Chapter 6
Equations and Inequalities in Context

In this chapter we will use the geometric relationships that we explored in the last chapter and combine them with the algebra we have learned in prior chapters. First we recall the methods developed in (the first section of) Chapter 3, but without the bar models, focusing on algebra as an extension of the natural arithmetic operations we have been performing until now in solving problems. We will then focus on how to solve a variety of geometric applications and word problems. We will also encounter inequalities and extend our algebraic skills for solving inequalities. By this chapter's end, we will be able to set up, solve, and interpret the solutions for a wide variety of equations and inequalities that involve rational number coefficients.

This chapter brings together several ideas. The theme throughout however is writing equations or inequalities to represent contexts. In the first section students work with ideas in geometry and represent their thinking with equations. Also in that section students solidify their understanding of the relationship between measuring in one-, two-, and three-dimensions. In the second section, students will be writing equations for a variety of real life contexts and then finding solutions. The last section explores inequalities. This is the first time students think about solutions to situations as having a range of answers.

In Chapter 3 students learned how to solve one-step and simple multi-step equations using models. In this chapter students extend that work to more complex contexts. In particular they build on understandings developed in Chapter 5 about geometric figures and their relationships. Work on inequalities in this chapter builds on Grade 6 understandings where students were introduced to inequalities represented on a number line. The goal in Grade 7 is to move to solving simple one-step inequalities, representing ideas symbolically rather than with models.

Throughout mathematics, students need to be able to model a variety of contexts with algebraic expressions and equations. Further, algebraic expressions help shed new light on the structure of the context. Thus the work in this chapter helps to move students to thinking about concrete situations in more abstract terms. Lastly, by understanding how an unknown in an expression or an equation can represent a "fixed" quantity, students will be able to move to contexts where the unknown can represent variable amounts (i.e. functions in Grade 8.)

Use variables to represent quantities in a real-world or mathematical problem, and construct simple equations and inequalities to solve problems by reasoning about the quantities. 7.EE.4

Consider the sentences $9 + 4 = 13$ and $8 + 6 = 12$. The first sentence is true, but the second sentence is false. Here is another sentence: $\diamond + 9 = 2$. This sentence is neither true nor false because we don't know what number the symbol $\diamond$ represents. If $\diamond$ represents -6 the sentence is false, and if $\diamond$ represents -7 the sentence is true. This is called an *open sentence*.

A symbol such as $\diamond$ is referred to as an *unknown* or a *variable*, depending upon the context. If the context is a specific situation in which we seek the numeric value of a quantity defined by a set of conditions, then we will say it is an "unknown." But if we are discussing quantifiable concepts (like length, temperature, speed), over a whole range of possible specific situations, we will use the word "variable."

So, for example, if I am told that Maria , who is now 37, three years ago was twice as old as Jubana was then, then I would write down the equation $37 - 3 = 2(J - 3)$, where J is the unknown age of Jubana. But if I write that $C = 2\pi r$ for a circle, C and r are the measures of circumference and radius (in the same units) for *any* circle. In this context, C and r are "variables."

Various letters or symbols can be used, such as $\diamond, x, y, a, b$, and c. Variables are symbols used to represent any number coming from a particular set (such as the set of integers or the set of real numbers). Often variables are used to stand for quantities that vary, like a person's age, the price of a bicycle, or the length of a side of a triangle. But, in a specific instance, if we write $\diamond + 9 = 2$, $\diamond$ is an unknown, and the number -7 makes $\diamond + 9 = 2$ true, so is a correct value of the unknown, and is called a *solution* of the open sentence.

Understand that rewriting an expression in different forms in a problem context can shed light on the problem and how the quantities in it are related. 7.EE.2

In Chapter 3, we moved seamlessly from pictorial models of arithmetic situations to algebraic formulations of those models, called expressions, without having said what an expression is. Let's do that now: an *expression* is a phrase consisting of symbols (representing unknowns or variables) and numbers, connected meaningfully by arithmetic operations. So $2x - 3(5 - x)$ is an expression as is $(4/x)(x^2 + 3x)$.

It is often the case that different expressions have the same meaning: for example $x + x$ and $2x$ have the same meaning, as do $x - x$ and 0. By *the same meaning* we mean that a substitution of any number for the unknown x in each expression produces the same numerical result. We shall call two expressions *equivalent* if they have the same meaning in this sense: any substitution of a number for the unknown gives the same result for both expressions.

Since checking two expressions for every substitution of a number will take a long time, we need some rules for equivalence. These are the laws of arithmetic: $2x + 6$ and $2(3 + x)$ are equivalent because of the laws of distribution and commutativity. In the same way, $2x + 5x$ is equivalent to $7x$; $-2(8x - 1)$ is equivalent to $2 - 16x$, and so forth. Reliance on the laws of arithmetic is essential: to show that two expressions have the same meaning, we don't/can't check every number; it suffices to show that we can move from one expression to the other using those laws. Also, to show that, for example $3 + 2x$ and $5x$ are not equivalent, we only have to show that there is a value of x that, when substituted, does not give the same result. So, if we substitute 1 for x we get 5 and 5. But what if we substitute 2 for x: we get 7 and 10. The expressions are not equivalent.

> To show that two expressions are equivalent, we must show how to get from one to the other by the laws of arithmetic. However, to show two expressions are not equivalent, we need only find a substitution for the unknown that gives different results for the two expressions.

Open sentences that use the symbol '=' are called *equations*. An equation is a statement that two expressions on either side of the '=' symbol are equal. The mathematician Robert Recorde invented the symbol to stand for 'is equal to' in the 16th century because he felt that no two things were more alike than two line segments of equal length. These equations involve certain specific numbers and letters. We refer to the letters as unknowns, that is they represent actual numbers which are not yet made specific. Indeed, the task is to find the values of the unknowns that make the equation true. If an equation is true for all possible numerical values of the unknowns (such as $x + x = 2x$), then the equation is said to be an equivalence. It is an important aspect of equations that the two expressions on either side of the equal sign might not actually always be equal; that is, the equation might be a true statement for some values of the variables(s) and a false statement for others.

For example, $10 + 0.02n = 20$ is true only if $n = 500$; and $3 + x = 4 + x$ is not true for any number x; and $2(a + 1) = 2a + 2$ is true for all numbers a. A solution to an equation is a number that makes the equation true when substituted for the variable (or, if there is more than one variable, it is the number for each variable). An equation may have no solutions (e.g. $3 + x = 4 + x$ has no solutions because, no matter what number x is, it is not

true that adding 3 to x yields the same answer as adding 4 to x. An equation may also have every number for a solution (e.g. $2(a + 1) = 2a + 2$). An equation that is true no matter what number the variable represents is called an identity, and the expressions on each side of the equation are said to be equivalent expressions. So, $2(a + 1)$ and $2a + 2$ are equivalent expressions.

