Chapter 7
Probability and Statistics

In this chapter, students develop an understanding of data sampling and making inferences from representations of the sample data, with attention to both measures of central tendency and variability. They will do this by gathering samples, creating plots, representing the data in a variety of ways and by comparing sample data sets, building on the familiarity with the basic statistics of data sampling developed over previous years. They find probabilities, including those for compound events, using organized lists, tables, and tree diagrams to display and analyze compound events to determine their probabilities. Activities are designed to help students move from experiences to general conjectures about probability and number. They compare graphic representations of data from different populations to make comparisons of center and spread of the populations, through both calculations and observation.

Statistics begins in the middle of the nineteenth century during the Crimean War, when Florence Nightingale (a nurse with the British Army) began to notice the excess of British soldiers dying in the hospital of "complications from their injuries" but not from the injuries themselves. Nurse Nightingale suspected that this could be attributed to the sterile (lack thereof) conditions of the field hospitals, and not to the nature of the injuries themselves. She began to gather data, both from field hospitals that attempted to maintain sanitary conditions and those that did not. Her goal was not just to uncover causes, but to suggest remedies that can be implemented instantly. She studied the data, correlating hospital practices with patient mortality, and concluded that there was one rule that, if applied faithfully, would significantly change the result: physicians should wash their hands. When she presented this suggestion to the high command, the response did not meet the urgency she felt. So, she appealed to Queen Victoria: she invented bar graphs to present these data so that the Queen could visualize the significance of her suggestion. It worked: the Queen issued an order that physicians should wash their hands, and that was the beginning of scientific statistics and modern medical practice.

Probability and Statistics are intricately entwined, but historically, the origins are quite distinct. Probability questions arise naturally in games of chance, and over the centuries gamblers placed their faith (and money) on rules that, with or without any foundation, had become folklore. Recall from Chapter 1 that, n the mid-seventeenth century, the Chevalier de Méré asked the mathematician Blaise Pascal about a rule in a game of dice that, unfortunately, did not work for him. Pascal began a correspondence with Pierre Fermat (two of the leading mathematicians of the century), and between them a theory of probabilities developed that accounted for de Méré's misfortune.Today that theory is fundamental in the study of many processes, particularly biological and economic ones, where there is the possibility of random influences on the sequence of events.

Section 1 begins with an exploration of basic probability and notation, using objects such as dice and cards. Students will develop modeling strategies to make sense of different contexts and then move to generalizations. In order to perform the necessary probability calculations, students work with fraction and decimal equivalents. These exercises should strengthen students' abilities with rational number operations. Some probabilities are not known, but can be estimated by repeating a trial many times, thus estimating the probability from a large number of trials. This is known as the Law of Large Numbers, and will be explored by tossing a Hershey's Kiss many times and calculating the proportion of times the Kiss lands on its base.

Section 2 investigates the basics of gathering samples randomly in order to learn about characteristics of populations, in other words, the basics of inferential statistics. Typically, population values are not knowable because most populations are too large and their characteristics too difficult to measure. “Inferential statistics” means that samples from the population are collected, and then analyzed in order to make judgments about the population. The key to obtaining samples that represent the population is to select samples randomly. This is not always easy to do, and an important part of this chapter is to think about what “random sample” means. Students will gather samples from real and pretend populations, plot the data, perform calculations on the sample results, and then use the information from the samples to make decisions about characteristics of the population.

Section 3 uses inferential statistics to compare two or more populations. In this section, students use data from existing samples and also gather their own data. They compare plots from the different populations, and then make comparisons of center and spread of the populations, through both calculations and visual comparisons.

This unit introduces the importance of fairness in random sampling, and of using samples to draw inferences about populations. Some of the statistical tools used in Grade 6 will be practiced and expanded upon as students continue to work with measures of center and spread to make comparisons between populations. Students will investigate chance processes as they develop, use, and evaluate probability models. Compound events will be explored through simulation, and by multiple representations such as tables, lists, and tree diagrams.

The eighth grade statistics curriculum will focus on scatter plots of bivariate measurement data. Bivariate data are also explored in Secondary Math I. However, statistics standards in Secondary Math I, II, and III return to exploration of center and spread, random probability calculations, sampling and inference.

The student workbook begins with the anchor problem: the game “Teacher always wins!” The purpose of this particular activity is to start thinking about what kind of data are needed to resolve a problem (in this case, the apparent unfairness of the game), and secondarily, to illustrate that such resolution is not always as simple as it originally seems. This point is made again towards the end of the first section with the Monty Hall problem. In the meantime, the problem introduces the student to all of the fundamental ideas of this chapter. In order to illustrate these goals, as will be done in the succeeding sections, let’s first look at this type of problem in the context of simple games.

What is a fair game?

A *game*, actually, a “simple game,” has *players* (2 or more); a *tableau*: the field on which the game is played; *moves*: the set of actions that the players can make on the tableau; and *outcomes*: the set of end positions on the tableau. Finally there is a rule to decide who is the winner. This may be described as a rule that assigns, to each outcome, one of the players as winner. In the language introduced in Chapter 1, for each player *A*, the statement “A is the winner” is an *event*. The entire set of outcomes is partitioned into the events *A is a winner* over all players *A*. We note that in many casino games, each player *A* decides on the event “A is the winner.” by placing a bet on a certain event (as in “red” or “even” on the roulette table), In this case, since the events may overlap, the game is not “simple,” but compound. A game is called *fair* if all the outcomes are equally likely. An example is the game of rolling dice, as discussed in Chapter 1. In the following we explore another example: *spinner games*.

a. First game: Two spinner game. This is a game for two players: each has a spinner partitioned into five sectors of equal areas, and each sector has one of the numbers $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ in it, and there is no number common to both spinners. A move consists of a spin by both players, and the winner is the spinner with the higher number.

Is this a fair game? We see right away that it need not be: if player A has $\{0, 1, 2, 3, 4\}$ and player B has $\{5, 6, 7, 8, 9\}$, then player B always wins. So, is there a configuration that is fair? For example, if player A has all the odd digits, and B has all the even ones, is this fair?

To answer such questions we first list all the possible outcomes, and then divide that set into two pieces: *A* where player A wins, and *B* where player B wins. If these sets have the same number of outcomes, then it is a fair game.

An outcome of this game is a pair of numbers (a, b), where the A needle lands on the sector marked a, and the B needle lands on b. If $a > b$, then (a, b) goes in the set A; otherwise $a < b$ and (a, b) goes in the set B.

Example 1.

Suppose that player A's spinner has the odd digits and B's spinner has all the even digits. Is this a fair game? Is there any configuration that gives rise to a fair game?

Solution. The sample space of outcomes consists of all pairs of numbers (a, b) where a is an odd digit, and b is an even digit. Since there are 5 even digits and 5 odd digits, the number of pairs (a, b), with a odd and b even is 25. Since 25 is odd, we cannot split the sample space into two sets, both with the same number of outcomes. So this cannot be a fair game. But which player has the edge, A or B? The answer to the second question is "no" by the same reasoning, although sometimes A may have the edge, and sometimes B.

Example 2.

Suppose that A's spinner has 6 sectors, marked {0, 2, 4, 6, 8, 10} and B's spinner has the 5 sectors {1, 3, 5, 7, 9}. Is this a fair game?

Solution. This time there are $6 \times 5 = 30$ possible outcomes, so this could be a fair game. The event "A wins" consists of all pairs (a, b) with $a > b$. If $a = 0$, A loses to all of B's spins; if $a = 2$, A loses to 4 of B's spins, and if $a = 4$, A loses to 3 of B's spins, and so forth. So, in all A loses in $5 + 4 + 3 + 2 + 1 = 15$ outcomes. Since there are 30 outcomes, A also wins in 15 outcomes, so this is a fair game

b. Second game: Player B always wins. In this game the tableau consists of four spinners, Red, Blue, Green, Yellow, each with three sectors marked with these numbers:

$$\text{Red} : \{3, 3, 3\} \quad \text{Blue} : \{4, 4, 2\} \quad \text{Green} : \{5, 5, 1\} \quad \text{Yellow} : \{6, 2, 2\}$$

First, player A selects a spinner and then player B selects a spinner from among those remaining. Now, they spin the spinners, and the player who shows the higher number wins. In case of a tie, they spin again. Let's analyze one set of choices: suppose A picks Blue, $\{4, 4, 2\}$, and B picks Yellow, $\{6, 2, 2\}$. There are nine outcomes, that is all pairs (a, b) where $a$is the number spun by player A, and b is a the number spun by player B. Since there are repetitions, let us distinguish the pairs by their places, so A has $\{4_1.4_2, 2\}$, and B has $\{6, 2_1.2_2\}$. Now we can count the wins:

$$\text{A wins} : \ (4_1, 2_1), \ (4_1, 2_2), \ (4_2, 2_1), \ (4_2, 2_2) \ ;$$

$$\text{B wins} : \ (4_1, 6), \ (, 4_2, 6), \ (2, 6) \ ;$$

$$\text{Tie} : \ (2, 2_1), (2, 2_2) \ .$$

Since a tie leads to the same scenario, where A has more wins than B, no number of ties will compensate for the fact that the odds favor an A win. So a this is not a fair game.

