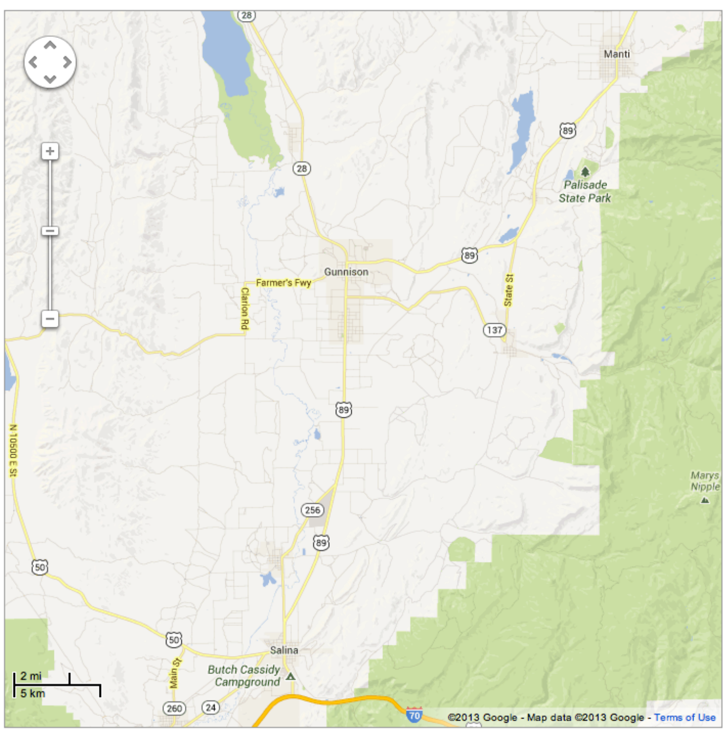


Figure 16: Google maps - Salina, Utah.

Section 5.3. Solving Problems with Circles

In this section circumference and area of a circle will be explored from the perspective of scaling, that is, we start by measuring the diameter and circumference of various circles and noting that the ratio of the circumference to the radius is constant (2π). This leads to discussions about all circles being scaled versions of each other and eventually "developing" an algorithm for finding the area of a circle using strategies used throughout mathematical history. In these explorations, two big ideas are discovered: 1) cutting up a figure and rearranging the pieces preserves area, and 2) creating a rectangle is a convenient way to find area. Additionally, we will connect the formula for finding the area of a circle (πr^2) to finding the area of a triangle where the base is the circumference of the circle and the height is the radius ($A = Cr/2$).

The section will close by applying what was learned to problem situations. Chapter 6 will use ideas of how circumference and area are connected to write equations to solve problems but in this section students should solve problems using informal strategies to solidify their understanding.

Know the formulas for the area and circumference of a circle and use them to solve problems; give an informal derivation of the relationship between the circumference and area of a circle. 7.G.4 .

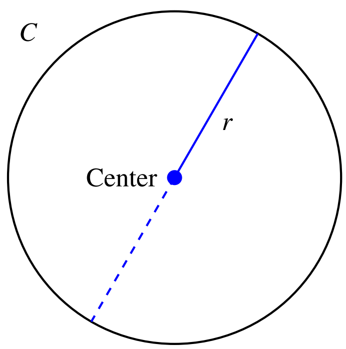


Figure 17

Recall that a *circle* is the set of all points equidistant from a center. We draw circles with a compass by fixing the point of the compass at the center, using a fixed angle at the compass hinge, and rotating the pencil point around to mark the circle. Draw a line segment from the center to a point on the circle; this is a *radius* of the circle. The plural of "radius" is "radii." By definition, all radii are of the same length. If we take any radius, and extend that line segment from the center in the opposite direction of the given radius, we get a *diameter* of the circle (Figure 17).