Statements involving the symbols '>', '<', '≥', '≤' or '≠' are called *inequalities*. Let's review the meanings of these symbols:

- $x > 3$ means "x is greater than 3."
- $7 < 4$ means "7 is less than 4." (Note that these are simply statements, with truth or falsity to be determined.)
- $x \geq y$ means "x is greater than or equal to y."
- $x \leq x^2$ means "x is less than or equal to x^2."
- $a \neq b + c$ means "a is not equal to $b + c$."

To *solve* an equation or inequality means to determine whether or not it is true, or for what values of the unknown it is true. Once found, those values are called the *solutions* of the problem.

EXAMPLE 1.

Find the solutions of these open sentences by inspection. How many solutions are there?

a. $\heartsuit - 4 = 1$.

b. $x + \frac{1}{4} = \frac{7}{4}$.

c. $5x = 12$.

d. $2x = 0$.

e. $0x = 5$.

f. x is a whole number and $x < 4$.

g. $x = 2x - x$.

SOLUTION.

a. $\heartsuit = 5$: one solution;

b. $x = \frac{6}{4} = \frac{3}{2} = 1\frac{1}{2}$: one solution;

c. $x = \frac{12}{5} = 2\frac{2}{5}$: one solution;

d. $x = 0$: one solution;

e. any number x would still make the left side equal zero; and $0 \neq 5$, therefore no solutions exist;

f. the solutions are 1, 2, and 3: three solutions;

g. Any value of x makes this sentence true, that is, every number is a solution.

Variables, equations, and inequalities empower us to write down ideas in concise ways.

EXAMPLE 2.

Sofia is in third grade and loves mathematics. She was thinking about numbers one day, and she wrote

$$2 + 2 = 4 + 4 = 8 + 8 = 16 + 16 = 32 + 32$$

on her paper and was eager to continue. Apollo was impressed, but confused. "Wait, something's wrong," he told her. " $2 + 2$ is **not** equal to $4 + 4$. " Sofia agreed and corrected her writing like so

$$2 + 2 = 4. \quad 4 + 4 = 8. \quad 8 + 8 = 16. \quad 16 + 16 = 32. \quad \text{etc...}$$

An equation is a sentence. The symbol '=' means 'is equal to,' and if you run all your thoughts together, you may say something false.

Writing mathematics has a grammar of its own. Just like a sentence has punctuation marks, mathematics makes use of *grouping symbols* to help the reader understand the meaning. Equality and inequality symbols are grouping symbols. They divide an equation or inequality into a left hand side (LHS) and a right hand side (RHS). Parentheses and fraction bars are also used as grouping symbols.

To better understand what is meant by *expression*, *equation*, and *inequality* examine the examples in the table below. Note that equations and inequalities are statements, while expressions are more like mathematical phrases.

expressions	equations	inequalities
$\dfrac{(20-2)}{6}$	$\dfrac{(20-2)}{6} = 3$	$\dfrac{(20-2)}{6} > 0$
$5x - 3$	$5x - 3 = 22$	$5x - 3 \leq 22$
$x^2 + 2x - 21$	$x^2 + 2x - 21 = 0$	$5 \neq -5$
$2l + 2w$	$A = \dfrac{1}{2}bh$	$|AB| + |BC| > |AC|$
$\dfrac{\pi r^2}{2\pi r}$	$m(\angle 1) + m(\angle 2) = 90°$	$\dfrac{a}{b} < \dfrac{c}{d}$

Equations are said to be *equivalent* if they have the same meaning. For expressions, this meant that substitution of a number for the unknown in each expression gives the same answer. For equations, it is the same, but as an equation represents a statement, the criterion is *true* or *false*. The test is this: the substitution of any number for the unknown in the two equations must give the same answer to the question: "True or False?" Another way of putting this is this: a *solution* of an equation is a number that, when substituted for the unknown, gives the response "true." To make the definition more precise, two equations are *equivalent* if they have the same set of solutions.

As with expressions, this is an impossible criterion to apply: we cannot test every number. So, we look for laws of arithmetic that do not change the solution set of an equation. For example, $2x = 10$ and $x = 5$ are equivalent equations: because one equation is double the other, so they have the same solution set. Also, $3(x + 9) = 72$ and $4x - 10 = 50$ are equivalent, because they have the same solution ($x = 15$).

EXAMPLE 3.

Write down the sequence of laws of arithmetic the take us from one equation to the other.

SOLUTION. We note first that the following equations are equivalent

$$3x + 6 \;=\; 15 \tag{1}$$
$$3x + 8 \;=\; 17 \tag{2}$$
$$3x \;=\; 9 \tag{3}$$
$$12x \;=\; 36 \tag{4}$$
$$x \;=\; 3 \tag{5}$$

The above five equations are all equivalent because they all have the same solution ($x = 3$). But we can also see that the equations are equivalent because they are related by operations that do not change the solution set:

- If we add 2 to both sides of equation (1), we obtain equation (2).

- If we subtract 8 from both sides of equation (2), we obtain equation (3).

- If we multiply both sides of equation (3) by 4, we obtain equation (4).

- If we divide both sides of equation (4) by 12, we obtain equation (5).

This example illustrates the most important operations on equations that do not change the solution set. They are:

1. By adding (or subtracting) the same number on both sides of an equation, the new equation is equivalent to the original equation.

2. Multiplying (or dividing) the same *nonzero* number on both sides of an equation, the new equation is equivalent to the original equation.

Note that when we 'multiply or divide both sides of an equation by a number,' we must apply that operation *to every term* on both sides of the equation. For example: solve

$$3x + 9 = 87 \,.$$

Our goal is an equation of the form "$x = \ldots$," so first let us divide both sides by 3 to get:

$$x + 3 = 29 \,.$$

Every term in the equation has been divided by 3. Subtract 3 from both sides to get $x = 26$. We also could have first subtracted 9 from both sides to get $3x = 78$, and then divided by 3. The order of operations does not matter; what is important is that we employ only operations that can be reversed: if we divide all terms in an equation by 3, we can go back to the original equation by multiplying all terms by 3. If we subtract 9 from both sides of the equation, we can go back by multiplying by 9.

EXAMPLE 4.