Example 3.

Once A has made a choice, is there a particular choice for B that favors B winning? Hint: we wouldn't ask the question if the answer weren't "yes."

Solution. Another clue is the label of this game. In fact, the odds favor player B if B always chooses the spinner listed directly after the spinner chosen by A! (This means, for example, that if A picks the yellow spinner, B should pick the red).

c. Third game: Four spinning players. Now let's have four players: Red, Blue, Green and Yellow, one for each spinner. Each player spins, and the highest number wins.

EXAMPLE 4.

Is this a fair game?

SOLUTION. There are $3 \times 3 \times 3 \times 3 = 81$ outcomes, since each of the four spinners can produce three numbers. For this to be a fair game, we would have to partition these 81 outcomes into four events, all with the same number of outcomes. Since $81/4$ is not an integer this cannot be done. Wait! Maybe there is a tie. Well, there are no ties, for the only duplicated number is 2, and if Blue and Yellow show a 2, Red always shows a 3, so Red wins if Green shows a 1, otherwise Green wins. To see if this is a fair game: count the number of outcomes that produce a win for each player.

The result is this: Red wins in 6 outcomes, Blue in 12, Green in 36 and Yellow in 27. At first it seems surprising that when only two spinners are used, we can have a bias toward any color, but when all four are spun, Green wins by a long shot.

In the above discussion we used the term *odds*.This term is used to describe the ratio among a set of events that partition a sample space. This is not an easy phrase to digest, so let us illustrate. In flipping a fair coin, the odds of "heads" to "tails" are 1:1. In rolling a fair die, the odds of the numbers turning up is 1:1:1:1:1:1. However, the odds of getting a number divisible by 3 are 2:4 since the set {1,2,3,4,5,6} has two numbers divisible by 3, and four that are not. We could also say that the odds are 1:2, since these both describe the same ratio.

In Example 1 the odds of A winning are 15:10, or 3:2; in Example 2, the odds vary, depending upon the choice A makes, but (if B makes the right choice), the odds always favor B (that is the odds are $a : b$, with b always larger than a. In example 3, the odds are 6:12:36:27, or (since all numbers are divisible by 3), 2:4:12:9.

The message here is simply this: uncovering biases in a game is not easy; one must choose outcomes and assign them to players as wins according to the given rules.

Section 7.1: Analyze Real Data and Make Predictions using Probability Models.

The importance of an understanding of probability and the related area of statistics to becoming an informed citizen is widely recognized. Probability is rich in interesting problems and provides opportunities for using fractions, decimals, ratios, and percent. For example, when we ask "what are the chances it will snow today?" or "what are the chances I will pass my math test?" or any question filled with the phrase "what are the chances?" we are really asking "what is the probability that something will happen." The subject of probability arose in the eighteenth century in connection with games of chance. But today it is an essential mathematical tool in much of science; in any phenomenon for which there is a random element (such as life), probability theory naturally comes up. For example, the basic concept in the life insurance business is to understand, in any given demographic, the probability of a person of age X living another Y years. In business and finance, probability is used to determine how best to allocate assets or premiums on insurance. In medicine, probability is used determine how likely it is that a person actually has a certain disease, given the outcomes of test results.

This section starts with a review of concepts from Chapter 1 Section 1 and then extends to a more thorough look at probability models. A complete probability model includes a sample space that lists all possible outcomes, together with the probability of each outcome. The probability may be considered as the relative frequency of the model (that is, in an ideal situation, the fraction of times the given outcome comes up in a given number of trials. The sum of the probabilities from the model is always 1 (reflecting the fact that any experiment always leads to an outcome in our sample space). A *uniform probability model* will have relative frequency probabilities that are the same, namely $1/n$ where n is the number of distinct outcomes. An event is any set of possible outcomes.

A probability model of a chance event (which may or may not be uniform) can be approximated through the collection of data and observing the long-run relative frequencies to predict the approximate relative frequencies. Probability models can be used for predictions and determining likely or unlikely events. There are multiple representations of how probability models can be displayed. These include, but are not limited to: organized lists, including a list that uses set notation, tables, and tree diagrams.

Approximate the probability of a chance event by collecting data on the chance process that produces it and observing its long-run relative frequency, and predict the approximate relative frequency given the probability. For example, when rolling a number cube 600 times, predict that a 3 or 6 would be rolled roughly 200 times, but probably not exactly 200 times. 7.SP.6

In Chapter 1 we discussed the experiments of John Kerrich with coin tosses. If we wanted to reproduce his experiment, we'd have to accept that it takes a considerable amount of time to toss a coin 10,000 times as he did. We might want to look for an easier way of reproducing the same situation and run that. This is what is called a *simulation*: a parallel experiment that has all the same properties as the coin toss. Such is possible using a spreadsheet on a computer. Spreadsheets (such as Excel) have a *random number generator*; it could be arranged to produce a 0 or a 1 (or an H or a T) at random. The spreadsheet can be programed to do this 10,000 times in just a few seconds, and tabulate the results. We could also simulate the spinner games by programming the random number generator to produce of on the first six whole numbers at random.

At this point, it will help to think of a *random process* as an experiment (to be repeated many times) whose outcome is one of a well-defined set of outcomes, all of which are equally likely. Actually, this is a *uniform random process*; we may have some reason to assign different probabilities to each outcome; this would be a *non-uniform* random process. Repeated tosses of a fair coin, repeated twirls of a fair spinner, (one in which all sectors have the same central angle), repeated tosses of a fair die; these are all uniform random processes. However, in more complicated situations, it depends upon what is seen as the outcome. For example, suppose the experiment is to toss a pair of dice, and the outcomes are the possible sums. Then the possible set of outcomes is the set of whole numbers between 1 and 12. Clearly, the outcomes are not equally likely: a 1 is impossible, and a 2 is far less likely than a 7, and so forth.

Example 5.

The athlete Maria wants to understand the randomness in her success at making baskets from the free throw line. She knows that she has about 50% accuracy, but that doesn't mean that she scores every other time she shoots the basket. Sometimes she can make two or three in a row; and other times she can miss as often. Maria would like to understand if this is about her, or a natural consequence of the randomness. So, she decides to simulate shooting 25 free throws. Using a coin (or a computer program) she lets heads represent the ball going into the basket and tails represent the ball missing the basket. Each toss of the coin represents a shot at the free throw line.

Maria begins by making a table or filling in the one below. She will mark an '*x*' in the appropriate column for each toss of the coin. Here are her first ten entries:

Toss number	Made the basket (heads)	Missed the basket (tails)
1	x	
2		x
3		x
4	x	
5	x	
6		x
7	x	
8	x	
9	x	
10		x

Questions for Maria to consider:

1. How many free throws went into the basket?
2. How many free throws missed the basket?
3. What outcome does she expect in the next free throw?
4. What is the theoretical probability that the next toss of the coin will show success (that is, come up heads)?
5. Suppose that we replace Maria's estimate of her rate of accuracy being 50% by her actual experience, that in the long run, she makes the basket two times out of three. Can you devise a new model based on this assumption of Maria's ability?

SOLUTION. The answers to the first two questions are 6 and 4 respectively. Questions **3** and **4** are more interesting. Since she knows that, on average, she gets one basket out of every two attempts, her expectation is that she will make the next basket. If instead she misses, her expectation of a "make" on the next try is even stronger, and so forth. However (in answer to **4**), the theoretical probability of a "make" on any attempt is *always* 50%, no matter how many misses precede the attempt. Probability theory tells us that in the long run, she'll have successes to make up for a sequence of misses, but we have no way of knowing on which attempt this will start. As for **5**, one possible simulation is to roll a die: a 1 or a 2 represents a missed basket; otherwise, she scores.

The objective for students in this section is to collect data from an experiment representing a random process. It may be a simulation, as in the above example, or it may be based on actual trials (in the case of the example, Maria actually shooting free throws). The goal is to recognize that as the number of trials increase, the experimental probability approaches the theoretical probability. This tendency is called the *Law of Large Numbers* In this standard we focus on relative frequency: the ratio of successes to the number of trials. The Law of Large Numbers tells us that, as the number of trials increases, this ratio should get closer and closer to the actual (or theoretical) probability. So, in the case of Maria's simulation, those ratios (expressed as decimals) are

$$1.0, 0.5, 0.33, 0.5, 0.6, 0.5, 0.57, 0.62, 0.66, 0.6\ .$$

Read this as follows. Maria makes the first toss, so her average after one toss is 1.0. She misses the next, so after two tosses her average is 0.5. She misses the next, so she has made 1 out of 3 and her average is 0.33. She makes the next, so now she has made 2 out of 4 tosses, and her average is once again 0.5. And so on...

Toward the end. we see a tendency toward 0.5,(she has even odds of making the free throw), but there haven't been enough trials to distinguish the result from the theoretical probability of the simulation to Maria's experience that she should make 2 baskets out of 3.

