Chapter 10
Geometry: Angles, Triangles and Distance

In section 1 we begin by gathering together facts about angles and triangles that have already been discussed in previous grades. This time the idea is to base student understanding of these facts on the transformational geometry introduced in the preceding chapter. As before, here the objective is to give students an informal and intuitive understanding of these facts about angles and triangles; all this material will be resumed in Secondary Mathematics in a more formal and logically consistent exposition.

Section 2 is about standard 8G6: *Explain a proof of the Pythagorean Theorem and its converse*. The language of this standard is very precise: it does not say *Prove...* but it says *Explain a proof of...*, suggesting that the point for students is to articulate their understanding of the theorem; not to demonstrate skill in reciting a formal proof. Although Mathematics tends to be quite rigorous in the construction of formal proofs, we know through experience that informal, intuitive understanding of the "why" of a proof always precedes its articulation. Starting with this point of view, the student is guided through approaches to the Pythagorean theorem that make it believable, instead of formal arguments. In turn, the student should be better able to explain the reasoning behind the Pythagorean theorem, than to provide it in a way that exhibits form over grasp.

In Chapter 7, in the study of tilted squares, this text suggests that by replacing specific numbers by generic ones, we get the Pythagorean theorem. We start this section by turning this suggestion into an "explanation of a proof," and we give one other way of seeing that this is true. There are many, for example:

`jwilson.coe.uga.edu/EMT668/emt668.student.folders/HeadAngela/essay1/Pythagorean.html`

The converse of the Pythagorean theorem states this: if $a^2 + b^2 = c^2$, where a, b, c are the lengths of the sides of a triangle, then the triangle is a right triangle, and the right angle is that opposite the side with length c. The Euclidean proof of this statement is an application of SSS for triangles. Although students played with SSS in Chapter 9, here we want a more intuitive and dynamic understanding. Our purpose is to encourage thinking about dynamics, which becomes a central tool in later mathematics. We look at the collection of triangles with two side lengths a and b with $a \geq b$. As the angle at C grows from very tiny to very near a straight angle, the length of its opposing side steadily increases. It starts out very near $a-b$, and ends up very near $a+b$. There is only on triangle in this sequence where $a^2 + b^2$ is precisely c^2.

In the final section, we use the Pythagorean Theorem to calculate distances between points in a coordinate plane. This is what the relevant standard asks: it does not ask that students *know* the "distance formula." The goal here is that students understand the *process* to calculate distances: this process involves right angles and the Pythagorean theorem and students are to understand that involvement. Concentration on the formula perverts this objective.

Section 10.1: Angles and Triangles

Use informal arguments to understand basic facts about the angle sum and exterior angle of triangles, about the angles created when parallel lines are cut by a transversal, and the angle-angle criterion for similarity of triangles. 8G5

In this section, we continue the theme of the preceding chapter: to achieve geometric intuition through exploration. We start with geometric facts that students learned in Grade 7 or earlier, exploring them from the point of view of rigid motions and dilations.

(1) Vertical angles at the point of intersection of two lines have the same measure.

The meaning of *vertical* is in the sense of *a vertex*. Thus, in Figure 1, the angles at V with arcs are vertical angles, as is the pair at V without arcs.

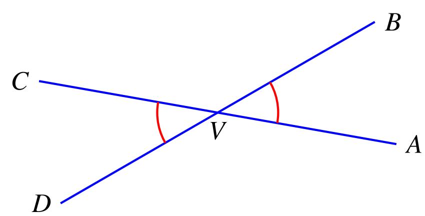

Figure 1

Rotation with vertex V through a straight angle (180°) takes the line AC into itself. More specifically, it takes segment VA to VC and VB to VD, and so carries $\angle AVB$ to $\angle CVD$. This rotation therefore shows that the angles $\angle AVB$ and $\angle CVD$ are congruent, and thus have the same measure.

The traditional argument (and that which appears in grade 7) is this: both angles $\angle AVB$ and $\angle CVD$ are supplementary to $\angle BVC$ (recall that two angles are *supplementary* if they add to a straight angle), and therefore must have equal measure. However, in grade 8 we want to understand measure equality in terms of congruence, and congruence in terms of rigid motions.

(2) If two lines are parallel, and a third line L cuts across both, then corresponding angles at the points of intersection have the same measure.

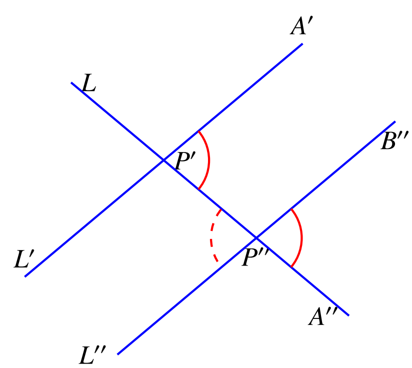

Figure 2

In Figure 2, the two parallel lines are L' and L'', and the corresponding angles are as marked at P' and P''. The translation that takes the point P' to the point P'' takes the line L' to the line L'' because a translation takes a line to another one parallel to it, and by the hypothesis, L'' is the line through P'' parallel to L'. Since translations also preserve the measure of angles, the corresponding angles as marked (at P' and P'') have the same measure. Because of **(1)** above, that opposing angles at a vertex are of equal measure, we can conclude that the angle denoted by the dashed arc is also equal to the angles denoted by the solid arc.

This figure also demonstrates the converse statement:

(3) Given two lines, if a third line L cuts across both so that corresponding angles are equal, then the two lines are parallel.