Explain why it is important to say *nonzero* number in the second rule above, but not in the first rule above.

SOLUTION. Suppose you add or subtract 0 from both sides of an equation. This doesn't change the equation at all. Adding 0 is the reverse of subtracting 0, so these operations can be undone.

Suppose now that you multiply both sides of an equation by 0. For example, if we start with $2x - 7 = 15$, multiplying by 0 gives us $0 = 0$. Certainly a simplification, but not a valuable one: there is no way to go from $0 = 0$ back to $2x - 7 = 15$.

Dividing by zero is not allowed because it simply doesn't make sense. Recall that division can be thought of as the inverse of multiplication. But we just saw that if we multiply all terms of *any* equation by 0, we get to $0 = 0$. There is no way of reversing this: to get from the equation $0 = 0$ to any equation.

Finally, if an expression in an equation is replaced by an equivalent expression, then the equations are equivalent. As an example:

$$3(x + 2) \quad = \quad 15 \tag{6}$$
$$3x + 6 \quad = \quad 15 \tag{7}$$

are equivalent equations.

Solve multi-step real-life and mathematical problems posed with positive and negative rational numbers in any form (whole numbers, fractions, and decimals), using tools strategically. Apply properties of operations to calculate with numbers in any form; convert between forms as appropriate; and assess the reasonableness of answers using mental computation and estimation strategies. 7.EE.3.

To 'solve' or 'find solutions' of a given equation, perform the previously mentioned operations to obtain equivalent equations until the *unknown* is alone on one side of the equation. We also call this process 'isolating' the unknown on one side of the equation. It can be helpful to simplify expressions on each side of an equation before or while solving the equation. Let's look at some examples.

EXAMPLE 5.

Solve $4x - 7 = 9$.

SOLUTION.

$$
\begin{aligned}
4x - 7 &= 9 \\
4x - 7 + 7 &= 9 + 7 \quad \text{(Add 7 to both sides)} \\
4x &= 16 \\
\frac{1}{4}(4x) &= \frac{1}{4}(16) \quad \text{(Multiply both sides by } \frac{1}{4}\text{)} \\
x &= 4
\end{aligned}
$$

Therefore, the solution is 4.

Every time we write a new line, we have an algebraic reason for doing so. Recall that multiplying by $\frac{1}{4}$ is the same as dividing by 4.

In chapter 3 we learned various properties of addition and multiplication. Here we study examples with focus on how those properties appear in each step as we solve equations.

EXAMPLE 6.

Solve $\frac{3}{5}\left(x - \frac{5}{6}\right) = \frac{7}{4}$.

SOLUTION.

$$\frac{3}{5}\left(x - \frac{5}{6}\right) = \frac{3}{4}$$

$$\frac{5}{3} \cdot \frac{3}{5}\left(x - \frac{5}{6}\right) = \frac{5}{3} \cdot \frac{3}{4} \quad \text{(Multiply both sides by } \frac{5}{3}\text{)}$$

$$\left(x - \frac{5}{6}\right) = \frac{5}{4}$$

$$x - \frac{5}{6} + \frac{5}{6} = \frac{5}{4} + \frac{5}{6} \quad \text{(Add } \frac{5}{6} \text{ to both sides)}$$

$$x = \frac{15}{12} + \frac{10}{12} = \frac{25}{12}$$

Therefore, the solution is $\frac{25}{12}$ or $2\frac{1}{12}$.

Another solution method:

$$\frac{3}{5}\left(x - \frac{5}{6}\right) = \frac{3}{4}$$

$$\frac{3}{5}x - \frac{3}{5} \cdot \frac{5}{6} = \frac{3}{4} \quad \text{(Apply the distributive property on the LHS.)}$$

$$\frac{3}{5}x - \frac{1}{2} = \frac{3}{4} \quad \text{(Simplify the expression } \frac{3}{5} \cdot \frac{5}{6}\text{)}$$

$$\frac{3}{5}x - \frac{1}{2} + \frac{1}{2} = \frac{3}{4} + \frac{1}{2} \quad \text{(Add } \frac{1}{2} \text{ to both sides.)}$$

$$\frac{3}{5}x = \frac{3}{4} + \frac{2}{4}$$

$$\frac{3}{5}x = \frac{5}{4}$$

$$\frac{5}{3} \cdot \frac{3}{5}x = \frac{5}{3} \cdot \frac{5}{4} \quad \text{(Multiply } \frac{5}{3} \text{ to both sides.)}$$

$$x = \frac{25}{12}$$

Again the solution is $\frac{25}{12}$ or $2\frac{1}{12}$.

EXAMPLE 7.

This process works even with tricky examples. Here we have positive and negative decimal coefficients $0.2x - 0.4 = -3.4$.

SOLUTION.

$$0.2x - 0.4 = -3.4$$

$$0.2x - 0.4 + 0.4 = -3.4 + 0.4 \quad \text{(Add 0.4 to both sides.)}$$

$$0.2x = -3.0$$

$$\frac{1}{0.2}(0.2x) = \frac{1}{0.2}(-3.0) \quad \text{(Multiply both sides by } \frac{1}{0.2}\text{)}$$

$$x = \frac{-3.0}{0.2} = -\frac{3.0}{0.2} = -\frac{30}{2} = -15$$

Therefore, the value of x is -15.

In the next to last step, we could have said we divided both sides by 0.2. This is the same as multiplying both sides by $\frac{1}{0.2}$, the multiplicative inverse of 0.2. Notice also that $\frac{1}{0.2} = \frac{10}{2} = 5$. So, another way to finish solving this equation is by multiplying both sides by 5.

Here is another solution method for this same example:

$$0.2x - 0.4 = -3.4$$
$$2x - 4 = -34 \quad \text{(Multiply both sides by 10, now the decimals are removed.)}$$
$$2x = -30 \quad \text{(Add 4 to both sides.)}$$
$$\frac{2x}{2} = \frac{-30}{2} \quad \text{(Divide both sides by 2.)}$$
$$x = -15$$

There are many ways to get to the right solution. It could be advantageous to multiply each side of an equation by a number just to remove fractions and decimals.

An equation of the form $px + q = r$, where p, q, and r are any numbers, is called a *first order equation*. All of the examples illustrated here are equations that can be written in this form (with $p \neq 0$).