A *fair game* is one in which each player has an equal chance of winning the game. Tossing a coin is considered a fair game, since there is an equal chance that a head or a tail will come up. Maria shooting baskets alternately with the point guard of her school basketball team is probably not a fair game (unless Maria is the point guard). Keep in mind, because a game is fair, this doesn't mean that in any set of repetitions the wins will be equal; one could toss a fair coin six times and get six heads.

Develop a probability model and use it to find probabilities of events. Compare probabilities from a model to observed frequencies; if the agreement is not good, explain possible sources of the discrepancy.

Develop a uniform probability model by assigning equal probability to all outcomes, and use the model to determine probabilities of events. For example, if a student is selected at random from a class, find the probability that Jane will be selected and the probability that a girl will be selected.

Develop a probability model (which may not be uniform) by observing frequencies in data generated from a chance process. For example, find the approximate probability that a spinning penny will land heads up or that a tossed *paper cup will land open-end down. Do the outcomes for the spinning penny appear to be equally likely based on the observed frequencies? 7.SP.7*

Example 6.

The Addition Game. Roll two dice (also called *number cubes*) 36 times. On each roll: if the sum of the two faces showing up is odd, player #1 gets a point; If that sum is even, player #2 gets a point.

The winner is the one with the most points after 36 rolls. Is this a fair game?

a. Play the game. Based on your data, what is the experimental probability of rolling an odd sum? An even sum?

P(odd) = ...   P(even) = ...

b. Find all the possible sums you can get when rolling two dice. Organize your data.

c. What is the theoretical probability of rolling an odd sum? An even sum?

P(odd) = ...   P(even) = ...

d. Do you think the addition game is a fair game? Explain why or why not.

Let us pause at this point to review the basic concepts of the probability theory as it has been developed so far, continuing from Chapter 1. First of all, in every example, be it tossing a coin, rolling a die or twirling a spinner, we are concerned with an *experiment*: an activity that can result in one of a set of *outcomes*. In our context - we may have defined "success" by a certain subset of outcomes. For a tossed coin, the set of outcomes is $\{H, T\}$; and if our interest is obtaining a head, then the success outcome is H. For a rolled die, the outcomes are the faces: $\{1, 2, 3, 4, 5, 6\}$. If success is defined as getting an even number, then the set of outcomes of interest is $\{2, 4, 6\}$. For any experiment, we need to list all possible outcomes: this list is called the *sample space*. A subset of the sample space is called an *event*. In a given context, a certain subset of the sample space is the event we are aiming for: the "success." The experiment is usually run many times in an attempt to discover what the probability is of success. This leads to what we call the *experimental probability*. In contrast, the *theoretical probability* (in a uniform random process) is the quotient of the number of outcomes in the success event divided by the total number of events. Often the set of possible outcomes is so large, that we estimate the probability of success by experiments, or by modeling. For example, instead of doing what John Kerrich did, we can now model 10,000 coin tosses by a con-tossing machine, or by a computer program that randomly chooses H or T successively for many times.

Example 7.

Suppose there are 100 balls in a bag. Some of the balls are red and some are yellow, but we don't know how many of each color ball are in the bag. Bailey reaches into the bag 50 times and picks out a ball, records the color, and puts it back into the bag. Here the sample space is the set of balls in the bag, and the "success" event is the set of yellow balls in the bags. If Bailey picked 16 yellow balls, then the experimental probability of picking a yellow ball is:

$$16/50 = 32\% .$$

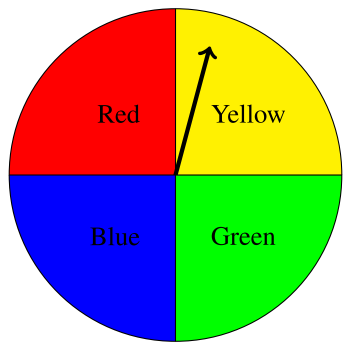
This 32% is likely to be close to the theoretical probability of picking a yellow ball, since 50 is quite a large number, relative to 100. However, if she had only picked 5 balls, three of which were yellow, then her experimental probability would have been 60%. Clearly Bailey will have more confidence in the experimental probability of the larger sample.

Find probabilities of compound events using organized lists, tables, tree diagrams, and simulation. Understand that, just as with simple events, the probability of a compound event is the fraction of outcomes in the sample space for which the compound event occurs.

Develop a probability model (which may not be uniform) by observing frequencies in data generated from a chance process. 7.SP.8.

EXAMPLE 8.

In some games that use spinners, the spinner is equally likely to land in red, yellow, blue, or green. If Sarah is allowed two spins, what are all the possible outcomes?



SOLUTION. We could make an organized list, table or tree diagram to show all the possible outcomes.

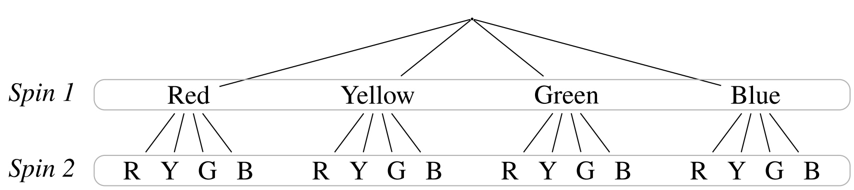
Organized List

Red, Red	Yellow, Yellow	Green, Green	Blue, Blue
Red, Yellow	Yellow, Red	Green, Red	Blue, Red
Red, Green	Yellow, Green	Green, Yellow	Blue, Yellow
Red, Blue	Yellow, Blue	Green, Blue	Blue, Green

Table

		Spin 2			
		Red	**Yellow**	**Green**	**Blue**
Spin 1	**Red**	RR	RY	RG	RB
	Yellow	YR	YY	YG	YB
	Green	GR	GY	GG	GB
	Blue	BR	BY	BG	BB

Tree Diagram (Vertical)



To determine how many different possible outcomes there are to this two-stage experiment first observe that there are 4 possible outcomes for the first spin (Spin 1) and four possible outcomes for the second spin (Spin 2). Each of the methods show that there are 16 different paths, or outcomes, for spinning the spinner twice. Rather than count all the outcomes, we can actually compute the number of outcomes by making a simple observation. Notice that there are four colors for the first spin and four colors for the second spin. We can say there are 4 groups of 4 possible outcomes which gives 4×4 (or 16) possible outcomes for the two-stage experiment of giving this spinner two spins.

> **Fundamental Counting Principle**
>
> If an event A can occur in m ways and event B can occur in n ways, then events A and B can occur in $m \cdot n$ ways. The Fundamental Counting Principle can be generalized to more than two events occurring in succession.

Example 9.

What is the probability that in 2 spins, the spinner will land first on blue and then on yellow? Table 2 below shows the outcomes both as a fraction F, and a percent P, of spinning the spinner.

	RR	**RY**	**RG**	**RB**	**YR**	**YY**	**YG**	**YB**	**GR**	**GY**	**GG**	**GB**	**BR**	**BY**	**BG**	**BB**
F	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$
P	6.25 %	6.25%	6.25%	6.25%	6.25%	6.25%	6.25%	6.25%	6.25%	6.25%	6.25%	6.25%	6.25%	6.25%	6.25%	6.25%

Table 2

Since each of the 16 spin outcomes is equally likely, and spinning a blue and then a yellow is just one event, its probability 1/16=6.25%. Note that the sum of the probabilities of all outcomes is one, indicating that all possible outcomes are shown.

We may also ask: what is the probability of spinning a blue and a yellow, in either order? Looking at the list, this event has two possible outcomes : YB and BY, so its probability is (1/16) +(1/16) = 1/8.

Example 10.

Roll two dice and examine the top faces.

- **a.** What is the probability of rolling two dice and getting two threes?
- **b.** What is the probability of getting any pair?
- **c.** What is the probability of at least one die showing a three?
- **d.** What is the probability of getting a three and an even number?

Solution.

- **a.** Here the sample space is all possible outcomes of rolling two dice: that is, all pairs (a, b) where a and b each run through the integers 1,2,3,4,5,6. There are 36 such pairs, and only one is (3,3). Thus the theoretical probability is 1/36.

- **b.** Since there are 6 doubles, the probability is 6/36, or one-sixth.

- **c.** We look at all pairs $(3, b)$: there are six of them. Now look at all pairs $(a, 3)$: here are another six. However, the pair (3,3) has been counted twice, so the number of pairs with at least one three is 6+6-1 = 11, and the probability of rolling at least one three is 11/36. Students may want to perform this experiment 36 times to see if they get an experimental probability close to this theoretical probability.

- **d.** From **c** we know that there are 11 outcomes with at least one three. From these listed outcomes, the ones that show a 3 on one face and an even on the other are $\{(3, 2), (3, 4), (3, 6), (2, 3), (4, 3), (6, 3)\}$. Thus there are six outcomes in the success event: a three and an even number. The theoretical probability then is 6/36, or 1/6.