To show this, we again draw Figure 2, but now the hypothesis is that the marked angles at P' and P'' have the same measure. Since translations preserve the measure of angles, the translation of P' to P'' takes the angle $\angle A'P'P''$ to $\angle B''P''A''$, and so the image of L' has to contain the ray $P''B''$, and so is the line L''. Since the line L'' is the image of L' under a translation, these lines are parallel.

(4) The sum of the interior angles of a triangle is a straight angle ($180°$)**.**

In Grade 7, students saw this to be true by drawing an arbitrary triangle, cutting out the angles at the vertices, and putting them at the same vertex. Every replication of this experiment produces a straight angle. This experiment is convincing that the statement is true, but does not tell us why it is true.

We have two arguments to show why it is true. The first has the advantage that it uses a construction with which the student is familiar (to find the area of a triangle) and thus reinforces that idea. The second has the advantage that it can be generalized to polygons with more sides. First, draw a triangle with a horizontal base (the triangle with solid sides in Figure 3). Rotate a copy of the triangle around the vertex B for $180°$, and then translate the new triangle along the segment BC until it is in the position shown (by the dfashed lines in Figure 3. We have indicated the corresponding angles with the greek letters α, β, γ. Since the angles $\angle ABC$ and $\angle C'B'A$" have the same measure (β), the lines AB and $B'A'$ are parallel. Since they are parallel, the angles $B'A'C'$ and $\angle A'C'E$ have the same measure Now look at the point $B = C'$: the angles α, β, γ form a straight angle.

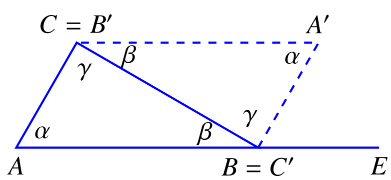

Figure 3

An alternative argument is based on Figure 4 below:

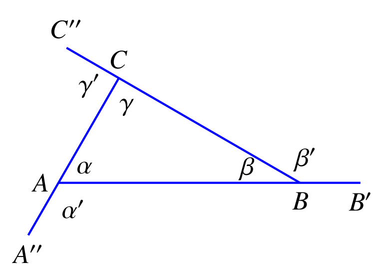

Figure 4

In this figure we have named the "exterior angles" of the triangle, α', β', γ', each of which is outside the triangle formed by the extension of the side of the triangle on the right. If we were to walk around the perimeter of the triangle, starting and ending at A looking in the direction of A', we would rotate our line of vision by a full circle, 360°. As this is the sum of the exterior angles, we have

$$\alpha' + \beta' + \gamma' = 360^\circ .$$

But each angle in this expression is supplementary to the corresponding angle of the triangle, that is, the sum of the measures of the angle is 180°. So, the above equation becomes

$$(180^\circ - \alpha) + (180^\circ - \beta) + (180^\circ - \gamma) = 360^\circ ,$$

from which we get $\alpha + \beta + \gamma = 180^\circ$.

We can generalize the second argument to polygons with more sides. Consider the quadrilateral in Figure 5.

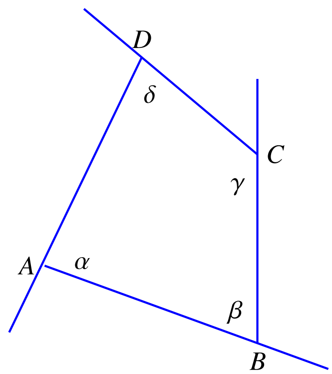

Figure 5

By the same reasoning as for the triangle, the sum of the exterior angles of the quadrilateral is also 360° and the sum of each interior angle and its exterior angle is 180°. But now there are four angles, so we end up with the equation

$$(180^\circ - \alpha) + (180^\circ - \beta) + (180^\circ - \gamma) + (180^\circ - \delta) = 360^\circ , \quad \text{or} \quad 720^\circ - (\alpha + \beta + \gamma + \delta) = 360^\circ .$$

(5) The sum of the interior angles of a quadrilateral is 360°.

Extension

The preceding argument is not yet complete. What if one of the interior angles is greater than a straight angle, as for an arrowhead. The same argument, based on counting the exterior angles on a tour around the perimeter of the figure works, but one now has to keep track of the signs of the angles. If, for example, the angle γ is great than 180°, then $180^\circ - \gamma$ is negative, meaning that the orientation of the angle is in a direction opposite that of the others. So, we keep that sign, and then the whole argument goes through. Can you now show that, for a five sided polygon, the sum of the interior angles of a pentagon is 540°? Can you go from there to the formula for an n-sided polygon?

(6) If two triangles are similar, then the ratios of the lengths of corresponding sides is the same, and corresponding angles have the same measure.

(7) Given two triangles, if we can label the vertices so that corresponding angles have the same measure, then the triangles are similar.

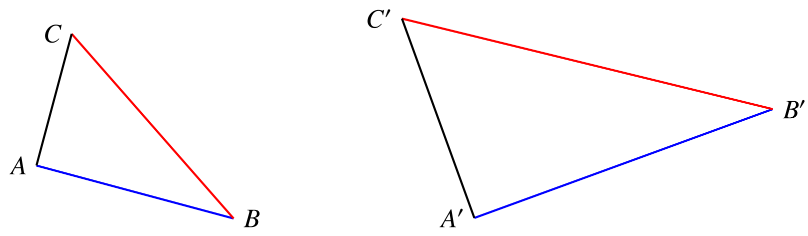

Figure 6a

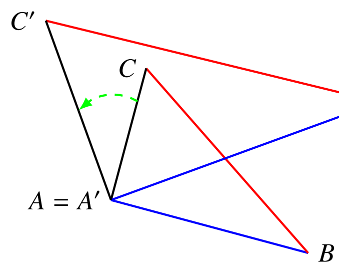

Figure 6b

We saw in Chapter 9 why **(6)** is true. Let us look more closely at statement **(7)**.