A *formula* tells us information about how different variables relate to each other (the plural form of 'formula' is 'formulae'). For example, the perimeter of a polygon is the sum of the lengths of its sides. Perimeter is usually denoted by the letter P. For a triangle of side lengths a, b, c, we express this by the formula $P = a + b + c$. So, if we are given a triangle of side lengths 19, 7, 21 units, we calculate perimeter as follows:

$$P = a + b + c, \quad a = 19, \quad b = 7, \quad c = 21$$
$$P = 19 + 7 + 21 = 47$$

Look over the table of mathematical formulas on the next page of the areas and perimeters of geometric objects. Remember that if length is given in certain units (ft., cm., . . .), then perimeter is measured in the same units, while area is measured in square units (sq. ft., sq. cm., . . .).

For example, if we have a rectangular lot that measures 72 yards in length and 72 yards in width, then to fence in the whole lot we need to calculate the perimeter and use the formula $P = 2l + 2w$, with $l = 72$ yards and $w = 40$ yards: $P = 2(72) + 2(40) = 224$ yards. To find the area of this lot we use the formula $A = l \cdot w$, so it has area $A = 72 \cdot 40 = 2880$ sq. yds.
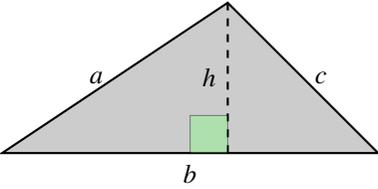
	$P = 2l + 2w$	$A = l \cdot w$
	$P = a + b + c$	$A = \frac{1}{2}b \cdot h$
	$P = b_1 + b_2 + s_1 + s_2$	$A = \frac{1}{2}(b_1 + b_2)h$
	$C = 2\pi r$	$A = \pi r^2$
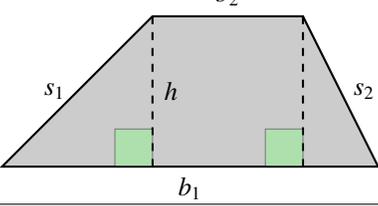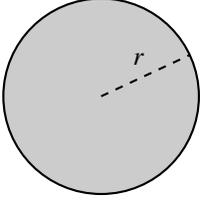
Section 6.1: Write and Solve Equations to Find Unknowns in Geometric Situations

This section builds upon what students learned about geometric relationships in chapter 5 and in earlier grades. We begin by applying the skills used in solving equations, to writing and solving one-step and multi-step equations involving finding missing measures of unknown values in contexts which involves various angle relationships with triangles, areas, perimeters, circles and scaling. In particular, we will pay close attention to the relationship between the structure of algebraic equations and expressions and the contexts they represent.

Use facts about supplementary, complementary, vertical, and adjacent angles to write and solve multi-step problems for an unknown angle in a figure. 7.G.5

In Chapter 5 we introduced some facts about relationships among angles, and illustrated their use in solving problems about angles. Here we will further develop the use of algebraic language to solve problems about angles. First, let us recall the basic facts from Chapter 5.

- The sum of the measures of interior angles in a triangle is 180°.
- Vertical angles have equal measure.
- Complementary angles add up to 90°.
- Supplementary angles add up to 180°.

EXAMPLE 8.

In the library shown in Figure 1, the pitch of the roof is 22° and the angle between the joist and the roof support beam is 38°. Find the measure of the remaining angles of the wooden support structure.
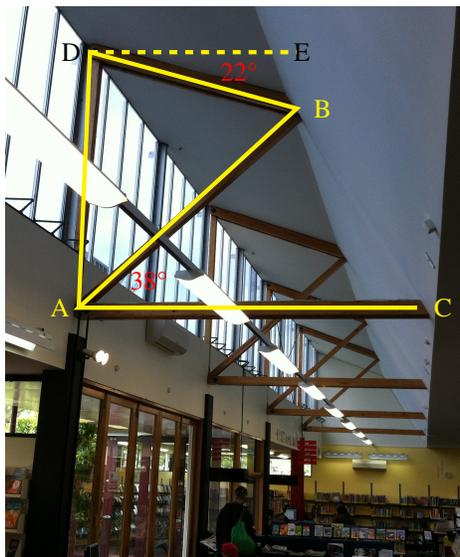
Figure 1

SOLUTION. We have used some technical language in phrasing this problem, so let's take a moment to clarify the language. Additionally, a scale drawing of the roof support structure connects the language with a visual.

The *pitch* of the roof is the angle it makes with the horizontal. So we have put a horizontal hashed line at the peak of the roof and indicate the angle with measure 22°. A *roof support beam* is a beam from a wall of the structure that makes a diagonal with the roof; in our case, this is the beam represented by the line segment *AB*. A *joist* is a horizontal floor beam (even if there is no floor); designated in our diagram by the segment *AC*, giving the angle measure 38° as labeled in Figure 1. Finally, although it has not been made explicit, we assume that the walls are perfectly vertical; in particular, the line segment *AD* is vertical.

Since *AD* is vertical and the hashed line is horizontal, the angle ∠*ADB* is complementary to the 22° angle, and so ∠*ADB* = 90 − 22 = 68°. Additonally, ∠*DAB* is complementary to ∠*BAC* whose measure is 38°. We conclude that ∠*DAB* has measure 52°. Given that the sum of the angles of a triangle is 180°, we find the measure of ∠*DBA*:

$$∠DBA = 180 − (68 + 52) = 60° .$$

EXAMPLE 9.

Determine if the statement below is *always, sometimes or never* true. It if is sometimes true, give an example when it is true and when it is false. If it is *always* or *never* true, provide a justificaton of the statement..

   a. Adjacent angles are also supplementary angles.
   b. Vertical angles have the same measure.

SOLUTION. First, we note that the terms counter example and non-example are interchangeable, that is, they represent an example that disproves a proposition. Examining examples and non-examples can help students understand definitions. Constructing an argument to disprove a statement only requires one counter example, while constructing an argument to "prove" something is more involved. In other words, one affirmative example does not prove a statement. In Grade 7 attention to precision in making statements is an important first step towards building arguments. So, for part b, in making the determination that it is always true, we look to explain why the statement is true, looking for statements that build on understanding of supplementary angles and transitivity.

a. Sometimes true. In Figure 1, $\angle CAB$ and $\angle BAD$ are complementary, not supplementary. However, in Figure 2 (below), $\angle ATD$ and $\angle DTB$ are adjacent and supplementary. Note that the converse is always true: supplementary angles are always adjacent.

b. Always true. Figure 2 is a generic picture of vertical angles: $\angle ATC$ and $\angle BTD$ are vertical angles, in the sense that they are opposing angles at a vertex. Angles $\angle CTB$ and $\angle DTA$ are vertical angles as well. Vertical angles have equal measure, since they are both supplementary to the same angle. For exampe, in Figure 2, $\angle ATC$ and $\angle BTD$ are supplementary to $\angle DTA$, so they must have the same measure.
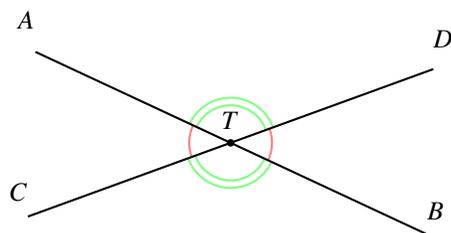
Figure 2

EXAMPLE 10.