There is another way of doing part **a**. The proposed experiment is the same as that of rolling one die twice in a row. In order to get two threes we must get a three on the first roll, and then a three on the second. The chance of getting a three in the first roll is 1 in 6 (and thus a probability of 1/6). After that there is again a 1 in 6 chance of getting a three on the second roll. Therefore, the chance of getting a three on two consecutive rolls is 1/6 of 1/6 or $\frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$. We can now also answer the question: what is the chance of rolling three consecutive threes? Well, there is a 1 in 36 chance of getting two consecutive threes, and after that 1 in 36, a 1 in 6 chance of getting a third three. So altogether there is a (1/36)(1/6) = 1/216 chance of getting three consecutive threes.

An event of this type is called a *compound event*; that is, a compound event is an event that can be viewed as two (or more) simpler events happening simultaneously. If the simple events do not influence each other, they are called *independent events*. When rolling a die two times in a row, the two rolls are independent. In this case, we can calculate the probability of the compound event as the product of the probabilities of the simple events. To ilustrate: what is the probability of rolling a three and then an even number? We analyze the problem this way: the probability of rolling a three is 1/6, and that of rolling an even number 1/2. Thus the probability of rolling a 3 *and then* an even number is (1/6)(1/2) = 1/12.

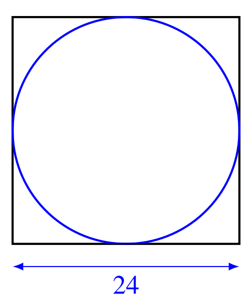
Example 11.

On average, Maria scores 10 or more points in 50% of the games, and Izumi does so in 40% of the games. What is the probability of both Maria and Izumi scoring 10 points in a game?

Solution. This is a compound event, so we look at it this way: the (experimental) probability that Maria scores 10 points or more is 0.5, and that for Izumi is 0.4 so the probability that both will happen is the product, (0.5)(0.4) = 0.2. Therefore, in 1/5 of the games, Maria and Izumi will each score 10 or more points.

Example 12.

Jamal is preparing for this competition: a square of side length 24 inches with an inscribed circle (see the diagram) is placed on the floor, and a line is drawn 8 feet away from the square. Competitors have to toss a small bean bag from behind the line, and get a point if the bag lands inside the circle. Jamal has honed his skills so that he knows that he can hit the square every time, but otherwise cannot affect where it lands. Given this he wants to determine the probability that he will strike the target somewhere within the circle.



Solution. Here the sample space is the square; due to Jamal's skill. Success is defined as landing in the circle, so the probability of success is the quotient of the area of the circle by the area of the square: the square has area 48 in $\times$ 48 in = 2304 in^2. The area of the circle is $\pi \cdot r^2 = \pi(24^2)$ or, approximately, 1809.56 in^2

The probability that Jamal would strike anywhere within the circle's target range would be

$$\frac{1809.56}{2304} = 0.785 = 78.5\% \ .$$

.

Now we move on to more complicated examples in order to demonstrate the value of tables, organized lists and tree diagrams in order to determine probabilities in compound experiments.

Example 13.

Ted and Mikayo are going to play a game with one die. At each toss Ted wins if the upturned face is even, and Mikayo wins if that face is odd.

Solution. The sample space consists of the set of all possible outcomes 1, 2, 3, 4, 5, 6. The event Even is the set of outcomes 2, 4, 6 and Odd is the set 1, 3, 5. Assuming a fair die, that is, all outcomes are equally probable, then since the sample space is partitioned evenly into the events "Ted Wins" and "Mikayo Wins," this is a fair game.

Example 14.

After a while, Ted and Mikayo get bored, and change the game. The die is rolled twice, and Ted wins if the sum is even, and Mikayo wins if the sum is odd. The sample space is now the set of all outcomes of two rolls of the die, that is (a, b), where *a* and *b* run through all positive integers less than or equal to 6. The event making Ted the winner is "a + b is even." and the event Mikayo wins by is "a + b is odd." Check that this is a fair game.

	1	**2**	**3**	**4**	**5**	**6**
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Example 15.

Now Ted and Mikayo turn to spinner games. They each take a spin, and Ted is the winner if there is at least one green; otherwise Mikayo wins. In Table 1, all the outcomes are equally possible. The event that Ted wins: "at least one green" has 7 outcomes, so the probability that Ted wins is 7/16, and that Mikayo wins is 9/16. Note that this is not a fair game!

So Ted suggests a new game: Ted wins if there is at least one green or one yellow, and Mikayo wins if there is at least one red or one blue in three spins. Sounds fair, but is it? Explain.

Example 16.

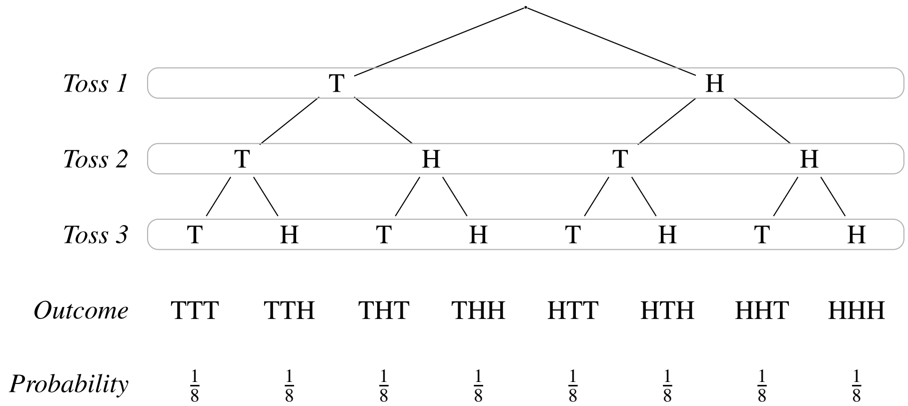
Kody tosses a fair quarter three times. What is the probability that two tails and one head in any order will result?

Solution. Tossing a coin repeatedly involves independent events. For example, the outcome of the first coin toss does not affect the probability of getting tails on the second toss. Kody's list of the sample space for the toss of three coins is written as

$$\{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

and can be displayed as a tree diagram, with accompanying outcome and probability.

This tree diagram makes clear that there are eight possible outcomes for the experiment "toss a coin three times," all of which are equally likely, so each has probability 1/8. We could also consider this a compound event, made up of the three successive events "toss a coin." Since the probability of each outcome of each event is 1/2, we multiply the probabilities, and again find that each outcome of three tosses has $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$ probability.



> Now we can solve the problem. The event of two tails and one head in any order consists of the three outcomes: HTT, THT, and TTH, so the probability of this event is 3/8 The event "at least one tail" consists of all outcomes but for HHH, thus has seven outcomes, and its probability 7/8.

This example illustrates certain important principles for finding probabilities. First of all, given an experiment to be performed, we first determine the set of all possible outcomes. The outcomes must be mutually exclusive, that is, it cannot happen that two outcomes can happen simultaneously. In rolling a pair of dice, we cannot have one outcome be "the sum of the dice is 7", and another be, "one of the dice is a three," because the roll (3,4) is in both of these outcomes. It is important that the outcomes be most elementary observations that could be made. In the case of rolling a pair of dice, the outcomes are pairs of integers between 1 and 6. Then "the sum of the dice is 7" is the event consisting of all pairs, the sum of whose faces is 7. With this understanding, the probability of an event is the sum of the probabilities of the outcomes in that event.

To illustrate with the example of the roll of three dice, the event is "2 tails and 1 head," and it consists of the three outcomes HTT, THT, TTH, so has probability 3/8.

More can be said: if have two events in a given experiment that have no outcomes in common, then the probability of either event happening is the sum of the probabilities of the two events. Consider, for example, the probability of getting either precisely two heads or precisely two tails in three coin tosses. Since there are only three coins, we cannot have both two heads and two tails, so there is no outcome common to both. Thus the probability of either precisely two heads or precisely two tails is 3/8+3/8 = 3/4.

When do we add probabilities and when do we multiply them? If an event can be viewed as either of two events *in the same experiment* happening, and the two events have no outcomes in common then we add their probabilities to get the probability of the main event. If an event can be viewed as two events in *different experiments* happening simultaneously, we multiply the probabilities of the component events to find the probability of the main event. This is analogous to working with lengths: when we add lengths we get another length, but when we multiply lengths we get area.

Section 7.2: Use random sampling to draw inferences about a population

In this section students will be looking at data of samples of a given population, and then making inferences from the samples to the population. Students will utilize graphs of data along with measures of center and spread to make comparisons between samples and to make an informal judgment about the variability of the samples. After examining the samples, then students make conclusions about the population.

It is important that students think about the randomness of a sample as well as how variations may be distributed within a population. These ideas are quite sophisticated. Activities within this section are designed to surface various ideas about sampling.

Understand that statistics can be used to gain information about a population by examining a sample of the population; generalizations about a population from a sample are valid only if the sample is representative of that population. Understand that random sampling tends to produce representative samples and support valid inferences.7.SP.1

EXAMPLE 17.