Observe that any two circles are scale drawings of each other, where the scale factor is the quotient of the lengths of the radii of the two circles. Therefore, if we know the length of the circumference of a circle of radius 1 unit, then the length of the circumference of a circle of radius r units is r times that length. Similarly, for area, except that the scale factor of is r^2. We can see that as follows: Draw two circles on a grid, one with twice the radius of the other (Figure 18).

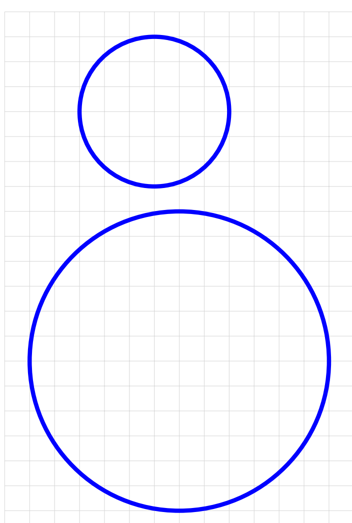


Figure 18

Now count the number of squares of the grid that are more than half inside the circle (use symmetry to make the counting easier). In our case those numbers are 32 for the smaller circle, and 120 for the larger one. This is about 4 times as large. This should be the same for all circles drawn by students: the number for the larger circle is about four times as large as the number for the smaller circle. If we took a different ratio, or a finer grid, the answer will be the same; if the radius is multiplied by some number a, then the area is multiplied by a^2. It follows that, if we know the area of a circle of radius 1 unit, then the area of a circle of radius r units is r^2 times that area. So, the ratio of the area of a circle and the square of its radius is a constant, and that constant is the area of circle of radius 1 unit, and is designated by.the greek letter *pi*, written π. Thus we get this formula for the area A of a circle in terms of its radius r: $A = \pi r^2$.

As a side note, methods for computing the area of simple polygons was known to ancient civilizations like the Egyptians, Babylonians and Hindus from very early times in Mathematics. But computing the area of circular regions posed a challenge. Archimedes (287 BC - 212 BC) wrote about using a method of approximating the area of a circle with polygons, as worked through 5.3b Classwork Activity.

The formula $A = \pi r^2$ doesn't help us much until we know the numerical value of π, or at least a good approximation of that numerical value. The exercise we have just done gives us an estimate for π by the count made in the Figure 19: for a circle of radius 6, we counted 120 squares more than half inside circle. So, $120 = \pi 6^2$, from which we get the estimate $120/36 = 3.33$ as an approximate value for π. Now, if we took an even larger circle on the same grid, we'd get a better approximation, and presumably we can get as good an approximation as we want in this way. It would be interesting for each student in the class to make such a calculation, and then take the average as a statistical experimental estimate for π.

Let's turn to circumference of a circle. By the same reasoning (that is, estimating the perimeter of the figure consisting of all the squares counted for area), we can give good evidence that the ratio of the length of its circumference to the length of the radius is constant. That this constant is related to π is amazing; we will now demonstrate this using an ancient Egyptian argument.

First, draw a circle of radius r, and let us denote its area by A and the length of the circumference by C. Fill the circle with a coiled rope, starting at the center and circling around until the circumference is reached (Figure 19).

Figure 19

Place a straight edge along a radius and cut the rope all the way through along that straight edge (see figure 20).

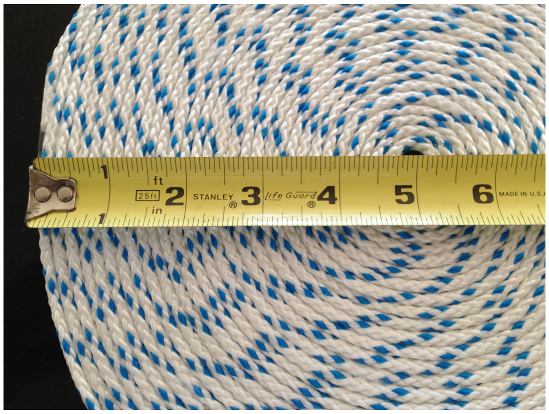


Figure 20

Flatten out the pieces of the rope so that each piece is a horizontal line with the outside piece at the bottom, and the center at the top, to get the isosceles triangle in Figure 21