Find the values of x and y in Figure 3.
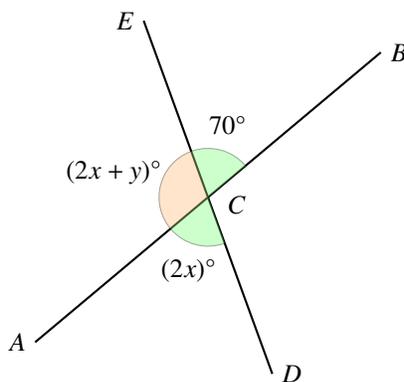
Figure 3

SOLUTION.

$$2x = 70° \quad \text{(Vertical angles are congruent)}$$
$$x = 35° \quad \text{(Multiply both sides by1/2)}$$
$$2x + y + 70 = 180° \quad \text{(Supplementary angles add to 180)}$$
$$2x + y = 110° \quad \text{(Add the opposite of 70 to both sides)}$$
$$70 + y = 110° \quad \text{(since } 2x \text{ is70)}$$
$$y = 40° \quad \text{(Add the opposite of 70 to both sides)}.$$

EXAMPLE 11.

The perimeter of a rectangle is 54 cm. Its length is 6 cm. What is its width?

SOLUTION. . Since we know that the perimeter of a rectangle is equal to 2 times the length plus 2 times the width, we can start by writing the formula $P = 2l + 2w$. In this example the perimeter is 54 cm and the length is 6 cm. Placing the information into our equation yields a new equation with just one unknown quantity, w:

$$54 = 2(6) + 2w$$

Now, we can proceed with solving the equation. We get:

$$54 = 12 + 2w$$
$$54 - 12 = 12 + 2w - 12$$
$$42 = 2w$$
$$\frac{42}{2} = \frac{2w}{2}$$
$$21 = w.$$

We have found that $w = 21$, so the width of the rectangle must be 21 cm.

EXAMPLE 12.

A trapezoid has a perimeter of 47 cm. and an area of 132 sq. cm. The longer of the two parallel sides has length 13 cm and the height of the trapezoid is 11 cm. What is the length of the shorter of the two parallel sides?

SOLUTION. We refer to the table of formulae above: the perimeter of the trapezoid is given by the formula

$$P = b_1 + b_2 + s_1 + s_2$$

where the b's refer to the parallel sides, and the s's refer to the other sides. Putting information into the formula gives us the equation

$$47 = b_1 + 13 + s_1 + s_2,$$

and we want to find b_1. But we don't know s_1 and s_2; what we do know is the height of the parallelogram, but that does not tell us the length of the sides (draw several parallelograms with the given dimensions but with differing lengths for s_1 and s_2. Maybe the area formula gives us some hope:

$$A = \frac{1}{2}(b_1 + b_2)h .$$

Putting in the given data gives us the equation:

$$A = 132 = \frac{1}{2}(13 + b_2)(11) ,$$

which we can solve for b_2. We multiply both sides by 2 and divide both sides by 11 to get $22 = 13 + b_1$, to find $b_2 = 9$.

EXAMPLE 13.

The ratio of the length to width of a rectangular photograph is 2 : 5. The longer side is 15 inches.

   a. What is the length of the shorter side?

   b. If the area of the photo is quadrupled, what will the new dimensions of the photo be?

SOLUTION.

   a. Given that the ratio of shorter to longer is 2 : 5, we are told that the longer side is 3 times the ratio. So, the shorter side must also be 3 times the ratio, so is $3 \cdot 2 = 6$ in.

   b. The given photograph has dimensions 6 by 15, so is a rectangle of area 90 in^2. The quadrupled photograph then has to have an area of 360 in^2. So, we seek two numbers in the ratio 2 : 5 whose product. Clearly if we double the dimensions, we quadruple the area, so the size of the photograph must be 12×30 in. See Figure 4 for a graphic of this computation.

15 in

6 in
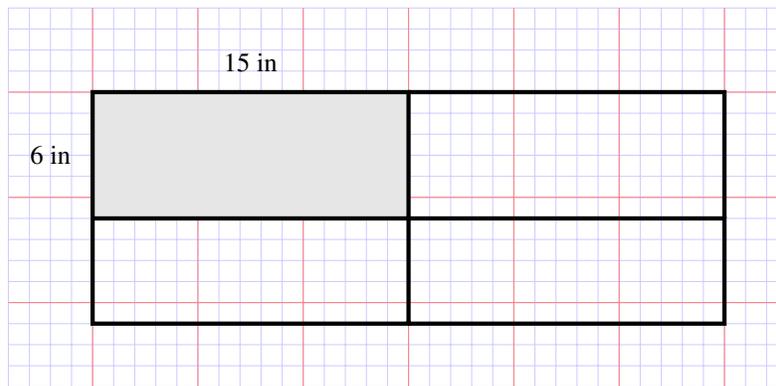
Figure 4

EXAMPLE 14.

Find the measure of ∠C and ∠B in Figure 5.
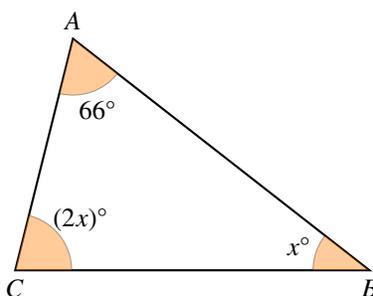
Figure 5

SOLUTION. We want to stress that the figure is a representation to help us think about the problem, it is not necessarily to scale. Additionally we stress that "x" is an unknown, it can take on any value when we write the expression $x + 2x + 70$. Once we set it equal to something, we can find a value for x to make the equation true. This is the very beginning of thinking about "x" as a variable rather than an unknown.

$$x + 2x + 66 = 180$$
$$3x + 66 = 180 \quad \text{(Combine like terms)}$$
$$3x = 114 \quad \text{(Add the additive inverse of 66 to both sides)}$$
$$x = \frac{114}{3} = 38 \quad \text{(Multiply the multiplicative inverse of 3 to both sides)}.$$

Therefore the measure of ∠B = 38°, and so the measure of ∠C = 2 · 38° = 76°.