Figure 6a shows a possible configuration of the two triangles. By a translation, we can place point A on top of point A' to get Figure 6b. Move the smaller triangle by a rotation with center $A = A'$, so that the point C lands on the segment $A'C'$. Since the angles $\angle CAB$ and $\angle C'A'B'$ have the same measure, the rotation must move line segment AB so that it lies on $A'B'$. Now, since $\angle ACB$ and $\angle A'C'B'$ have the same measure, the line segments CB and $C'B'$ must be parallel (by Proposition 2). The dilation with center A that puts point C on C', puts triangle ABC onto triangle $A'B'C'$, so they are similar.

The argument is not fully completed, for the configuration of Figure 6a is not the only possibility. In Figure 6a, C is between A and B if we traverse the edge of the triangle in the clockwise direction, and the same is true for $\Delta A'B'C'$. However, this is not true in the configuration of Figure 7. Now, how do you find the desired similarity transformation?

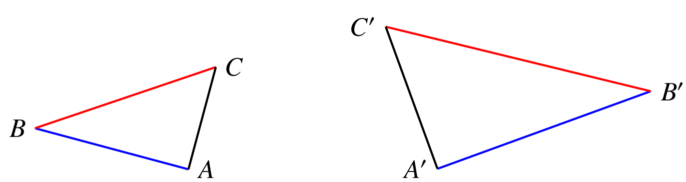

Figure 7

Again, translate so that the points A and A' coincide. But now, the rotation that puts the line segment AB on the same line as $A'B'$ doesn't lead to the configuration of Figure 6b and the smaller triangle cannot be rotated so that corresponding sides lie on the same ray. This is corrected by reflecting the smaller triangle in the line containing the segment AB and $A'B'$, and now we are in the configuration of Figure 6b. Indeed, we could have started with a reflection of the small triangle in the line through AC, and then followed the original argument.

Have we covered all cases? The answer is yes: the difference between Figure 6a and that of Figure 7 is that of orientation. If we start again with the two triangles ABC and $A'B'C'$ with corresponding angles of the same measure, then we should first ask: is the orientation $A \to B \to C$ the same as the orientation $A' \to B' \to C'$ (both clockwise or both counter clockwise)? If so, we are in the case of Figure 6a. If not, after reflection in *any* side of triangle ABC puts us in the case of Figure 7.

End Extension

Section 10.2 The Pythagorean Theorem.

Explain a proof of the Pythagorean Theorem and its converse. 8G6

In Chapter 7 we constructed "tilted" squares of side length c whose area c^2 is a specific integer. If c is not an integer, we have observed that it cannot be expressed as a rational number (a quotient of integers). At the end of the discussion we mentioned that these specific examples generalized to a general theorem (known as the *Pythagorean theorem*) relating the lengths of the sides of a right triangle. This mathematical fact is named after a sixth century BCE mathematical society (presumed to be led by someone named *Pythagoras*). It is clear that this was known to much earlier civilizations: the written record shows it being used by the Egyptians for land measurements, and an ancient Chinese document even illustrates a proof. But the Pythagorean Society was given the credit for this by third century BCE Greek mathematicians. The Pythagorean Society is also credited with the discovery of constructible line segments whose length cannot be represented by a quotient of integers (in particular, the hypotenuse of a triangle whose legs are both the same integer.). The legend is that the discoverer of this fact was sacrificed by the Pythagoreans. We mention this only to highlight how much the approach to mathematics has changed in 2500 years; in particular this fact, considered "unfortunate" then, is now appreciated as a cornerstone of the attempt to fully understand the concept of number and its relationship to geometry.

Let's pick up with the discussion at the end of Chapter 7, section 1. There we placed our tilted squares in a coordinate plane so as to be able to more easily see the relationship between the areas of the squares and its associated triangles. Here, in order to stress that the understanding of the Pythagorean theorem does not involved coordinates, we look at those Chapter 7 arguments in a coordinate free plane. For two positive numbers, a and b, construct the square of side length $a + b$. This is the square bounded by solid lines in Figure 8, with the division points between the lengths a and b marked on each side. Draw the figure joining these points - this is the square with dashed sides in Figure 8.

In Chapter 7 we observed that this is a square; now, let us check that this is so. First of all, triangle I is congruent to triangle III: translate triangle I horizontally so that the side labeled "a" lies on the side of triangle II labeled "b." Now rotate triangle I (in its new position) clockwise by 90°. Then the sides labeled a and b on the two triangles coincide. Since the included angle on both triangles is a right angle, the triangles also coincide, so are congruent. In the same way we can show that trianlge III is congruent to triangle II, and II is congrurent to IV.

These congruences tell us that the dashed lines are all of the same length. For the dashed figure to be a square, we have to show that all the vertex angles are right angles. Now by rotation of the whole figure around its center by 90° at a time, we see that the vertex angles are all congruent. Since they add up to 360°, they all have to be right angles. Alternatively, by the congruences of triangles I and III, we see that the angles outside of the vertex angle at A add up to a right anlge. So the vertex angle at A is also a right angle.

