Chapter 2
Exploring Linear Relations

In the preceding chapter we completed the topic of finding solutions of a linear equation in one unknown. In chapter four we will turn to this study of techniques to find solutions for a pair of linear equations in two unknowns. But now we want to turn to another thread started in previous grades, that of representing and understanding linear relations between two variables. Notice the change in language: from *equation* and *unknown* to *relation* and *variable*. This is a significant change in objective: from that of finding specific numbers that satisfy given conditons, to that of understanding how conditions on the relation of two variables determine how they behave with respect to one another. In seventh grade students studied the properties of a proportional relation between two variables; in this chapter we turn to linear relations between two variables. A significant tool is the graphical representation of a linear relation by a straight line, leading to the correspondence between *rate of change* (for the relation) and *slope* (of the line).

There are two ways to bring together the study of proportional relations and the solution of linear equations in order to understand linear relations, one emphasizing geometric aspects and the other emphasizing the algebra. Algebraically, linear relations are generalizations of proportional representations: we replace the equation $y = mx$ by the equation $y = mx + b$. The commonality between these is that the *rate of change* of y with respect to x is a constant; the difference is that for a proportional relation, the quotient y/x is constant, and is called the *unit rate* of y with respect to x. For a linear relation, the quotient y/x is not constant unless $b = 0$ Here, b is considered the *initial value* of y; that is, the value of y corresponding to $x = 0$. Geometrically, linear relations and proportional relations are both represented by straight lines; the difference is that the graph of a proportional relation goes through the origin, while the graph of a linear relation goes through the point $(0, b)$, called the *y-intercept*. So, a proportional relation is a special case of a linear relation. In particular, if we slide the graph of $y = mx + b$ by the amount b, we get a line through the origin, and thus the graph of a proportional relation. This just realizes the fact that in the linear relation $y = mx + b$, the quantities $y - b$ and x are proportional, with m the constant of proportionality.

The facts that there are these two ways of developing the subject of linear relations, that both approaches are important, and that the differences are subtle, create a learning issue: the student has to assimilate the two approaches at the same time and appreciate the subtle differences between them. Our solution is to present both approaches, the geometric in the workbook and the algebraic in the foundation. This directly exposes the two approaches and gives the teacher the freedom, and obligation, to develop them simultaneously in a way that works best in that classroom.

Here we begin by continuing the study of proportional relations from seventh grade, focusing on the *unit rate* as a rate of change of one quantity with respect to the other. There will be a shift in language as we move from calculating values of quantities in a proportional relation, to the study of the relation itself. For example, we now consider the unit rate as the *constant of proportionality* of the relationship in order to emphasize that it is what remains *constant* while the measure of the quantities vary, and therefore are called *variables*. We observe that the graph of y vs. x, when y and x are in a proportional relation, is a straight line through the origin.

After this review of proportionality, we return to the study of linear expressions begun in Chapter 1. Here we want to represent them graphically, so we introduce the relation $y = mx + b$ for specific numerical values of m and b, calculate the value of y corresponding to a collection of values of x and observe that the graph of that set of values (x, y) lie on a straight line that crosses the y-axis at the point $(0, b)$. By examination of tables of values and the graph, we observe that, although the variables are not proportional, the *changes* from one measurement to another are proportional: that is, the quotient of the change in y values with respect to the change in x values is constant (independent of the points chosen for the computation). This is called the *rate of change*. The constancy of the rate of change along the graph is a defining property of a straight line, as we shall see in section 3. The move from proportional relations to linear relations, and the accompanying shift from *unit rate* to *rate of change* is subtle and may be difficult for students to appreciate at this time. For this reason, we feel that it is essential to develop the subject in contextual examples, moving the abstract algebraic formulation to Chapter 3.

Although we have observed that the graph of a linear relation is (to be precise, *appears to be*, since we can only plot a finite number of values) a straight line, we still need to understand why this is so. In addition, we need to understand why a straight line is the graph of a linear relation. For this, we seek an algebraic characterization of a straight line, and to get there we have to begin with geometric ideas. Here we introduce *dilations*: transformations of the figures in the plane that retain "shape" but not "size." These properties will be examined in detail in chapter 9; for the present purpose it suffices to observe that a dilation takes a right triangle with horizontal and vertical legs to another such triangle, and that the lengths of the corresponding sides of the triangles are proportional.