EXAMPLE 15.

Pizzas are sold according to diameter. For example, a 6 inch pizza is a pizza with a diameter of 6 inches. At Francesco's pizzeria, there are two pizzas. Pizza A is a 12 inch, and Pizza B has an area of 4150 in^2. Which pizza is bigger? What is the percent of increase from the smaller pizza to the larger pizza?

SOLUTION. To compare Pizza A to Pizza B, let's determine the area of Pizza given the area of Pizza B is 450 in^2. Applying $A = \pi \cdot r^2$ to Pizza A (remember that the radius is half the diameter): $A = \pi \cdot 6^2 = 36\pi$ or about 113 in^2. Therefore, Pizza B is bigger.

The percent of increase from the smaller pizza, Pizza A to the larger pizza, Pizza B, if denoted by p (expressed as a fraction), leads to the equation

$$113 + 113p = 150$$

with the solution $p = 0.327$. Thus the percent increase is 32.7%.

Section 6.2: Write and Solve Equations from Word Problems

In this section we focus on working in two "different directions;" for example, in some sections we challenge students, given a context, find relationships and solutions, while in other sections, students will be given relationships and asked to write contexts. The goal is to help students understand the structure of context in relationship to algebraic representations.

In Chapter 3 we stressed that, in expressing a proportion, it is necessary to be precise about units. If we are thinking that one cup weighs 8 ounces, we might say that the ratio of volume (in cups) to weight (in ounces) is 1:8. Again, proper use of units was seen to be essential in understanding scaled drawings. Here again, precision in choice of units is necessary to understanding equations. When we write a statement that two expressions are equal, the units on each side of the equation must be the same. To answer the question, "how many cups of water makes 32 ounces?", we write an equation that expresses this question, in units on both sides of the equation. So, if x is the number of cups, we have to equate that in ounces to 32. x cups contain $8x$ ounces, so the equation to solve is $8x = 32$.

Concentration on the relevant units in a problem actually helps us to write an equation. For example, suppose someone drove from Layton to Salt Lake City in 30 minutes. If they traveled at a constant speed, how fast were they driving? Typically we talk about driving speeds in the unit of miles per hour. But in this case, we are given minutes, so it is a good idea to write 30 minutes as $\frac{1}{2}$ hour. Since we want to know an answer in miles per hour, we see that we are missing some information here. We need to know how many miles it is from Layton to Salt Lake City. A map shows that the distance is about 24 miles, so the drive was 24 miles in half an hour. In another half hour the driver will drive another 24 miles, so the speed is 48 miles per hour. If we solve this algebraically, we can let s be the speed in miles per hour and, what we are given is that s is 24 miles in 1/2 hour, leading to the equation

$$s = \frac{24}{\frac{1}{2}}.$$

The units match on both sides of the equation, so we don't need to write them down. We can perform arithmetic without them: $s = \frac{24}{1} \cdot \frac{2}{1} = 48$; the speed was 48 miles per hour.

Solve word problems leading to equations of the form $px + q = r$ and $p(x + q) = r$, where p, q, and r are specific rational numbers. Solve equations of these forms fluently. Compare an algebraic solution to an arithmetic solution, identifying the sequence of the operations used in each approach. 7EE4a.

EXAMPLE 16.

A youth group is going on a trip to the state fair. The trip costs $52. Included in that price is $11 for a concert ticket and the cost of 2 passes, one for the rides and one for the game booths. Each of the passes cost the same price. Write an equation representing the cost of the trip and determine the price of one pass.

SOLUTION. We start by identifying an unknown and use a variable to represent it. Since we want to determine the price of one pass, we will say x is the price of one pass.

Next, we identify what we do know. We know that the cost of the trip is $52. Both of these steps require careful reading of the problem, and translation of these ideas into mathematical language.

What connects the two pieces of information together? Well, the fact that the $52 trip covers the cost of an $11 item and two passes, that is: $52 is equal to $11 plus the cost of two passes.

We represent the price of each pass by x, we have $52 = 11 + 2x$, which is an algebraic expression of the statement: $52 is the price of an $11 item and 2 passes.

This diagram shown is also a helpful way to arrive at the given equation.

x	x	11
52		
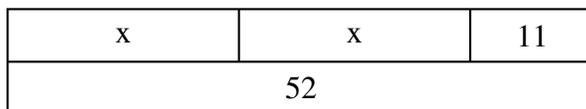
To solve this equation we may make algebraic steps to isolate x on one side, as follows

$$2x + 11 = 52$$
$$2x = 41$$
$$x = 20.5$$

Now that we have solved the equation, we interpret the result in context. The price of one pass must be $20.50.

Does that result make sense? Sure it does, because 2 passes at that price would be $41. Plus $11 for the concert ticket and we've got the total price of $52. Notice how this reasoning is represented and expressed in algebraic equations.

The above example could also be solved by arithmetic reasoning. We began with $52, $11 was spent on the concert ticket, subsequently the two passes together cost $52 − $11 = $41. Given that the cost of the additional 2 passes is the same price, they are each ? of the remaining balance of $41.00, or $20.50. Most importantly, the algebraic method encodes this reasoning.

The student may ask, "why are we learning algebraic methods, when the arithmetic method is so easy?" Algebra is a powerful problem-solving tool. However, in order to come to appreciate the value of algebra, students need to encounter problems that are not easily solvable using arithmetic calculations. If they are only asked to solve problems with algebra that could just as easily be solved with arithmetic calculations or by "guess and check," students will likely not see the point in using algebra.

To help bridge the gap between arithmetic and algebraic solution strategies we should put concerted efforts into helping students make connections between arithmetic and algebra, recognizing that formulating an equation is not an intuitive way for many students to represent a problem. We begin by making connections between pictorial and symbolic representations of unknown values, focusing on developing students' abilities to accurately represent word problems in equations.

Encouraging students to write sentences describing algebraic equations will help them learn to model using algebraic equations. For example, students who can translate $C = 2B$ to "There are twice as many carrots as bananas" are close to translating "there are twice as many carrots as bananas" to $C = 2B$.

EXAMPLE 17.

Brock ate 16 Girl Scout cookies in 5 days (he wasn't supposed to eat any cookies because they belonged to his sister.) The second day he ate 3 more than the first (he felt pretty bad about that.) The third day he ate half as much as the 1st day (he was able to get better control of himself.) The fourth and fifth days, he ate twice each day what he ate the first day (he really likes Girl Scout cookies.) How many cookies did he eat each day?