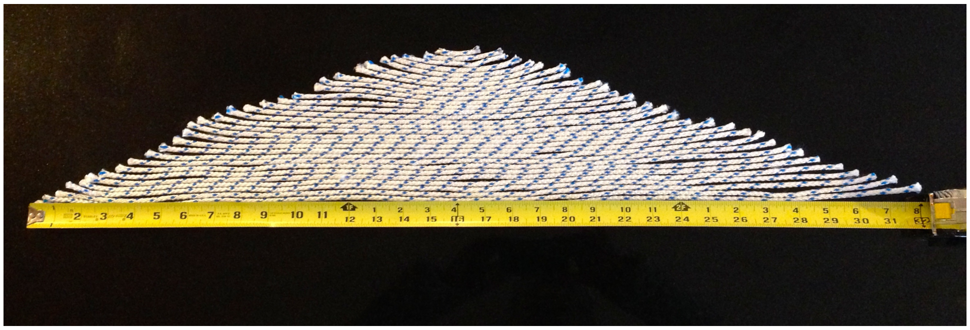


Figure 21

We know that the area of a triangle is one half the product of the base times the height. For this triangle, the base is the circumference of the circle, and the height the radius. Thus the triangle has area $\frac{1}{2}Cr$. But since the area is that filled in by the rope, whether it be coiled in the circle or flattened in the triangle, we conclude that this is the area of the circle:

$$A = \frac{1}{2}Cr\ .$$

If we replace A by πr^2, we can solve $\pi r^2 = \frac{1}{2}Cr$ for r to obtain the formula for circumference in terms of radius:

$$C = 2\pi r\ .$$

This "real" construction gives us a way of estimating π. For the particular circle in Figures 20-22, the radius is 5 inches, and the circumference (the base of the triangle in the above figure) is 32 inches. From $C = 2\pi r$ we can write $32 = 2\pi(5)$, so we get the estimate 3.2 for the value of π.

Because the rope has substantial thickness, this is a rough estimate, and probably larger than the true value of π. To do better, select a circular disc of some thickness. Measure the length of a radius, call it r. Now take a thin string and wrap it around the circumference of the disc and mark the length that just makes one full loop around the disc. Measure the length of this string, call it C. Then the ratio $C/2r$ is an approximation to the value of π.

The area formula provides another way to evaluate π. Inscribe a regular polygon with n sides, as in Figure 22 (where $n = 10$):

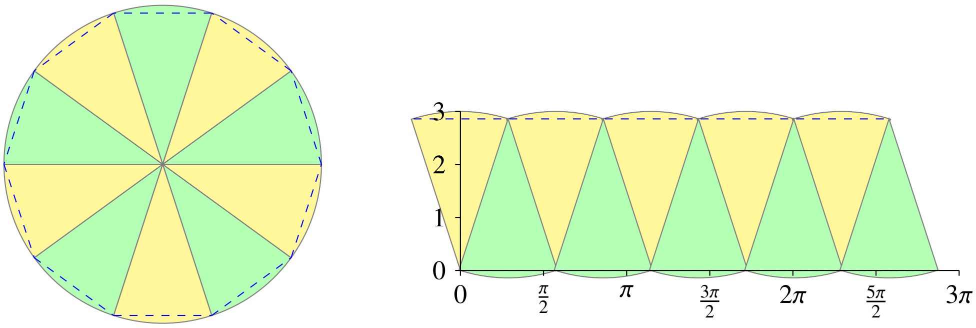


Figure 22

The area of one triangle can be estimated by measuring the base and altitude. Then the area of the polygon is n times that number, and this provides an estimate for the area of the circle.