If we draw a line in the plane (that is neither horizontal, nor vertical) and then pick two points on the line, the segment of the line between those two points is the hypotenuse of a right triangle with vertical and horizontal legs. This we call *the slope triangle* for that segment. If we now draw the slope triangle for another pair of points, we can exhibit a dilation that takes one triangle to the other. The fundamental property of dilation is this: the length of line segments and the length of the images are proportional, with constant of proportionality the *factor* of the dilation. We conclude from this that the slope of any slope triangle on a given line is constant and this is called the *slope* of a line. This constant is in fact the *rate of change* of y with respect to x.

This leads directly to a way to calculate the *equation of a line*, which is the algebraic expression of the relation between the variables y and x graphically expressed by saying that they lie on a line. The outcome, which will be explored in detail in the next chapter is this: for a line L and two points P and Q on L, construct the slope triangle whose hypotenuse is the segment PQ. Let m be the slope of that segment. Now, let $(0, b)$ be the point on the y-axis that lies on the line. Then the equation of the line is $y = mx + b$.

Figure 1 illustrates the geometry in this discussion. We show the cases for both positive and negative slope, to emphasize that slope is not the ratio of the lengths of the sides of the slope triangle, but the ratio of the *changes* in the variables. Thus, when the change in y is negative for the corresponding increase in x, then the slope will be negative.

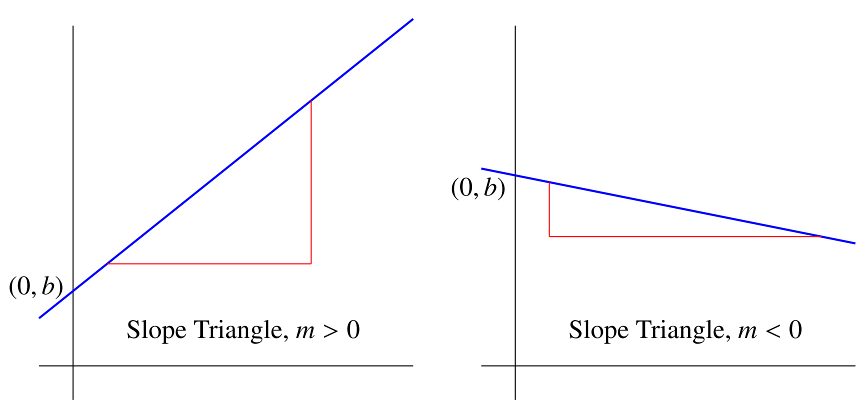


Figure 1

Section 2.1: Linear Patterns and Contexts

Proportional Relationships

Graph proportional relations, interpreting unit rate as the slope of the graph, which is a straight line. Compare two proportional relationships represented in different ways (tables, graphs, equations). 8.EE.5

The ideas of ratio and proportion were introduced in grade 6 and further developed in grade 7. In this section, after a brief review of the development of these ideas, we move on to the relation "proportional" (from the focus on the values of variables "in proportion." This complements the gradual move away from the language of "unknowns" and "equations" to that of "variables" and "relations."

In grade 6 the concept of *ratio* is introduced as "a way of describing a relation" between two collections of objects without reference to the actual size of those collections. So we may say that, in U.S. population, the ratio of minors to adults is 2:5, meaning that for every two minors there are five adults. This knowledge tells us, for example, that if we collect together a random group of people of size 140, we should expect 40 of them to be minors.

An old joke says that a shepherd keeps track of the herd by counting the legs and then dividing by 4. In terms of ratios, this expresses the fact that the ratio of sheep to sheep legs is 1:4. Actually, in Utah and Nevada where there are sheep herds numbering in the thousands, the shepherd keeps track of the herd by counting the black sheep and then multiplying by 40. This works for two reasons. First, sheep are social animals congregating in groups, so a count of sheep that estimates the actual number of the sheep within the size of a group has not missed any groups of sheep (maybe a stray lamb or two, but then the mother ewe will notify the shepherd of her distress). Second is that the ratio of sheep to black sheep is, for reasons of genetics, 40:1.