SOLUTION. The question illustrates the value of having a systematic way of encoding the information that leads to a straightforward algorithm for solving the problem. For such problems, algebra gives us a technique for discovering that some information is missing.

Some general guidelines for solving word problems are:

- Identify what is unknown and needs to be found. Represent this with a variable. x is the amount of cookies Brock ate on the first day

- Determine whether or not you have enough information to solve the equation. If so, write an equation that expresses the known information in terms of the unknown.

$$16 = x + (x + 3) + \frac{1}{2}x + 2x + 2x$$

- Solve the equation. $x = 2$

- Interpret the solution and ensure that it makes sense.

Brock ate 16 Girl Scout cookies in 5 days (he wasn't supposed to eat any cookies because they belonged to his sister.) The second day he ate 3 more than the first; that means he ate 5 cookies on the 2nd day. The third day he ate half as much as the 1st day; eating only 1 cookie. The fourth and fifth days, he ate twice each day what he ate the first day; eating 4 on both days.

$$16 = 2 + 5 + 1 + 4 + 4 .$$

Section 6.3: Solve and Graph Inequalities, Interpret Inequality Solutions

Grade 6 content included writing inequalities and graphing them on a number line. For example:

- write an inequality of the form $x > c$ or $x < c$ to represent a constraint or condition in a real-world or mathematical problem;
- recognize that inequalities of the form $x > c$ or $x < c$ have infinitely many solutions;
- represent solutions of such inequalities on number line diagrams.

Grade 7 develops further with solving and graphing one-step and multi-step inequalities using knowledge of solving one-step and multi-step equations. It is important that students understand the similarities and differences between finding the solution to an equation and finding solution(s) to an inequality, and the relationship of each to the real line.

In this section, language can become a difficulty for students. For example, in the preceding section we may have written "6 less 5," meaning 6-5. This is different from "6 is less than 5" (algebraically, $6 < 5$ which, incidentally, is false). So, whereas previously "less" and "greater" might have suggested operations, here the are to be taken as relations. When explicit numbers are replaced by unknowns, it is even more confounding: "x less y" means $x - y$ while "x is less than y" means $x < y$.

Let's begin by reviewing the symbols representing inequalities, and explore the work in simplifying inequalities between expressions.

Recall, from the discussion on equivalence at the beginning of this chapter: for a and b two numbers

- $a < b$ means "a is less than b," as in $\frac{1}{2} < \frac{3}{4}$.
- $a \leq b$ means "a is less than or equal to b," as in $\frac{10}{16} \leq \frac{5}{8}$.
- $a > b$ means "a is greater than b," as in $11 > 4$.
- $a \geq b$ means "a is greater than or equal to b," as in $11 \geq \frac{22}{2}$.

The realization of numbers as points on the line gives another very useful interpretation of these symbols: on the real line $a < b$ means that *a lies to the left of b*, and $a > b$ means that *a lies to to the right of b*.

Consider the statement $x < 4$. There isn't just one value of x that makes this open sentence true. The inequality would be true if $x = 3$ or if $x = 3.9$ or if $x = 0$ or $x = -2$ or $x = \frac{1}{8}$. In fact, it is impossible to list all the values that make this statement true. The solution set for this open sentence contains an infinite number of values. We can resolve this issue by representing solutions of inequalities graphically on the number line.

EXAMPLE 18.

Suppose $9x - 10 = 26$. Find the solution, and then represent the solution on a number line.

SOLUTION. Add 10 to both sides of the equation to get $9x = 36$. Now, divide both sides by 9 to get $x = 4$. To represent a single number on a number line, we draw a filled-in dot on that number, as in the following image:
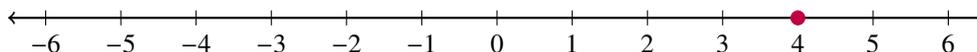
To show the solution set of the inequality $x \le 4$ on a number line, we fill in the point $x = 4$ and shade the region on the number line less than 4 like so:

The arrow indicates that the shading should continue forever, so numbers like -4 and even -100 are included.

Now, suppose we are given the relation

$$9x - 10 \le 26.$$

If we add 10 to both sides of the inequality, then, in the real line representation, everything is shifted to the right by ten units, so we have the relation $9x \le 36$, which has the same solution set. Now, if we divide by 9, we are just changing the scale by a factor of 1/9, so again we have the same solution set, but now written as $x \le 4$, with the same graph on the real line as above. Thus this is the graph of the solution set of the inequality $9x - 10 \le 26$.

Suppose now that we want to indicate $x < 4$ on the number line. We can read this inequality as "x is less than 4" or "x is strictly less than 4" to emphasize that $x = 4$ is not part of the solution set. We represent the solution on the number line as follows:

Notice that the dot over the number 4 is left open to show that the number 4 is not included in the set of values that makes $x < 4$ true.

For the most part we can work with inequalities in much the same way that we work with equations. But there are some important differences too.

As with equations, two inequalities are *equivalent* if they have the same solution set. When we work with equations, we can add (or subtract) the same number to both sides of the equation, and we can multiply (or divide) the same nonzero number to both sides of the equation, and the result is an equivalent equation. Do these same rules apply to inequalities? Let's go through this carefully, using the real line representation of the number system

- Add a number to both sides of an inequality: the solution set does not change. If the number is positive, the effect is to shift everything to the right by that number - by "everything" we mean both sides of the inequality. So if $a < b$, the $a + 10 < b + 10$, and $a - 3 < b - 3$, and so, if E and F are expressions, the solution set of $E < F$ is the same as the solution set $E + 10 < F + 10$ and $E - 3 < F - 3$, and so forth for any number replacing 10 and 3.

- Multiply both sides of an inequality by a positive number a: the solution set does not change. We can think of this as a rescaling: when we replace every number x by ax we are just replacing the unit length 1 by the unit length a. So the relation between two numbers remains the same, and so the solution set of $E < F$ is the same as the solutions set of $aE < aF$.