Thus the dashed figure is a square of area c^2, where c is the length of the hypotenuse of the four triangles. By reconfiguring the picture as in Figure 9, we see that this is also $a^2 + b^2$. In Figure 8 the part of the large square that is not in one of the four triangles has area c^2 and in Figure 9, that part is of area $a^2 + b^2$.

Thus

The Pythagorean Theorem:

$$a^2 + b^2 = c^2$$

for a right triangle whose leg lengths are a and b and whose hypotenuse is of length c.

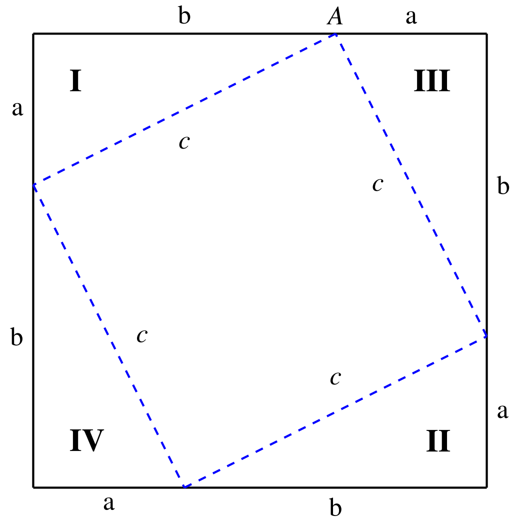

Figure 8

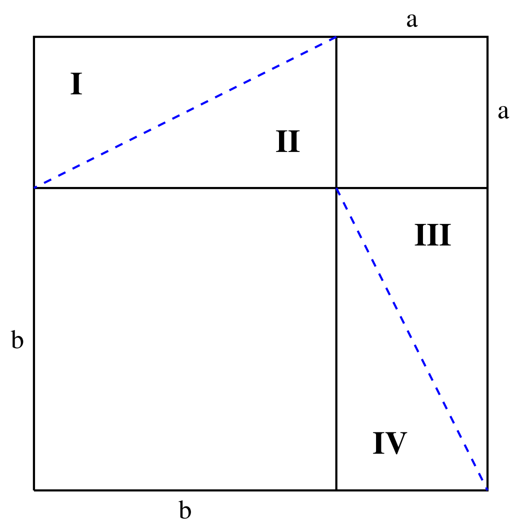

Figure 9

This is known as the *Chinese proof* of the Pythagorean theorem; records show that it precedes the Pythagorean Society by close to 1000 years.

It is useful to give another "proof without words" of the Pythagorean theorem; this one is due to Bhaskara, a 12th century CE mathematician in India. Start with the square of side length c (so of area c^2, and draw the right triangles of leg lengths a and b, with hypotenuse a side of this square, as shown in the top left of Figure 10. Now reconfigure these triangles and the interior square as in the image on the right. Since it is a reconfiguration, it still has area c^2. But now, by redrawing as in the lower image, we see that this figure consists of two squares, one of area a^2 and the other of area b^2.

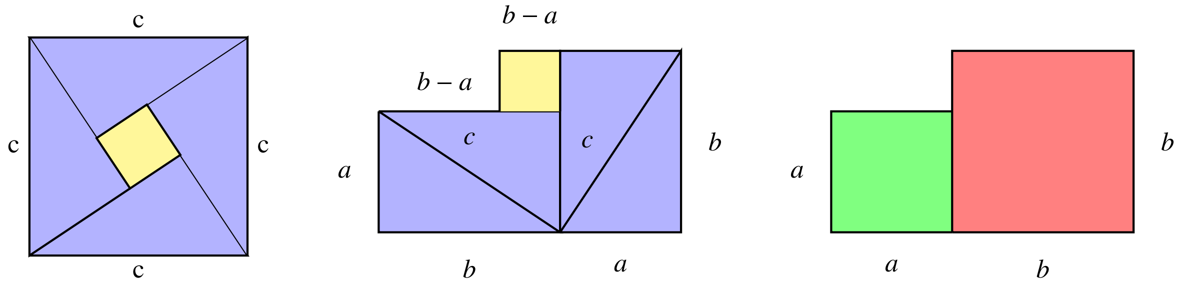

Figure 10

EXAMPLE 1.

a. A right triangle has leg lengths 6 in and 8 in. What is the length of its hypotenuse?

SOLUTION. Let c be the length of the hypotenuse. By the Pythagorean theorem, we know that

$$c^2 = 6^2 + 8^2 = 36 + 64 = 100\ ,$$

so $c = 10$.

b. Another triangle has leg lengths 20 in and 25 in. Give an approximate value for the length of its hypotenuse.

SOLUTION.

$$c^2 = 20^2 + 25^2 = 400 + 625 = 1025 = 25 \times 41\ .$$

So, $c = \sqrt{25 \times 41} = 5 \times \sqrt{41}$. Since $6^2 = 36$ and $7^2 = 49$, we know that $\sqrt{41}$ is between 6 and 7; probably a bit closer to 6. We calculate: $6.4^2 = 40.96$, so it makes good sense to use the value 6.4 to approximate $\sqrt{41}$. Then the corresponding approximate value of c is $5 \times 6.4 = 32$.

EXAMPLE 2.

a. The hypotenuse of a triangle is 25 ft, and one leg is 10 ft long. How long is the other leg?