Mrs. Moulton was curious to know what proportion of the 7th grade students at her school chose pizza as their favorite menu item for lunch from the school cafeteria. She asked her 7th grade 1st and 2nd periods to help her find an answer to this question. Mrs. Moulton's students realized that they will not be able to interview every 7th grader so instead, they considered the 1st and 2nd period classes to be 'random samples' and took a poll in each class to determine if their favorite lunch menu item was pizza, hamburger, salad, or if the question didn't apply, because either they brought a lunch from home or had no lunch at all. Once the data were gathered, the students were asked to answer the following questions:

a. Create a bar graph to view the sample data for each class, using percentage data.

b. Describe the differences and similarities between the data from the two classes.

c. Based on these two samplings, do you think your class data is representative of the 7th grade?

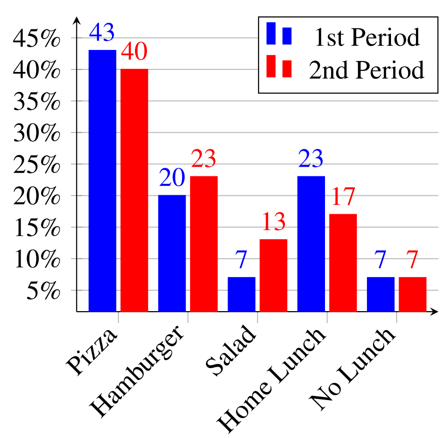
d. If the sampling is representative, what conclusions could you draw?

e. Create a bar graph of the combined sampling, using percentage data.

f. Compare your original class sample and the combined sampling, using percentage data.

SOLUTION.

a. This is the graph.



When comparing data from two populations, it is more useful to use percentage data (called a *relative frequency graph*) rather than the raw data. For example, if Mrs. Moulton's first class had 40 students, of whom 15 selected "pizza" and her second class had 60 students of whom 20 selected "pizza", then the raw data graph would suggest that pizza were more highly favored in the second class than in the first. But in fact, only 33% of the second class chose pizza, while over 37% in the first class did so. More importantly, Mrs. Moulton wants to predict the *proportion* of students favoring pizza, and so we should be looking at proportional, not raw, data.

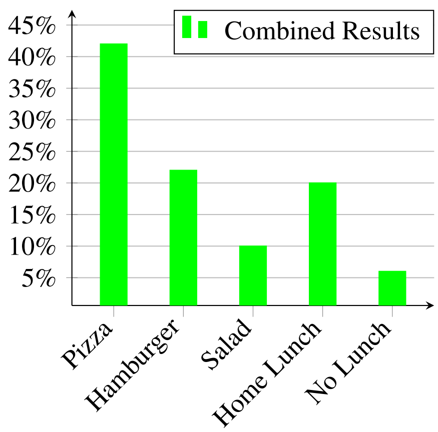
b. Some students may say that the graphs are almost the same, others may say that a greater percentage of students in the second class preferred salad. Others may say that there is no difference, for the ranking order (pizza, hamburger, home lunch, salad, no lunch) is the same for both classes. The question really is "are there *significant* differences?" and students should know that there are statistical measures of significance, but their use has to be justified in terms of the context. In this case, relative to the question posed by Mrs. Moulton, we would conclude that there is no significant difference.

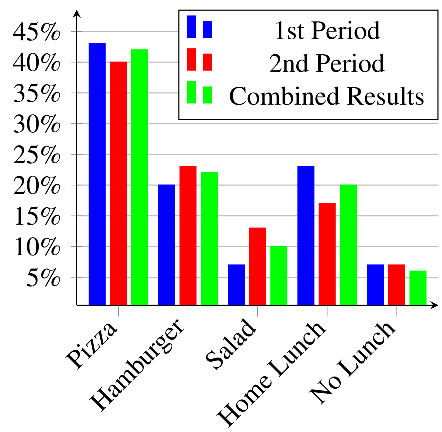
c. Of course this is a subjective question, but students should learn that there are statistical measures of the amount of confidence the researcher can put in these graphs, and again, that use of those measures has to be justified in terms of the context.

d. Accepting the sampling as representative, it is fair to conclude that more than 40% of 7th graders choose "pizza."

e. Graph 2 is a relative frequency graph for the combined data, and Graph 3 has put them all together.

f.





There is little doubt that, as far as the question (what proportion of 7th grade students prefer pizza?), the conclusion stated in part **d** is confirmed by either data set and the combined data set. One might go further and express confidence in the ordering of the preferences, and that the "significant" disagreement in the "salad" casts doubt on any prediction of the proportion of salad eaters in the class.

What do we mean by a random sample? A random sample is a subset of individuals (a sample) chosen from a larger set (a population). Each individual is chosen randomly and entirely by chance, such that each individual has the same probability of being chosen at any stage during the sampling process, and each subset of k individuals has the same probability of being chosen for the sample as any other subset of k individuals. A simple random sample is an unbiased surveying technique.

In the student workbook homework section 7.2a, students will work on a problem titled *Inquiring Students Want to Know!* The premise of the problem is to make a list of topics of student interest and to design a sampling method for collecting data from 10 or more randomly selected students. In the teacher notes a recommendation is made to have students ask every tenth student who comes into the school. This type of sampling is called systemic sampling.

Systematic sampling is an additional statistical method involving the selection of elements from an ordered sampling frame. The most common form of systematic sampling is an equal-probability method. In this approach, progression through the list is treated circularly, with a return to the top once the end of the list is passed. The sampling starts by selecting an element from the list at random and then every kth element in the frame is selected, where k, the sampling interval is calculated as $k = N/n$, where n is the sample size, and N is the population size.

Using this procedure each element in the population has a known and equal probability of selection. This makes systematic sampling functionally similar to random sampling and is typically applied if the given population is logically homogeneous, because systematic sample units are uniformly distributed over the population.

Drawing conclusions from data that are subject to random variation is termed *statistical inference*. Statistical inference or simply "inference" makes propositions (predictions) about populations, using data drawn from the population of interest via some form of random sampling during a finite period of time. The outcome of statistical inference is typically the answer to the question "What should be done next?"

Random sampling allows results from a sample to be generalized to the population from which the sample was selected. The sample proportion is the best estimate, given the constraints of the population proportion. Students should understand that conclusions drawn from random samples can then be generalized to the population from which the sample was appropriately selected. The sample result and the true value from the entire population are likely to be very close but not exactly the same. Understanding variability in the samplings allows students the opportunity to estimate or even measure the differences.

Use data from a random sample to draw inferences about a population with an unknown characteristic of interest. Generate multiple samples (or simulated samples) of the same size to gauge the variation in estimates or predictions. For example, estimate the mean word length in a book by randomly sampling words from the book; predict the winner of a school election based on randomly sampled survey data. Gauge how far off the estimate or prediction might be. 7.SP.2

The primary focus of 7.SP.2 is for students to collect and use multiple samples of data (either generated or simulated), of the same size, to gauge the variations in estimates or predication, and to make generalizations about a population. Issues of variation in the samples should be addressed by gauging how far off the estimate or predication might be.

Was Mrs. Mouton's technique for gathering data effective? To check on this, and to see how far off the estimate might be, Mrs. Moulton visits the school district's food services website and learns the actual percentages of 7th graders' consumption over the academic year: 40% favor pizza, 20% favor hamburgers, 10% favor salad, 20% bring a home lunch, and 10% are unaccounted for (which is calculated as having no lunch). This is pretty strong confirmation. Another approach would be to ask each student in each grade to pick a student at lunch at random and check what the choice was. This might work, but because of the group of students who bring their own lunch or have no lunch, it could be skewed. In summary, selection of random samples, and representation of the data by graphs are often questions of taste or common consent.

We often use bar graphs for comparing categorical data and making inferences about populations from the graphs. If comparisons are being made between unequal sized groups, percents give a better basis of comparison. However if the samples are equal in size, then either counts or percents give comparable graphs. For categorical data such as this, either bar graphs or pie charts are appropriate. If bar graphs are made, they should be called bar graphs, rather than "histograms" because the data are categorical. Histograms are used for graphs that display a range of numerical values, such as heights, or ages (which we will discuss further in Section 7.3). Although histograms can be used in drawing the graph, this may generate graphs that do not truly represent the data, unless each histogram is equal in size.

The variability in the samples can be studied by means of simulation. Although we have used this word before, in a colloquial sense, it is now desirable to work on making the concept somewhat more precise. A *simulation* is an experiment that models a real-life situation and helps students develop correct intuitions and predict outcomes analogous to the original problem. Mrs. Moulton wants to create a simulation that models the eating habits of 7th graders. Here is her model: she prepares a large non see-through bag, with 200 red skittles (representing pizza), 100 purple skittles (representing hamburgers), 50 green skittles (representing salad), 100 yellow skittles (representing home lunch) and 50 orange skittles (representing no lunch). These proportions are close to the proportions shown in Graph 2. The purpose is to create a population of 500 (representing the amount of students in the school) with 40% red skittles for pizza, 20% purple skittles for hamburgers, 10% green skittles for salad, 20% yellow skittles for home lunch and 10% orange skittles representing no lunch.