This second method was used in antiquity by Archimedes, with $n = 16$ to get an approximation for π correct to three figures. Figure 22 also gives us another way of determining the relation $A = \frac{1}{2}Cr$. For each triangle, the area of the triangle is $\Delta) = \frac{1}{2}br$, where Δ is the area of the triangle, and b is its base, and the height is the radius r. Now adding this over all the triangles, amounts to multiplying both sides of the equation by n, the number of triangles, giving us $A = \frac{1}{2}Cr$ for the circle.

Section 5.4: Angle Relationships

The closing section involves applications with angle relationships for vertical angles, complementary angles and supplementary angles. In addition, students will use concepts involving angles to relate scaling of triangles and circles. Practice with the skills learned in this section will be further developed in Chapter 6 when students write equations involving angles.

Use facts about supplementary, complementary, vertical, and adjacent angles to write and solve multi-step problems for an unknown angle in a figure. 7.G.5

First, let us recall some concepts having to do with combining angles.

The sum of the angles $\angle ABC$ and $\angle DEF$ is defined as follows (see Figure 23) : Move (by sliding and rotating) $\angle DEF$ so that the vertices B and E coincide, so that rays BA and ED coincide and so that ray EF is not on the same side of BC as BA. Then the sum of the given angles is the angle $\angle ABF$.

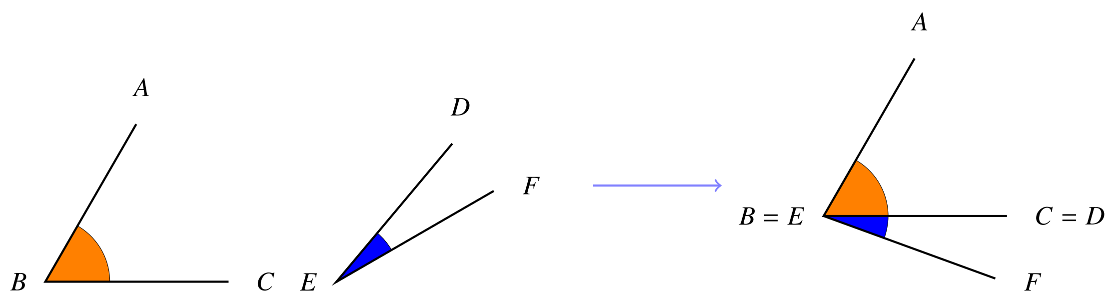


Figure 23

When two lines intersect at a point, four angles are formed. Figure 24 shows line AB intersecting line CD at T. Angles $\angle ATC$ and $\angle CTB$ are *adjacent angles* on a straight line, hence the sum of the measure of these two angles is 180°. Such angles are called *supplementary*. Angles $\angle ATC$ and $\angle BTD$ are called *vertical angles*, in the sense that they are opposing angles at a vertex. Angles $\angle CTB$ and $\angle DTA$ are vertical angles as well. Vertical angles are equal, since they are both supplementary to the same angle. That is, as in Figure 24, $\angle ATC$ and $\angle BTD$ are supplementary to $\angle DTA$, so are equal.

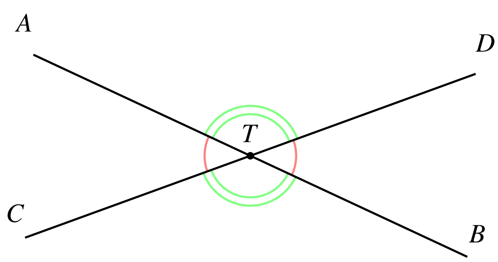


Figure 24

If all the angles in Figure 24 are equal, then they are all of measure 90°; that is, they are all *right* angles. In this case, the lines AB and CD are said to be *perpendicular*.