SOLUTION. Let b be the length of the other leg. We know that $10^2+b^2 = 25^2$, or $b^2+100 = 625$, and thus $b^2 = 525$. We can then write $b = \sqrt{525}$. If we want to approximate that, we first factor $525 = 25 \times 21$, sp $b = 5\sqrt{21}$ We can approximate $\sqrt{21}$ by 4.5 ($4.5^2 = 20.25$). and thus b is approximately given by $5 \times 4.5 = 22.5$.

b. An isosceles right triangle has a hypotenuse of length 100 cm. What is the leg length of the triangle.

SOLUTION. Referring to to the Pythagorean theorem, we are given: $a = b$ (the triangle is isosceles), and $c = 100$. So, we have to solve the equation $2a^2 = 100^2$. Since $100^2 = 10^4$, we have to solve $2a^2 = 10^4$, or $a^2 = 5 \times 10^3$. Write $5 \times 10^3 = 50 \times 10^2$, which brings us to $a = \sqrt{50 \times 10^2} = 10 \times \sqrt{50}$. Since $7^2 = 49$, we can give the approximate answers $a = 10 \times 7 = 70$.

EXAMPLE 3.

Figure 11 is that of an isosceles right triangle, ΔABC, lying on top of a square. The total area of the figure is 1250 sq. ft. What is the length of the altitude (CD) of the triangle? Note: This problem may be beyond the scope of 8th grade mathematics, but it still may be worth discussing, since it illustrates how the interaction of geometry and algebra works to solve a complex structural problem.

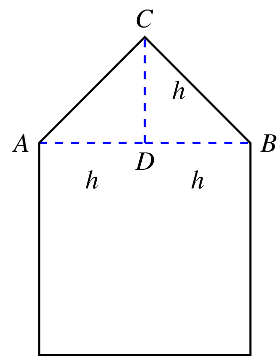

Figure 11

Solution. Since the triangle is isosceles, the measure of $\angle CAD$ and $\angle CBD$ are both 45°. The altitude of a triangle is perpendicular to the base; from which we conclude that the triangles ΔCAD and ΔCBD are congruent. Since we want to find the length of CD, let's denote that number by h. This gives us the full labelling of Figure 11. The information given to us is about area, so let's do an area calculation. Since the length of a side of the square is $2h$, its area is $4h^2$. The area of the triangle on top of the square (one-half base times altitude) is $\frac{1}{2}(2h)(h) = h^2$. So, the area of the entire figure is $4h^2 + h^2 = 5h^2$, and we are given that that is 1250 sq. ft. We then have

$$5h^2 = 1250 \quad \text{so} \quad h^2 = 250 = 25 \times 10 \quad \text{and} \quad h = 5\sqrt{10}\,.$$

The Pythagorean theorem describes a relation among the lengths of the sides of a right triangle; it is also true that this relation describes a right triangle. This is what is called the *converse* of the Pythagorean theorem. The idea of "converse" is important in mathematics. Most theorems of mathematics are of the form: Given certain conditions we must have a specific conclusion. The converse asks: if the conclusion is observed, does that mean that the given conditions hold? In our case, the converse asks: does the equation $a^2 + b^2 = c^2$ relating the sides of a triangle tell us that the triangle is a right triangle? The answer is *yes*:

Converse of the Pythagorean Theorem:

For a triangle with side lengths a, b, c if $a^2 + b^2 = c^2$, then $\angle ACB$ is a right angle.

In order to see why this is true, we show how to draw all triangles with two side lengths a and b. Let's suppose that $a \geq b$. On a horizontal line, draw a line segment BC of length a. Now draw the semicircle whose center is C and whose radius is b (see Figure 12). Then, any triangle with two side lengths a and b is congruent to a triangle with one side BC, and the other side the line segment from C to a point A on the circle.

As the line segment AC is rotated around the point C, the length c of the line segment BA continually increases. When AC is vertical, we have the right triangle, for which the length $c = \sqrt{a^2 + b^2}$. We conclude that for any triangle with side lengths a and b, the length c of the third side is either less than $\sqrt{a^2 + b^2}$ (triangle is acute), or greater than $\sqrt{a^2 + b^2}$ (triangle is obtuse) *except for the right triangle* (where the segment AB is vertical).

Example 4.

Draw a circle and its horizontal diameter (AB in Figure 13). Pick a point C on the circle. Verify by measurement that triangle ABC is a right triangle.

Solution. For the particular triangle the measures of the side lengths, up to nearest millimeter are:

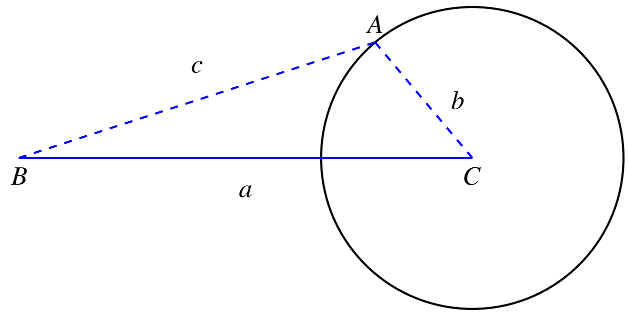

Figure 12

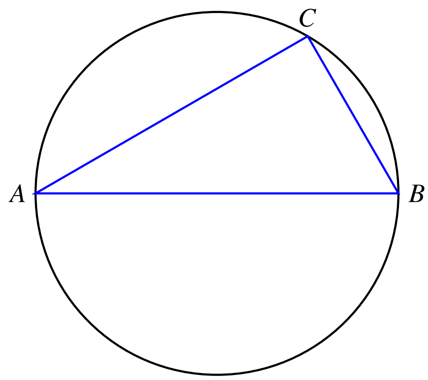

Figure 13

$AB = 44$ mm, $BC = 18$ mm, $AC = 40$ mm. Now, calculate: $BC^2 + AC^2 = 324 + 1600 = 1924$, and $AB^2 = 1936$. This is pretty close. If all students in the class get this close, all with different figures, then that is substantial statistical evidence for the claim that the triangle is always a right triangle.