Mrs. Moulton then has students randomly select 10 skittles at a time, repeated roughly five times, returning each group of 10 skittles to the bag each time. Notice that expected values are given. Because approximately 40% of the skittles in the bag are red, we should expect to average a draw of 4 skittles. The goal of the activity is that students notice that a random sampling gives close to 40% red skittles.

It is not necessary to replace each skittle individually after it is sampled. In sampling, if less than 10% of the population is sampled at a time, you do not have to sample with replacement. In this case, sampling 10 skittles will not change the ratio of, i.e., pizza:salad enough to matter. If you are sampling less than 10% of the population, and the sample is random, then independence can be safely assumed. This is called the 10% rule for sampling without replacement. Ten skittles are far less than 10% of the skittles in the bag; given the number of skittles in each bag are 500. Thus, samples of up to 50 without replacement would still be appropriate.

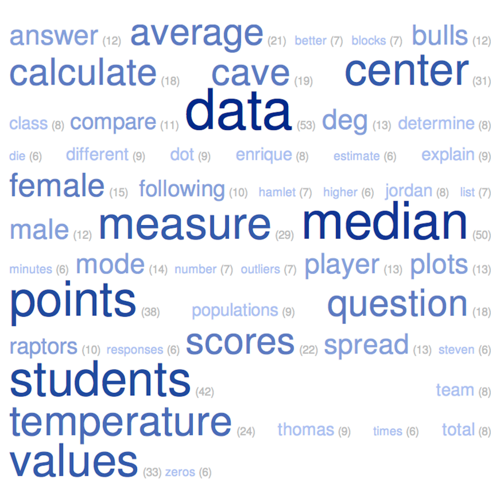
Let's consider one more scenario. Suppose Mrs. Moulton wishes to investigate the occurrence of pizza demand in her entire school district. A cluster sample could be taken by identifying the different school boundaries in her school district as clusters. Cluster sampling is a sampling technique where the entire population is divided into groups, or clusters, and a random sample of these clusters are selected. All observations in the selected clusters are included in the sample.

A sample of these school boundaries (clusters) would then be chosen at random, so all schools in those school boundaries selected would be included in the sample. It can be seen here then that it is easier to call, email, or visit several schools in the same chosen school boundary, than it is to travel to each school in a random sample to observe the occurrence of say pizza demand in the entire school district. The big question is: how is the demand for pizza at Mrs. Moulton's school similar or different from that of the school district as a whole?

Scientists use data from samples in order to make conclusions about the world. If using data from the samples, to come to an agreement on an estimate for the demand for pizza in the school district is calculated by averages, we give a cautionary note. If the sample sizes are different, then averaging the data gives more "weight" to the larger samples, so a method will need to be developed to come up with a way to adjust for the different sample sizes, that is, scaling it up to represent the population of the school district, such as multiplying the estimate by number of school boundaries in the district.

There potentially could be variability between all the samples taken, i.e. samples taken from a high school versus an elementary school. Variability is a measure of how much samples or data differ from each other. How could we accommodate for the variability? We would sample only schools that are most similar, i.e. just junior high schools within the chosen school boundary.

Sometimes data are examined to make a table of frequency of the entries. For example, if we want to study the height of 7th graders, we might collect data (from sufficiently large samples, and make a table showing the percentage of 7th graders in a given height range (say, counting by inches). In order to get a good estimate, we might take several samples of the same size. Another use is demonstrated in the course workbook (see section 7.2c of Chapter 7) of frequency of letters of the alphabet . Literary sites sometimes use word frequency of text to try to identify the author of the text, based on their knowledge of the word use of a collection of authors. A very nice piece of software allowing for quick visualization of use words is www.tagcrowd.com. A graphic of the 50 most common words in the first half of the workbook section 7.3, is displayed on the next page.



Section 7.3: Draw Informal Comparative Inferences about two Populations

Informally assess the degree of visual overlap of two numerical data distributions with similar variabilities, measuring the difference between the centers by expressing it as a multiple of a measure of variability. For example, the mean height of players on the basketball team is 10 cm greater than the mean height of players on the soccer team, about twice the variability (mean absolute deviation) on either team; on a dot plot, the separation between the two distributions of heights is noticeable. 7.SP.3

Use measures of center and measures of variability for numerical data from random samples to draw informal comparative inferences about two populations. For example, decide whether the words in a chapter of a seventh-grade science book are generally longer than the words in a chapter of a fourth-grade science book. 7.SP.4

The focus of 7.SP.3 and 7.SP.4 is on informal comparative inferences about two populations. In 7.SP.3 we informally assess the degree of visual overlap of two numerical data distributions with similar variabilities, measuring the difference between the centers by expressing it as a multiple of measure of variability. Practical problems dealing with measures of center are comparative in nature, as in comparing average scores on the first and second exams. Such comparisons lead to conjectures about population parameters and constructing arguments based on data to support the conjectures. If measurements of the population are known, no sampling is necessary and data comparisons involve the calculated measures of center. Even then, students should consider variability.

Specifically, students will calculate measures of center and spread of data sets, and then use those measures to make comparisons between populations and conclusions about differences between the populations. The workbooks have many problems and activities involving data sets that are easily handled without much computational sophistication, while adequately exposing the essential ideas to be assimilated.The real power of data analysis comes out when dealing with large data sets. Here we want to give a flavor of this through two examples, the first (Example 18) of which is worked out in detail, and the second is left for class exploration. As the detail in these examples go beyond the expectations of Standards, we describe the rest of this chapter as an **Extension**: to be used for deeper understanding of the material than required by the Standards. In particular, the first example goes into the comparison of two data sets through both box plots and the Mean Absolute Deviation, allowing for a comparison of the two methods.

Extension.

Sources of large data sets are readily available online, often with suggestions for classroom computer-based projects. Some examples are: the American Fact Finder (Census Bureau; Statistical Universe (LEXIS-NEXIS); Federal Statistics (FedStats); National Center for Health Workforce Analysis (HRSA); CDC Data and Statistics Page; CIA World Factbook; and locally at Utah Division of Wildlife Resources (UDWR; USGS Water Data for Utah; Utah Statistics (aecf.org) or NSA Utah Data Center; Hawkwatch. In the following, we study in detail a a comparison of two data sets that are large enough to illustrate the power of these methods. This is followed by a more exploratory investigation of data, given in maps.

Research into data sets provides opportunities to connect mathematics to student interests and other academic subjects, utilizing statistic functions in promethean boards, graphing calculators, or excel spreadsheets; most especially for calculations with large data sets. In 6th grade the measures of central tendency (mean, median, mode) were studied, as well as various techniques for summarizing the variability in the data (dot plots, five-number summary and box plots). Here we will introduce a numerical measure of spread: the mean absolute deviation (MAD). This concept is a little more intuitive and easier to calculate (by hand) than the standard deviation that will be discussed in grade 10. To illustrate the use of these concepts we will work with a particular data set over the next few pages.

EXAMPLE 18.

How much taller are the Utah Jazz basketball players than the students in Mr. Spencer's A3, 7th grade math class?

We will use this context over the next few pages as a vehicle to recall several important statistical measures of spread of data, and further developing their use started in 6th grade. These are: dot plots, histograms, five-number summary, boxplots and the mean absolute deviation (MAD).

Mr. Spencer wanted to compare the mean height of the players on his favorite basketball team (the Utah Jazz) and his A3 7th grade students in his mathematics class. He knows that the mean height of the players on the basketball team will be greater but doesn't know how much greater. He also wonders if the variability of heights of the Jazz players and the heights of the 7th graders in his A3 class is related to their respective ages. He thinks there will be a greater variability in the heights of the 7th graders (due to the fact that they are ages 12-13 and experiencing growth spurts) as compared to the basketball players. To obtain the data sets, Mr. Spencer used the online roster and player statistics from the team website, and the heights of his students in his A3 class to generate the following lists and then asked the following questions. First, the data:

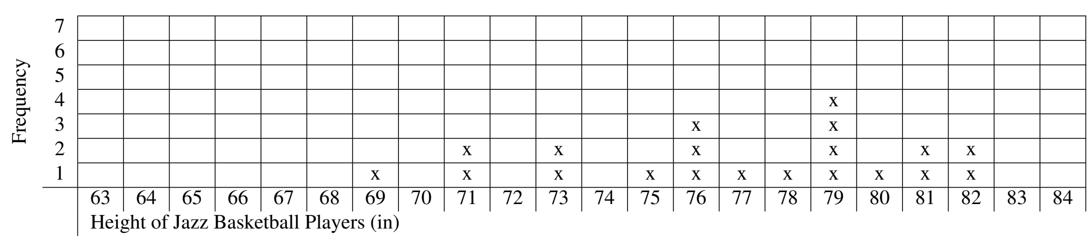
Utah Jazz 2013-14 Team Roster: Height of Players in inches
(Data pulled from www.nba.com/jazzroster):