Two lines are said to be *parallel* if they have no point of intersection. To be clear about this: not just “no point of intersection on our piece of paper,” nor in our line of vision, but “no point of intersection” *anywhere*. This definition supposes that we can imagine the whole plane, infinite in extent in all directions, and can see that the two lines called parallel indeed never intersect. This, then, is a theoretical, rather than an operational definition, and for that reason bothered the mathematicians in Alexander’s day (and for the next 2000+ years). The Greeks did the best they could with this problem, and postulated the intersection point in the *Elements of Geometry* as follows.

Given two lines L and L', draw a third line L” that intersects both given lines (see Figure 25).

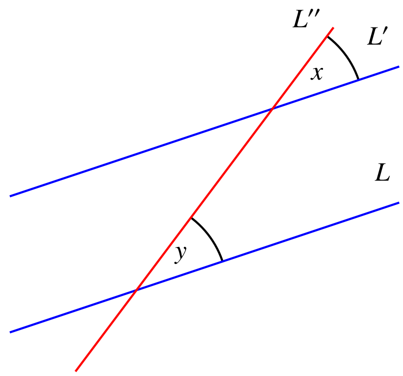


Figure 25

Focus on the corresponding angles x and y at those intersection points; corresponding in the sense that they are in the same position relative to the two points of intersection. The two lines L and L' are called *parallel* if the angles x and y are of the same measure. In our figure it appears that x and y are of the same measure, so the lines L and L' would be parallel by this definition. It also looks like lines L and L' also will never intersect, but since we cannot see the full extent of the whole plane, we cannot know. Confronted with this dilemma, Euclid could show that these two definitions of “parallel” were the same by turning the statement around: If lines L and L' intersect at some point V, then the angles x and y cannot be equal. This, however, required knowledge of an important fact about the angles of a triangle, to which we now turn.

Example 6.

The sum of the angles of a triangle is 180°. To see this, draw a triangle, cut out the angles and put all the vertices together, so that the angles are adjacent and not overlapping. You will get a straight angle (see Figure 26). If everyone in the class does this, they will all get the same result: a straight angle. This provides convincing experimental evidence that the statement is true, but it is important to keep in mind that, while experimental evidence might confirm a hypothesis, it does not explain why it is true. In mathematics, such explanations come out of *proof*. In grade 8 students will explore explanations of this fact.

Figure 26

Now, we can complete Euclid’s argument. Return to Figure 25 and suppose the lines L and L' are *not parallel*; that is, they intersect at some point V. Figure 27 portrays this; although the lines may have extended for millions of miles before arriving at that intersection, this image represent the situation. Since the angle at V has to have positive measure, the angles at A and B (of the triangle ΔAVB) have measures that add up to less than 180°, so those angles cannot be supplementary. Since the angle $\angle ABV$ and x are supplementary, x and y do not have the same measure.

It follows that if the measures of the angles x and y are equal, then the lines L and L' can never intersect. For if they did intersect we'd have a picture just like that of the lines L and L', and we just saw that if the lines intersect, the interior angles cannot be equal.

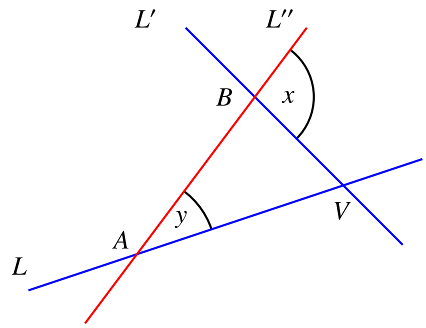


Figure 27

Example 7.

In Figure 30, the measures of two angles are given. Find the measures of the remaining angles.

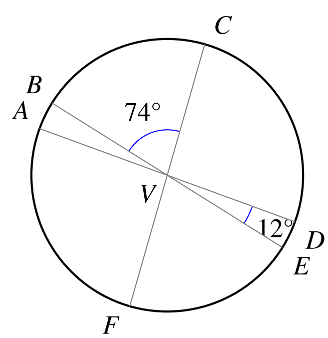


Figure 28

Solution. The sum of the two given angles and $\angle CVD$ is a straight angle, so has 180°. This tells us that $74 + \angle CVD + 12 = 180$, giving us the measure of $\angle CVD$, 94°. Each of the other angles is the vertical angle associated to one of these, so that gives us all the measures. Going clockwise around V, the measures are 74°, 94°, 12°, 74°, 94°, 12°.