Section 10.3 Applications of the Pythagorean Theorem.

Apply the Pythagorean Theorem to determine unknown side lengths in right triangles in real-world and mathematical problems in two and three dimensions. 8G7

Example 5.

What is the length of the diagonal of a rectangle of side lengths 1 inch and 4 inches?

Solution. The diagonal is the hypotenuse of a right triangle of side lengths 1 and 4, so is of length $\sqrt{1^2 + 4^2} = \sqrt{17}$.

Example 6.

Suppose we double the lengths of the legs of a right triangle. By what factor does the length of the hypotenuse change, and by what factor does the area change?

Solution. This situation is illustrated in Figure 14, where the triangles have been moved by rigid motions so that they have legs that are horizontal and vertical, and they have the vertex A in common. But now we can see that the dilation with center A that moves B to B' puts the smaller triangle on top of the larger one. The factor of this dilation is 2. Thus all length change by the factor 2, and area changes by the factor $2^2 = 4$.

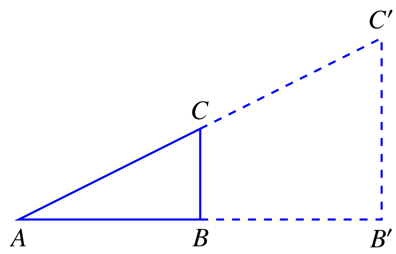

Figure 14

EXAMPLE 7.

An 18 ft ladder is leaning against a wall, with the base of the ladder 8 feet away from the base of the wall. Approximately how high up the wall is the top of the ladder?

SOLUTION. The situation is visualized in Figure 15. The configuration is a right triangle with hypotenuse (the ladder) of length 18 feet, the base of length 8 feet, and the other leg of length h. By the Pythagorean theorem, we have

$$h^2 + 8^2 = 18^2 \quad \text{or} \quad h^2 + 64 = 324 .$$

Then $h^2 = 260$. Since we just want an approximate answer, we look for the integer whose square is close to 260: that would be 16 ($16^2 = 256$). So, the top of the ladder hits the wall about 16 feet above the ground.

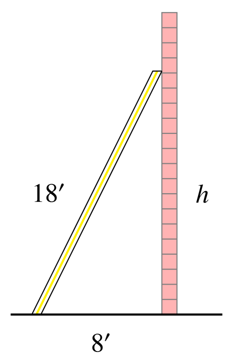

Figure 15

Extension

EXAMPLE 8.

A room is in the shape of a rectangle of width 12 feet, length 20 feet, and height 8 feet. What is the distance from one corner of the floor (point A in the figure) to the opposite corner on the ceiling?

SOLUTION. In Figure 16, we want to find the length of the dashed line from A to B. Now, the dashed line on the floor of the room is the hypotenuse of a right triangle of leg lengths 12 ft. and 20 ft. Its length is $\sqrt{12^2 + 20^2} = 23.3$. The length whose measure we want is the hypotenuse of a triangle ΔACB whose leg lengths are 23.3 and 8 feet. Using the Pythagorean theorem again we conclude that the measure of the line segment in which we are interested (AB) is $\sqrt{8^2 + 23.3^2} = 24.64$; since our original data were given in feet, the answer: 25 ft. should suffice.

EXAMPLE 9.

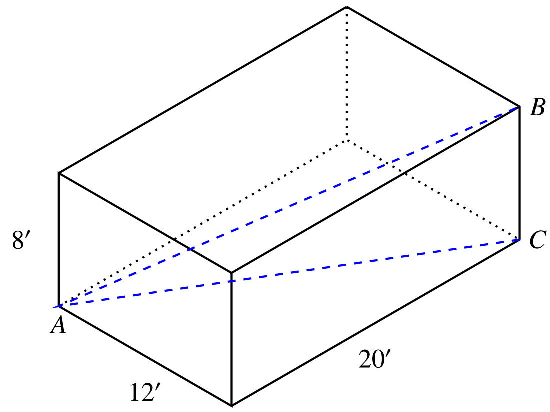

Figure 16

What is the length of the longest line segment in the unit cube?

Solution. We can use the same figure as in the preceding problem, taking that to be the unit cube. Then the length of the diagonal on the bottom face is $\sqrt{1^2 + 1^2} = \sqrt{2}$ units, and the length of the diagonal AB is $\sqrt{1^2 + (\sqrt{2})^2} = \sqrt{3}$.

These examples show that we can use the Pythagorean theorem to find lengths of line segments in space. Given the points A and B, draw the rectangular prism with sides parallel to the coordinate planes that has A and B as diametrically opposite vertices (refer to Figure 16). Then, as in example 8, the distance between A and B is the square root of the sum of the lengths of the sides.

Example 10.

What is the length of the longest line segment in a box of width 10", length 16" and height 8"?

$$\text{Length} = \sqrt{10^2 + 16^2 + 8^2} = \sqrt{100 + 256 + 64} = \sqrt{428} = \sqrt{4 \cdot 107} = 2\sqrt{107}$$

inches, or a little more that 20 inches.

End Extension

Example 11.