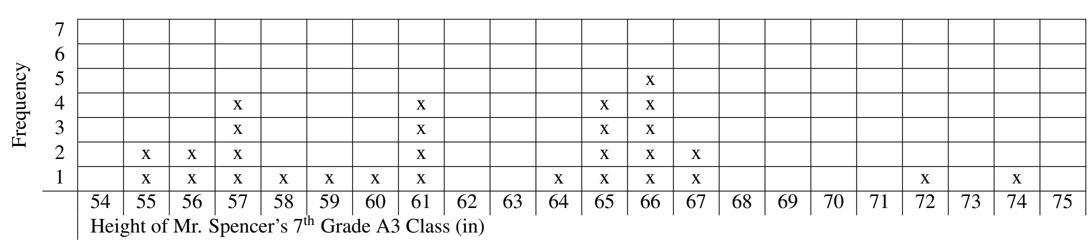
84, 73, 77, 75, 81, 81, 82, 84, 78, 80, 78, 75, 79, 83, 83, 71, 73, 81, 78, 81

Mr. Spencer's 7th grade A3 Class: Height of Students Fall 2013
(Data pulled from www.CDC.gov/growthcharts):

66, 65, 57, 56, 55, 54, 64, 71, 64, 58, 59, 56, 65, 65, 66, 65, 64, 65, 56, 73, 56, 64, 63, 60, 60, 55, 54, 60

a. To compare the data sets, create two dot plots on the same scale with the shortest individual 54 inches and the tallest individual 84 inches. A dot plot (Figure 1), is a graph for numerical variables where the individual values are plotted as dots (or other symbols, such as x). In contrast a histogram (Figure 2) is a graphical display of data using bars of different heights. There are many tools available to create both dot plots and histograms, such as promethean boards, graphing calculators, Excel spreadsheets, or in the case of the histograms, Geogebra. In many of the tools available in creating a histogram, a choice is made to determine the number of bins. In Figure 2, the choice was made to create 8 bins, but the number of bins can vary depending on the best way to summarize the data.





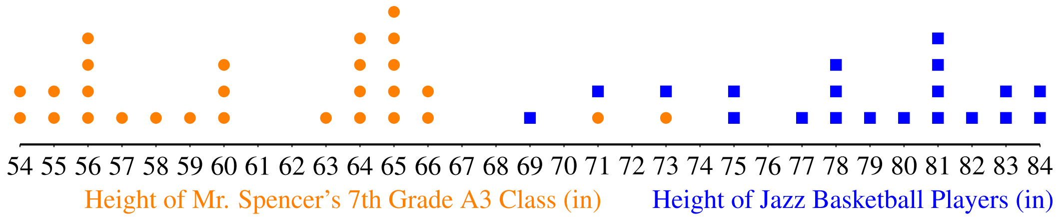


Figure 1

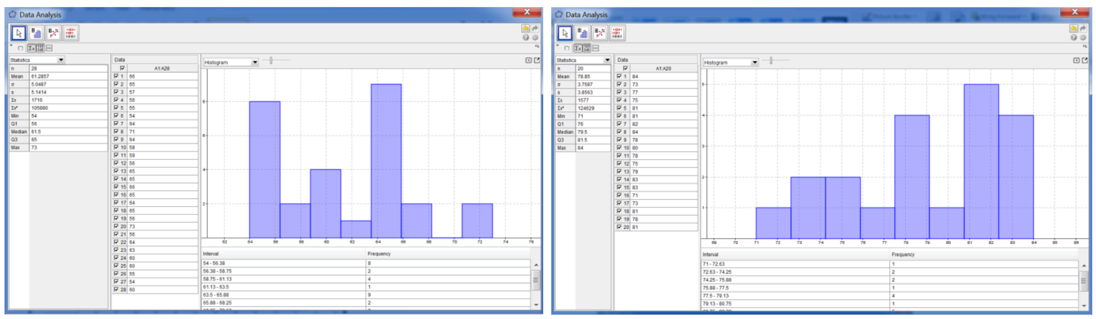


Figure 2

When we have a collection of numerical data it is especially helpful to know ways to determine the nature of the data. In particular, it is helpful to have a single number that summarizes the data. We are often interested in the measure of center and we commonly use the terms mean, median and mode as descriptors of the data. The mean, median and mode each provide a single-number summary of a set of numerical data; although we typically use the mean to make fair comparisons between two data sets. The mean is used for relatively normal data. The median is used for skewed distributions. It is not uncommon to have extreme values that will alter the mean of your data. In these situations, the median is a better measure of center. (see Figure 3). Mode is rarely used as a description for the measure of center.

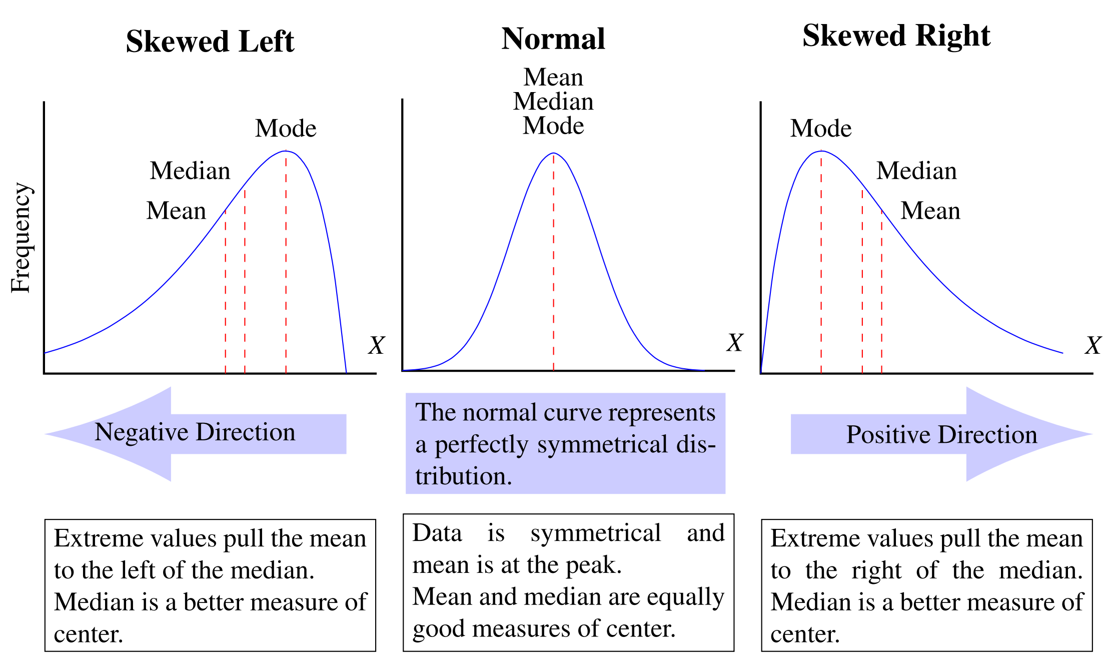


Figure 3

To calculate the mean, or the average, of a list of numbers, add all the numbers and divide this sum by the number of values in the list. For example, consider the data set {4, 9, 3, 6, 5}. The mean is

$$\frac{4+9+3+6+5}{5} = \frac{27}{5} = 5.4\,.$$

The mean is an important statistic for a set of numerical data: it gives some sense of the "center " of the data set. Two other such statistics, the median and the mode were discussed in 6th grade, but will not be considered here. A brief review is found in Section 7.3a homework.

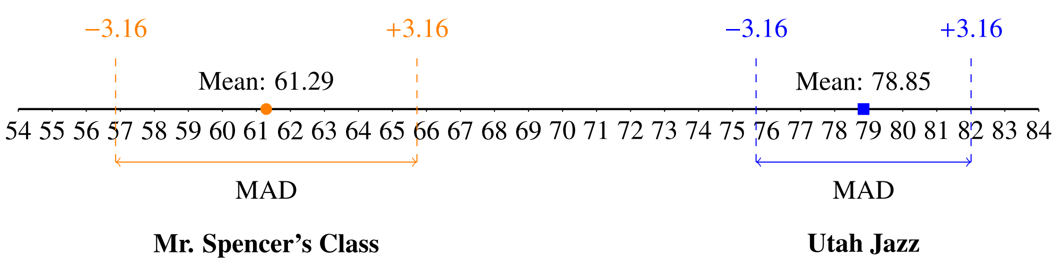
Once we have calculated the mean for a set of data, we want to have some sense of how the data are arranged around the mean: are they bunched up close to the mean, or are they spread out? There are several measures of the spread of data; here we concentrate on the mean absolute deviation (MAD). Methods for calculating center and MAD are standards from the Grade 6 curriculum and are being revisited in Grade 7 as they compare centers and spreads of different data sets. The mean absolute deviation (MAD) is calculated this way: for each data point, calculate its distance from the mean. Now the MAD is the mean of this new set of numbers. Let's do this calculation for the above set of numbers {4, 9, 3, 6, 5}, with mean 5.4.