The concept of ratio (used mostly in counting individuals in particular sets) leads to the concept of *proportion*, which is more convenient than ratio for quantities that take on all numerical values, not just integral values.

- Given two quantities x and y, they are said to be *proportional* if, whenever we multiply one by a factor r, the other is multiplied by the same factor, r. For example, if we double the variable x , then y also doubles.

If two quantities are measuring the same physical attribute, they are going to be proportional. When we measure the length of a rod, we may do so in yards (Y), or in feet (F). Since these are measures of the same physical characteristic, they have to be proportional: if the rod triples in size, then its measure in feet or in yards also triples in size. A yard is defined as being 3 feet long, so we say that the ratio of yards to feet is 1:3. This can be rephrased as a proportional relationship with the *unit rate*: 3 feet per yard. We can write this algebraically as $F = 3Y$.

- If quantities y and x are in proportion then the *unit rate* of y with respect to x is the amount of y that corresponds to one unit of x. If m is the unit rate, then for any value of x, the corresponding y value is mx. If we interchange the roles of y and x, we would speak of the unit rate of y with respect to x .

Since the unit rate of feet to yards is 3, $F = 3Y$ we can rewrite this as $Y = \frac{1}{3}F$, telling us that the unit rate of yards to feet is 1/3.

EXAMPLE 1.

There are 5280 feet in a mile. How many yards are in a mile?

SOLUTION. Since one foot is $\frac{1}{3}$ of a yard, 5280 feet are $\frac{5280}{3}$ yards:

$$1 \text{ mile } = 5280 \text{ feet } \times \frac{1 \text{ yard}}{3 \text{ feet}} = \frac{5280}{3} \text{yards} = 1760 \text{ yards}.$$

In seventh grade, the unit rate is reinterpreted as the *constant of proportionality*. This corresponds to the change of focus from specific instances of a proportional relationship to that of the relationship itself. This leads to the equation $y = mx$, where y and x are the quantities in the proportional relation, and m is the constancy of proportionality. When two variables are proportional, all we need to know is one specific pair of values (x_0, y_0) in the relation to be able to compute all such pairs of values, for the ratio y_0/x_0 gives us the value of m. Graphically this is clear: if we know a pair of values (x_0, y_0) in the relation, all pairs (x, y) in the relation lie on the line joining $(0, 0)$ to (x_0, y_0). So all we need to do is to draw the line joining the origin to the given point, (x_0, y_0).

The concepts of *ratio*, *constant of proportionality* and *unit rate* seem interchangeable, since they can all be represented by the same quotient; this could be a source of confusion with students. The way to address this is to first understand that these terms *are* interchangeable, but emphasize different ways of viewing a relation, depending upon the context. The second step in addressing this issue is to explore the differences among these concepts as different ways of looking at a problem in context, and that one has to learn how to decide which interpretation is relevant for a given context. Here is a way to illustrate this.

Example 2.

In basketball, it is necessary to have 12 players in a roster. In a particular district in Eastern Utah, the middle school basketball league has teams that are made up of boys and girls. For fairness, it is decided that each team must have 7 girls and 5 boys. This tells us that the ratio of girls to boys in the basketball league is 7:5. The relation, girls to boys in the basketball league is a proportional relationship, with constant of proportionality "girls to boys" equal to 7/5.

Question 1. The district decides to have 8 teams in the league. How many girls and boys are there in the competition? This problem guides us to think in terms of the ratio 7:5: since there are 8 teams, each of which has 7 girls and 5 boys, the total number of players are $8 \times 7 = 56$ girls, and $8 \times 5 = 35$ boys.

Question 2. There are 45 boys eligible for basketball. How many girls are needed to to complete the league? Here we want to think in terms of the constant of proportionality, which is 7/5. So the number of girls needed is 7/5 of the number of boys available; that is, $(7/5) \times 45 = 7 \times 9 = 63$.

Here is a different problem: my grandfather drives at exactly 30 miles per hour.