In the movie *Despicable Me*, an inflatable model of The Great Pyramid of Giza in Egypt was created by Vector to trick people into thinking that the actual pyramid had not been stolen. When inflated, the false Great Pyramid had a square base of side length 100 m. and the height of one of the side triangles was 230 m. What is the volume of gas that was used to fully inflate the fake Pyramid?

Solution. The situation is depicted in Figure 17. We know that the formula for the volume of a pyramid is $\frac{1}{3}Bh$, where B is the area of the base and h is the height of the pyramid (the distance from the base to the apex, denoted by h in the figure). Since the base is a square of side length 100 m., its area is 10^4 m^2. To calculate the height, we observe (since the apex of the pyramid is directly above the center of the base), that h is a leg of a right triangle whose other leg is 50 m. and whose hypotenuse has length 230 m. By the Pythagorean theorem $h^2 + 50^2 = 230^2$. Calculating we find $h^2 = 230^2 - 50^2 = 52900 - 2500 = 50400$. Taking square roots, we have $h = 225$ approximately. Then, the volume of the pyramid is

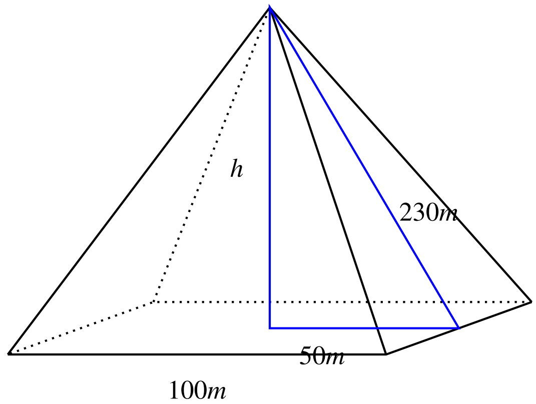

Figure 17

$$\text{Volume} = \frac{1}{3}(10^4)(2.25 \times 10^2) = .75 \times 10^5 = 75,000 \text{ m}^3.$$

Section 10.4 The Distance Between Two Points

Apply the Pythagorean theorem to find the distance between two points. 8G8.

For any two points P and Q , the *distance* between P and Q is the length of the line segment PQ.

We can approximate the distance between two points by measuring with a ruler, and if we are looking at a scale drawing, we will have to use the scale conversion. If the two points are on a coordinate plane, we can find the distance between the points using the coordinates by applying the Pythagorean theorem. The following sequence of examples demonstrates this method, starting with straight measurement.

Example 12.

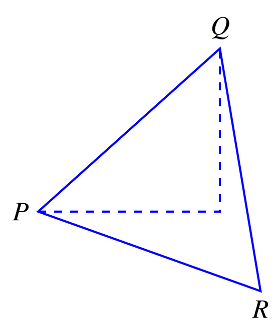

Figure 18

a. Using a ruler, estimate the distance between each of the three points P, Q and R on Figure 18.

Solution. The measurements I get are PQ = 39 mm; PR = 39 mm and QR = 41 mm. Of course, the actual measures one gets will depend upon the display of the figure.

b. Now measure the horizontal and vertical line segments (the dashed line segments in the figure), to confirm the Pythagorean theorem.

Solution. The horizontal line is 30 mm, and the vertical line is 27 mm. $30^2 + 27^2 = 900 + 729 = 1629$, and $39^2 = 1621$. This approximate values are close enough to confirm the distance calculation, and attribute the discrepancy to minor imprecision in measurement.

Example 13.

In the accompanying map of the northeast United States, one inch represents 100 miles. Using a ruler and the scale on the map, calculate the distance between

a. Pittsburgh and Providence

b. Providence and Concord

c. Pittsburgh and Concord

d. Now test whether or not the directions Providence → Pittsburgh and Providence → Concord are at right angles.

US: Northeast Region

Solution. After printing out the map, we measured the distances with a ruler, and found

a. Pittsburgh to Providence: Four and an eighth inches, or 412.5 miles;

b. Providence to Concord: 15/16 of an inch, or 93.75 miles;

c. Pittsburg to Providence: Four and a quarter inches, or 425 miles.

d. Since we are only approximating these distances, we round to integer values and then check the Pythagorean formula to see how close we come to have both sides equal. Let us denote these distances by the corresponding letters a, b, c, so that $a = 413$, $b = 94$, $c = 425$. We now calculate the components of the Pythagorean formula:

$$a^2 = 170,569, \; b^2 = 8,836, \text{ so } a^2 + b^2 = 179,405 \; ; \quad c^2 = 180,625 \; .$$

The error, 1220, is well within one percent of c^2, so this angle can be taken to be a right angle.

Example 14.

On a coordinate plane, locate the points $P(3,2)$ and $Q(7,5)$ and estimate the distance between P and Q. Now draw the horizontal line starting at P and the vertical line starting at Q and let R be the point of intersection. Calculate the length of PQ using the Pythagorean theorem.

Solution. First of all, we know the coordinates for R: (7,2). The length of PR is 4, and the length of QR is 3. By the Pythagorean theorem, the length of PQ is $\sqrt{4^2 + 3^2} = 5$. The measurement with ruler should confirm that.

Example 15.

Find the distance between each pair of these three points on the coordinate plane: $P(-3,2)$, $Q(7,7)$ and $R(2,-4)$.