Data Point	Mean	Deviation
4	5.4	1.4
9	5.4	3.6
3	5.4	2.4
6	5.4	0.6
5	5.4	0.4

Add the deviations: 1.4+3.6+2.4+0.6+0.4 = 8.4, and divide by the number of data points, 5, to get the MAD = 8.4/5 = 1.68.

Now, let's apply this technique to the two sets of data Mr. Spencer wants his class to consider. Before starting the computation Mr. Spencer asks, based on the representations in Figure 1 above, which of the data sets seems to have a larger mean absolute deviation (that is, a broader spread).

b. Once the MAD has been calculated, Mr. Spencer asks the class to put the means of the two data sets, and make marks of the place on both sides of the mean that are the MAD away from the mean. These insertions tell us directly which data set has the greater spread. The following image is the result of this work.



This display of the data confirms Mr. Spencer's hunch: that the data for the Utah Jazz are not spread out as much as the data for his class, and that the spread in his class is symmetric around the mean, while it is more spread in the lower end for the Jazz. What we see in the dot plot that we do not see in the MAD data is the two clusters of heights in Mr. Spencer's class.

When we use the mean and the MAD to summarize a data set, the mean tells us what is typical or representative for the data and the MAD tells us how spread out the data are. The MAD tells us how much each score, on average, deviates from the mean, so the greater the MAD, the more spread out the data are.

A *box plot* (or box-and-whisker plot) is a visual representation of the *five-number summary*, and tells us much more about the spread of the data, answering questions like: on which side of the mean is there more spread; how far away the extremes are. Recall these statistics from Grade 6. First, the *median* is the middle number: there are as many values below the median as there are above. The five number summary shows 1) the location of the lowest data value, 2) the 25th percentile (first quartile) or the center between the minimum and the median, 3) the 50th percentile (median), 4) the 75th percentile (third quartile) or the center between the median and the maximum, and 5) the highest data value. A box is drawn from the 25th percentile to the 75th percentile, and 'whiskers' are drawn from the lowest data value to the 25th percentile and from the 75th percentile to the highest data value. If there is no single number in the middle of the list, the median is halfway between the two middle numbers.

As an example we find the five-number summary for the data set: Height of the Utah Jazz Basketball Players 2013-14 season.

71 73 73 75 $\underbrace{75 \quad 77}_{\text{Quartile 1}}$ 78 78 78 $\underbrace{79 \quad 80}_{\text{Median}}$ 81 81 81 $\underbrace{81 \quad 82}_{\text{Quartile 3}}$ 83 83 84 84

Median is the average of 79 and 80; (79 + 80)/ 2 = 79.5

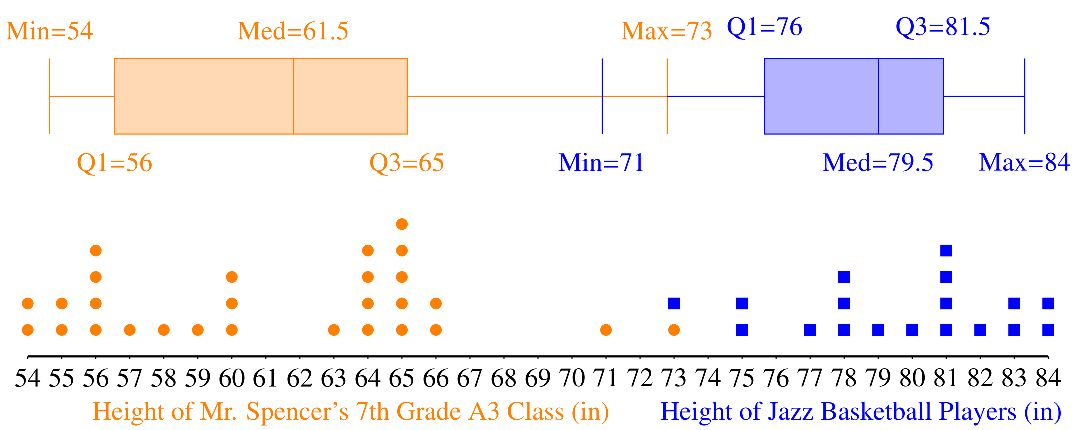
Quartile 1 (Q1) is the average of 75 and 77; (75 + 77) 2 = 76

Quartile 3 (Q3) is the average of 81 and 82; (81 + 82) 2 = 81.5

The five-number summary is:

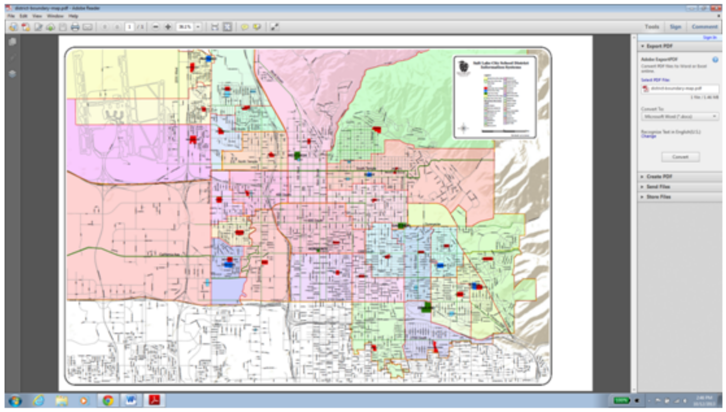
Min	Q1	Median	Q3	Max
71	76	79.5	81.5	84

Given the five number summary, we can make the box-plots. Here we show the box plots for both sets of data Mr. Spencer wants to compare, and below that the dot plots so that we can compare the information that can be obtained from each representation. For example, the box plots show greater spread in Mr. Spencer's class, and that in both cases the spread of the second quartile is greater than the spread of the third quartile.

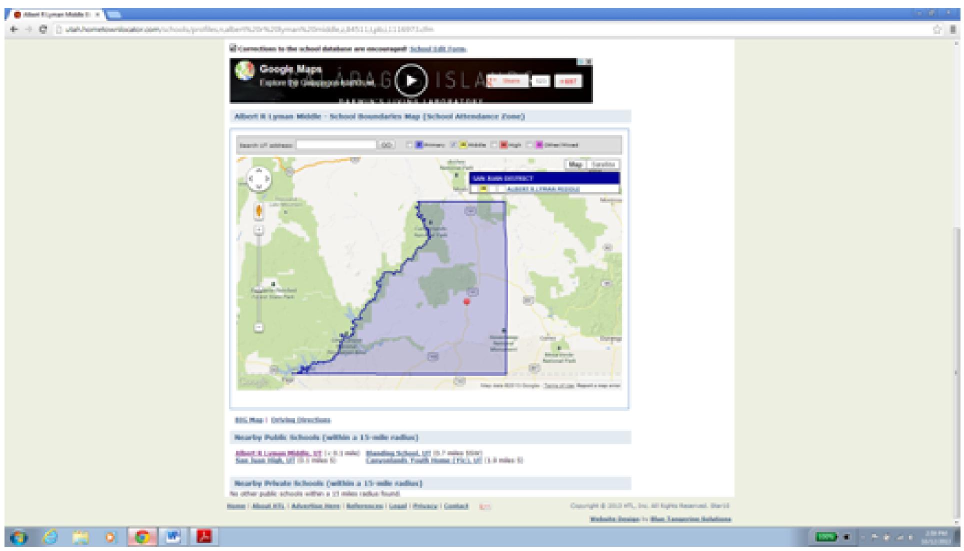


Example 19.

Glendale Middle School is located in the heart of Salt Lake School District. It is one of five middle schools in the district, and has approximately 835 students recorded in attendance, (data from the 2009-2010 school year). The map shows the boundary of the Salt lake School District, and the arrow pointing to the small blue area is the boundary of the middle school, and the region for the Glendale Middle School is the blue area denoted by the arrow.



Albert R. Lyman Middle School is located in the San Juan School district. In the 2009-2010 school year there were approximately 312 students in attendance. The school is located in Blanding, Utah and is the only middle school in the district. The blue portion highlighted is the San Juan School District.



The State School Board wants to determine how far students travel to school and picked two schools; Albert R. Lyman Middle in San Juan School District and Glendale Middle in Salt Lake City School District. Ten students each, at both schools, were chosen at random and were asked how far they traveled to school. The responses are below:

Glendale Middle
0.1
0.3
0.4
0.6
0.7
0.8
1.2
1.6
2.8
5

Albert R. Lyman Middle
0.5
1.5
4
5
10
12
18
24
30
65

The State School Board asked the students to answer the following questions.

a. What is the mean of "distance traveled" for each school, and what does the mean represent?

b. What is the mean absolute deviation (MAD) for both schools? Create a table for each data set to help with the calculations. Describe what the mean absolute deviation represents?

c. To compare the data sets, create two dots plots on the same scale. What conclusions can be made from these two data sets? Note: We are only working with data from 10 students and conclusions need to be cautiously represented. Conclusions cannot be made without having an adequate sample size and confirming that the students were chosen at random.