Example 8.

Consider the rectngle in Figure 31. There are 9 labeled angles. The figure provides the measure of three of these angles. Find the measure of the remaining angles.

Solution. We see a complex geometric diagram that shows many angle relationships, so we should proceed carefully using the basic angle relationships: vertical angles are equal, supplementary angles add to 180°, the sum of the angles of a triangle is 180°, and transversals intersect parallel lines at equal angles.

- $\angle g = 67°$. Since we are given the measure of two angles in ΔCEG, we use the fact that the sum

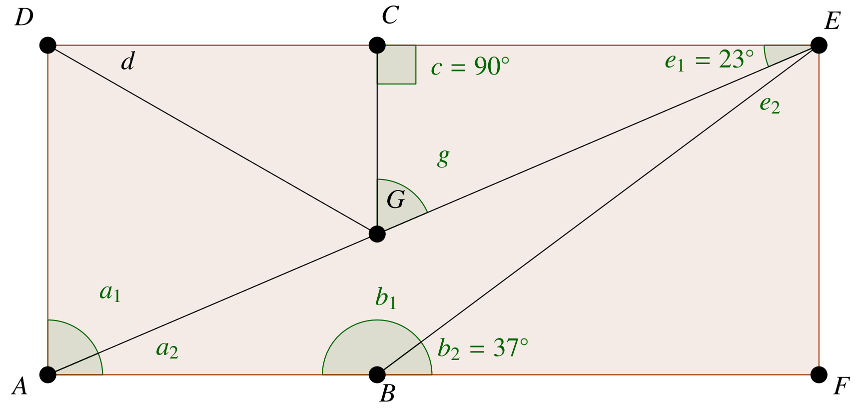


Figure 29a

of the angles of a triangle is 180°.

- $\angle b_1 = 143°$, because $\angle b_1$ is supplementary to $\angle b_2$ of 87°.
- $\angle a_2 = 23°$. AE is transveral to the lines DCE and ABF, so $\angle a_2$ has the same measyer as $\angle e_1$.
- $\angle a_1 = 67°$ since ii is complementary to $\angle a_2$.
- $\angle e_2 = 63°$ since ii is complementary to $\angle b_2$.
- $\angle d$ cannot be determined. One way to see that is to give up trying. A more convincing way is this: notice that the vertical line segmentCG can be moved horizontally without changing any of the given angles. However, $\angle d$ cdoes change as CG moves.

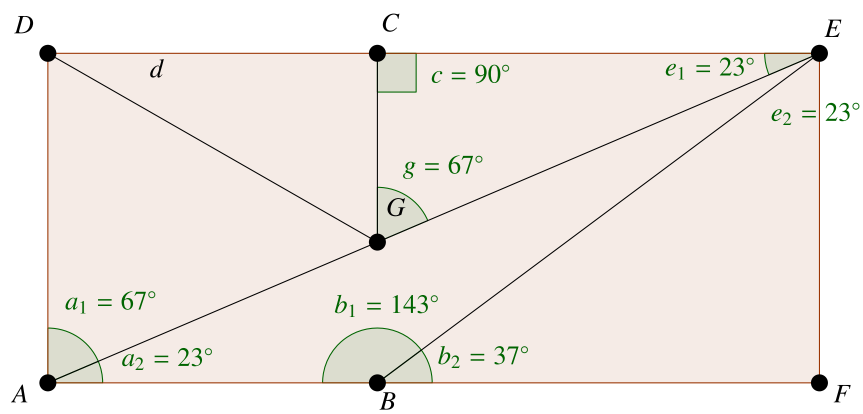


Figure 29b

Extension. The fact that the sum of the angles of a triangle is a straight angle is a powerful fact in geometry, leading to many interesting observations. Students interested in the application of algebra to geometry might find the following examples of interest.