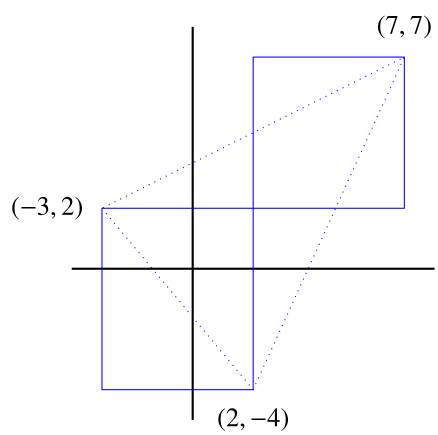

Figure 19

Solution. In Figure 19 have drawn the points and represented the line joining them by dotted lines. To calculate the lengths of these line segments, we consider the right triangle with horizontal and vertical legs and PQ as hypotenuse. The length of the horizontal leg is 7-(-3) = 10, and that of the vertical leg is 7-2 = 5. So

$$|PQ| = \sqrt{10^2 + 5^2} = \sqrt{5^2(2^2+1)} = 5\sqrt{5} \; .$$

For the other two lengths, use the slope triangles as shown and perform the same calculation:

$$|QR| = \sqrt{11^2 + 5^2} = \sqrt{121 + 25} = \sqrt{146} \; ,$$

$$|PR| = \sqrt{5^2 + 6^2} = \sqrt{61} \; .$$

These examples show us that the distance between two points in the plane can be calculated using the Pythagorean theorem, since the slope triangle with hypotenuse the line segment joining the two points is a right angle. This can be stated as a formula, using symbols for the coordinates of the two points, but it is best if students understand the protocol and the reasoning behind it, and by no means should memorize the formula. Nevertheless, for completeness, here it is.

The Coordinate Distance Formula:

Given points $P : (x_0, y_0),\ Q(x_1, y_1)$

$$|PQ| = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}\ .$$

Example 16.

Figure 20

Figure 20 is a photograph, by Jan-Pieter Nap, of the Mount Bromo volcano on the island Java of Indonesia taken on July 11 2004.

`http://commons.wikimedia.org/wiki/File:Mahameru-volcano.jpeg`

From the bottom of the volcano (in our line of vision) to the top is 6000 feet. Given that information, by measuring with a rule, find out how long the visible part of the left slope is, and how high the plume of smoke is.

Solution. Using a ruler, we find that the height of the image of the volcano is 12 mm, the length of the left slope is 21 mm, and the plume is 10 mm high. Now we are given the information that the visible part of the volcano is 5000 feet, and that is represented in the image by 12 mm. Thus the scale of this photo (at the volcano) is 12 mm : 6000 feet, or 1 mm : 500 feet. Then the slope is $21 \times 500 = 10,500$ feet, and the plume is $10 \times 500 = 5000$ feet high.

Example 17.

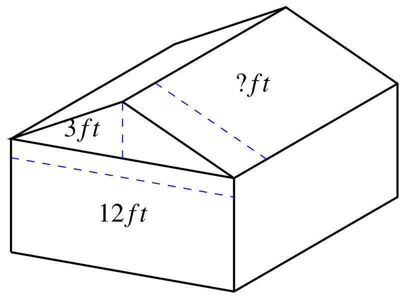

Figure 21

In my backyard I plan to build a rectangular shed that is 12’ by 20’, with a peaked roof, as shown in Figure 21. The peak of the roof is 3’ above the ceiling of the shed. How long do I have to cut the roof beams?

Example 18.

Figure 22 is a detailed map of part of the Highline Trail, courtesy of www.christine@lustik.com.

Using the scale, find the distance from Kings Peak to Deadhorse Pass as the crow flies. Now find the length of the trail between these points. In both cases, just measure distances along horizontal and vertical lens and use the Pythagorean theorem. Measuring with a ruler on the scale at the bottom, we

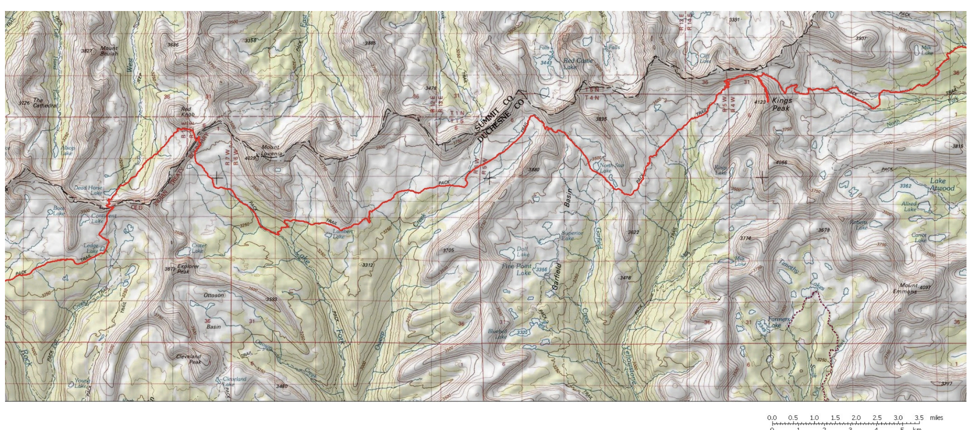

Figure 22

find that the scale is 20 mm : 5 km, or 4 mm/km. Measuring the distance from King’s Peak to Deadhorse Pass, we get 108 mm. Then the actual distance in km is

$$(108 \text{ mm}) \cdot \frac{1 \text{ km}}{4 \text{ mm}} = 27 \text{ km} .$$

Now, to find the length of the trail, you will have to measure each straight length individually and add the measurements. Alternatively, you can go to

`http://lustik.com/highline_trail.htm`

and read an entertaining account of the hike.