Question 1. If Gramps drives 5 hours, how far does he go? Here, we think of unit rate: the rate of miles per hour is 30. Since

$$\text{miles} = \frac{\text{miles}}{\text{hours}} \times \text{hours} ,$$

he traveled $30 \times 5 = 150$ miles. Question 2: Gramps wants to drive to St. George from Logan; that is 440 miles. How long will it take him at that rate. Here we want to convert to minutes, and the concept of ratio: the ration of minutes to miles is 2:1. So to drive 440 miles, takes Gramps 880 minutes, or 880/60 = 14. 6667, or 14 hours and 40 minutes.

Extension

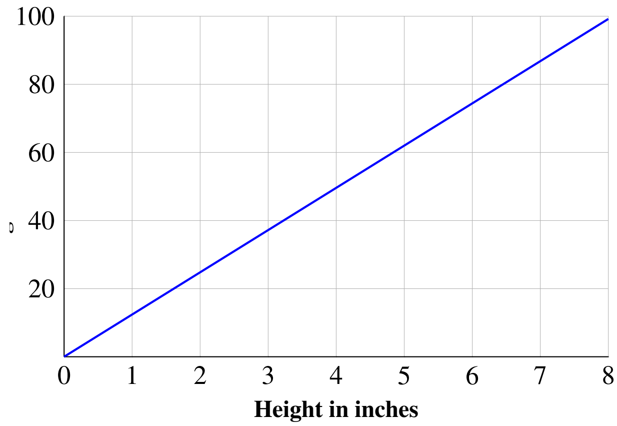
Example 3.

To illustrate the development of a proportional relationship, consider measuring the amount of water in a cylindrical container (a glass or can). If we put a quantity of of water in the cylinder, we record the height of the column of water, H, and the weight W of the column of water. Both of these are measures of the amount of water: if we double the amount of water, both the height and the weight double.

Suppose that this experiment is done with a quantity of water, filling the cylinder a bit at a time, and each time, measuring both H and W. A table of the data would look something like this:

Height	0	2	3	4	6	8	inches
Weight of Container	2.5	2.5	2.5	2.5	2.5	2.5	ounces
Measured Weight	2.5	27.3	39.7	52.1	76.9	101.7	ounces
Weight of Water	0	24.8	37.2	49.6	74.4	99.2	ounces

Notice that, we have accounted for the weight of the container itself by first measuring it empty, and then subtracting that weight from the weight of the container and water at each measurement. We now graph the these data, plotting height along the horizontal axis and weight on the vertical: The graph appears to be a straight line, giving confirmation of our hypothesis that the height of the column of water and its weight are proportional. We can calculate the unit rate of change using any one of the measurements: for example, 4 inches of the water weighs 49.6 ounces, so we have 12.4 ounces per inch of water. This is expressed by the relation $W = 12.4H$. In an actual experiment there always will be slight variations due errors or estimation, given the accuracy of the instruments used. So, the rates computed from each measurement may differ slightly. We will return to this in the statistics chapter.



Graph of Example 3 Data

In summary, the height of the column of water in the container and its weight are two different ways of measuring the volume of the column of water. Since the volume of the water is the same no matter how it is measured, the measurements are related. Similarly, yards, feet, inches and meters are different ways of measuring lengths; ounces, pounds, grams are different ways of measuring weight. All these relations have the property that a doubling or halving of the object (volume of water or length of stick) has the effect of doubling or halving the measure. In fact if the amount of the object is changed by the factor a, then any measure of the object also changes by the factor a. When quantities are related in this way, we say that they are *proportional*.

End Extension

Example 4.

If we are told that x and y are in the relationship $y = 7x$, then (1,7) , (2.5, 16.5), (8,56) are all in this relationship, because the ratio of the y value to the x value is always 7.

Example 5.

There are 5280 feet in a mile, so Feet/Miles = 5280, or Feet = 5280×Miles. To find out how many feet are in a quarter mile, let f represent that number of feet. Then we have $f = 5280(1/4) = 1320$ feet. In yards, that is 1320/3 = 440 yards.

Example 6.

We have made measurements of two quantities, and formed this table:

x	0	2	4	5	7	8	10
y	0	3.6	7.2	9	12.6	14.4	18

When we put this points on a graph, they appear to lie on a straight line through the origin. This suggests a proportional relationship: Notice that whenever the value of x doubles, so does the value of y, and that a change in x of 1 unit is accompanied by a change in y of 1.8 units. Finally, when we calculate the quotient y/x for any pair of points, we always get the value 1.8. This can be phrased this way: the proportional relationship $y = 1.8x$ models the given data.