Example 9.

Draw a circle and label the center O. In the circle draw a diameter, labeling the endpoints A and B. Now select a point C on the circle, different from A and B and draw the triangle ABC. See Figure 30:

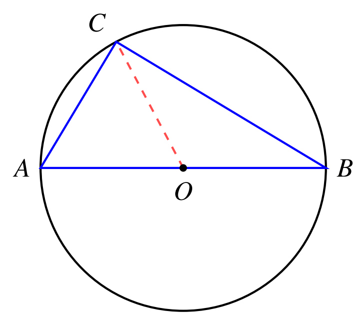


Figure 30

With a protractor, ask students to find the measure of $\angle ACB$. All answers from the class will be that the measure is is 90°, On the basis of this experiment, the class can hypothesize: If a triangle is circumscribed by a circle so that one side of the triangle is a diameter, then the triangle is a right triangle. Students may want to go further: given a right triangle, the circumscribing circle has the hypotenuse as a diameter. The following text is intended to provide a guide to this exploration.

Draw the ray from O to C. Lines AO,BO, CO are all of equal length since they are all radii of the circle. In particular, triangles AOC and BOC are isosceles triangles, and so the base angles are equal (a property of isosceles triangles). That tells us that the measure of $\angle ACO$ is also x and the measure of $\angle BCO$ is also y, so that the measure of $\angle ACB$ is $x + y$. But, since the sum of the measures of the angles of a triangle is 180°, we have

$$x + \angle ACB + y = x + (x + y) + y = 180\ ,$$

so $2(x + y) = 180$, or $x + y = 90$ and thus $\angle ACB$ has measure 90°.

The student will remember, from 5th grade, the fact that for an isosceles triangle (a triangle with two sides equal), the base angles are equal. This can be demonstrated as follows: draw an isosceles triangle AVB, as in the following figure, with the lengths of AV and BV equal. Now fold the triangle along a line that goes through the vertex V, so that the line segment AB folds over onto itself. The student will see that in fact the triangle on one side is perfectly superimposed on the triangle on the other side, so that the angle at A lies directly over the angle at B, and so they are of equal measure.

Example 10.

Here is another interesting fact that follows from the basic fact about the sum of the angles of a triangle. Draw a circle and select three points A, V, B on the circle and draw the angle AVB. Suppose that the center of the circle O lies inside $\angle AVB$, as in the figure below. Draw the line segments AO and BO. Then the measure of $\angle AVB$ is half the measure of $\angle AOB$.

Solution. To see this, label the measures of the angles with the letters x, y, z, w, u as in the diagram. Because the lines AO, VO, BO are all radii, and thus of the same lengths, angles with the same letter indeed do have the same measure. Now, look at all the triangles involved to get:

$$2x + w = 180°\ , \qquad 2y + z = 180°\ , \qquad u + w + z = 360°\ .$$

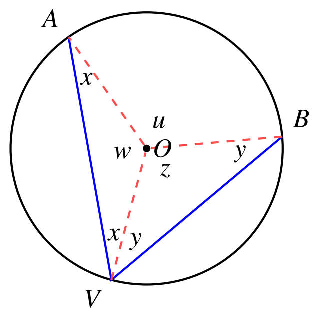


Figure 31

Add the first two equations to get

$$2x + 2y + w + z = 360°$$

and use the third equation to get $2x + 2y + 360 - u = 360°$, and now conclude that $u = 2(x + y)$. But u is the measure of $\angle AOB$ and $x + y$ is the measure of $\angle AVB$.

Example 11.

By drawing Figure 31 to solve Example 10 we assumed that the given angle had its rays on opposite sides of the center of the triangle. However, the statement is still true for any angle with vertex on the circle. Students may want to try to figure out why.

End Extension