- Multiply both sides of an inequality by -1 and reverse the inequality: the solution set does not change. Recall that multiplication by -1 takes any point on the line to the point on the other side of 0 of the same distance from 0. 7 is two units further away from 0 as is 5, so the same must be true of -7 and -5. But whereas 7 is 2 units to the right of 5, -7 (being two units further away from 0) must be 2 units to the left of 5. Consider the statement: $-x > 3$. This decries the set of all points whose opposite is greater than 3. Now if $-x$ is to the right of 3, then x is to the left of -3; that is $x < -3$. So, we see , using the number line, that multiplication by -1 changes *left of* to *right of* and *right of* to *left of*. $5 < 7$ says that 7 is two units to the right of 5, and $-7 < -5$ says that -7 is two units to the left of five. In general, if $a < b$, then $-a > -b$; that is, for any a and b, both statements are true at the same time, or false at the same time. So, this is true of expressions: $E < F$ and $-E > -F$ have the same solutions set: if a number is inserted for the unknown in these expressions, the two statements are either both true or both false.

Consider for example the statement
$$15 > 10$$

If we multiply both sides by -2, we get $-2 \times 15 = -30$ on the left-hand side and $-2 \times 10 = -20$ on the right-hand side. If we think of the number line or temperature we can see that -20 is to the right of, or is 10 degrees higher than, -30 degrees. So, to get a true statement we have to reverse the "greater than" sign resulting in

$$-30 < -20 \,.$$

Solve word problems leading to inequalities of the form $px + q > r$ or $px + q < r$, where p, q, and r are specific rational numbers. Graph the solution set of the inequality and interpret it in the context of the problem. 7EE.4b.

EXAMPLE 19.

Solve and graph the solution set of the following inequality:

$$8(10 - x) < 20$$

SOLUTION.

$$(10 - x) < \frac{20}{8}$$
$$10 - x < \frac{5}{2}$$
$$10 - x + (-10) < \frac{5}{2} + (-10)$$
$$-x < \frac{5}{2} - \frac{20}{2}$$
$$-x < -\frac{15}{2}$$
$$x > \frac{15}{2}$$

And the solution set looks like:
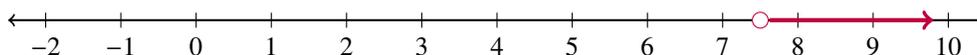
EXAMPLE 20.

Andy has $550 in a savings account at the beginning of the summer. He wants to have at least $200 in the account by the end of the summer. He withdraws $25 each week for food, clothes, and movie tickets. How many weeks will his money last?

SOLUTION. Recall there is not just one way to get to the solution of an equation, that is, many routes are possible provided each algebraic step is justifiable. The same can be said of inequalities.

Let's consider our known information; Andy starts with $550, takes away $25 each week, wants at least $200 in the end.

Let w = number of weeks the money can last. Our relationship describes the money Andy has, which needs to be more than (greater) or equal to $200.

$$550 - 25w \geq 200$$
$$-25w \geq -350$$
$$w \leq 14 \ .$$

Andy's money will last at most, 14 weeks.

EXAMPLE 21.

As a salesperson, you are paid $50 per week plus $3 per sale. This week you want your pay to be at least $100. Write an inequality for the number of sales you need to make, and describe the solutions.

Let x be the number of sales made in the week. This week you want,

$$50 + 3x \geq 100.$$

Solving this,

$$50 + 3x \geq 100$$
$$3x \geq 50$$
$$x \geq \frac{50}{3} = 16\frac{2}{3}$$

So, the conclusion is you will need to make at least $16\frac{2}{3}$ sales this week, and since $\frac{2}{3}$ of a sale isn't possible, you will need to make at least 17 sales to exceed $100 of earnings.

Extension

EXAMPLE 22.

A triangle has two sides of length 13 in and 6 in. What can be the length of the third side?

SOLUTION. The restrictions are given by the triangle inequality: the sum of the lengths of any two sides must be greater than the length of the third side. Letting a represent the length of the unknown side, this gives the restrictions:

$$a + 13 > 6, \qquad a + 6 > 13, \qquad 13 + 6 > a.$$

Since a must be positive, the first equation is satisfied. Subtracting 6 from both sides of the second equation gives the condition $a > 7$, and the third condition is $a < 19$. So the length of the third side is greater than 7 in and less than 19 in.

EXAMPLE 23.

A triangle has a perimeter of 20 units. One side is 9 units long. What is the length of the shortest side?

Let a be the variable to represent the shortest side. Here is a picture.
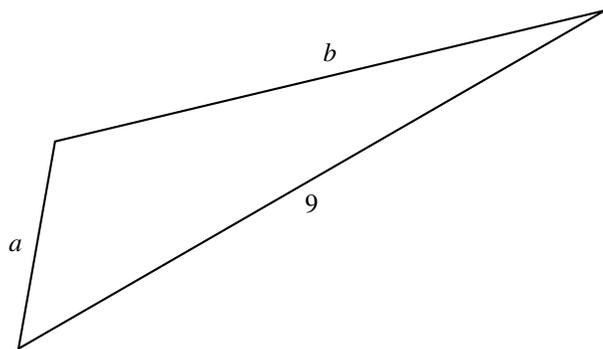
We know that $a > 0$ because it is a length. Also it is true that $a < 9$ since a is the shortest side. We will see later, that this is not so important.

If the perimeter of the triangle is 20 units, then we must have $a + 9 + b = 20$. So the third side b must be $20 - 9 - a$ units, or $11 - a$ units.

Since a is the shortest side, it must be less than the third side, $11 - a$. So,

$$a < 11 - a$$
$$a + a < 11 - a + a$$
$$2a < 11$$
$$a < \frac{11}{2}$$

If a is less than $\frac{11}{2} = 5.5$, then it automatically is also less than 9 units. So this condition that $a < \frac{11}{2}$ is a stronger restriction than $a < 9$.

In conclusion, $a > 0$ and $a < 5.5$. We can write this as $0 < a < \frac{11}{2}$, and it means the value of a lies between 0 and 5.5.

End Extension

Summary

To solve a given equation, we perform a sequence of steps that will replace the equation with an equivalent equation, such that the unknown is isolated on one side of the equation. The following actions result in equivalent equations and may be performed to solve an equation:

- Add/subtract the same quantity on both sides of the equation.

- Multiply/divide the same *nonzero* quantity on both sides of the equation.

- Replace an expression or quantity in an equation by an equivalent expression or quantity.

The process of setting up and solving equations is an art. There is not just one way to get to the solution of an equation. Many routes are possible provided each algebraic step is justifiable.

The same process works for solving inequalities. We must remember however, that if we multiply both sides of an inequality by a *negative* number, the direction of the inequality reverses.

Equations and inequalities arise in a variety of contexts in life including financial problems and geometry. If we carefully read the problem and write a corresponding equation or inequality, the properties of numbers and operations can help us do the rest of the work to solve many kinds of problems.