- If quantities y and x are in proportion then the graph of pairs (x, y) in this relation will be a straight line through the origin. That line is characterized by the assertion that y/x is constant, and in fact, is the constant of proportionality. In terms of the graph, we call this its *slope*.

Linear relationships

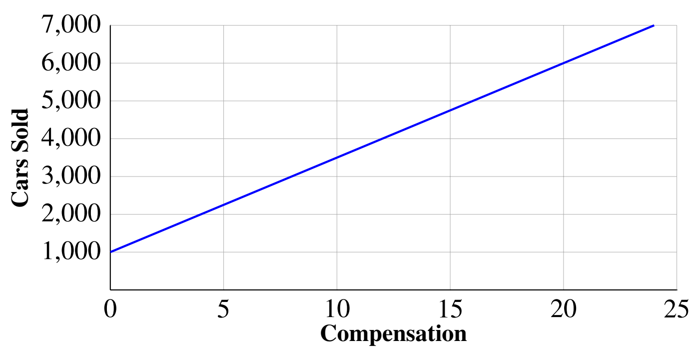
Construct a function to model a linear relationship between two quantities. 8.F.4

In Chapter 1 we concentrated on solving linear equations of the form $y =$ *linear expression*, where y is either a number or another linear expression. We also compared two linear expressions by graphing them (see the figure on page 8 of Chapter 1)), and found that the graph of each linear expression is a line. In this chapter our goal is to see why there is this correspondence between linear expressions and lines. Let's start by taking another look at Example 14 of Chapter 1.

Example 7.

A salesman at the XYZ car dealership receives a base salary of \$1000/month and an additional \$250 for each car sold. How many cars should he sell each month so as to earn a specified amount each month? There we ended up with this formula: $C = 1000 + 250N$, where N is the number of cars sold in a month, and C is the compensation received. Let's make a table for some possible values of N and then graph the result:

N	0	4	8	12	16	20
C	1000	2000	3000	4000	5000	6000



Cars Sold and Compensation

The graph (see the figure above) is a straight line that does not go through the origin: even if the salesman sells no cars, he receives the base salary of \$1000. Also note that to each increment of 4 cars sold, the salesman receives an increase of \$1000. In particular we can say that *the increase in income is to the increase of number of sales as 1000:4* giving us a unit rate of \$250 in compensation per unit of cars sold. This is just the coefficient of N in the equation $C = 1000 + 250N$.

Let us focus on what is learned by this example. Here the number 250 expresses a relation between the variables N and C, even though those variables are not proportional. C is the total compensation of the salesperson, and N is the number of increases by \$250 (above the base salary of \$1000). The actual *change* in compensation is $250N$. If we write the relation as $C - 1000 = 250N$, then we see this algebraically: the change in compensation from the base of \$1000 is *proportional* to the number (N) of cars sold.

Let's look at a few more examples to emphasize this point and to see how students should be able to extend this idea.

Example 8.

At the statewide championship game, each player on each team receives five complimentary tickets, and can buy additional tickets at \$20 each. Carlos wants 8 tickets and Louis wants 16 tickets. How much does each pay for the full set of tickets?

Solution. One might to say that, since Louis is getting twice as many tickets, he has to pay twice as much. But that would be a mistake, the cost is not proportional to the number of tickets, but cost *is* proportional to the number of tickets *in excess of 5* . In this situation, they each get 5 complimentary tickets, so Carlos pays for 3 tickets and Louis pays for 11 tickets. At \$20 apiece, Carlos pays \$60 and Louis pays \$220.

By applying this thinking to the general case, we can write down a formula for the cost C of N tickets for any player. If a player wants N tickets, he gets 5 free and pays \$20 each for the remaining tickets. There are $N - 5$ remaining, so the cost is $C = 20(N - 5)$ or $C = 20N - 100$. The form of these equations tell different things, both interesting. The first ($C = 20(N - 5)$) tells us that the cost is proportional to the excess of tickets above the first 5. The second tells us that the cost is \$20 per ticket, less \$100 for the free 5 tickets. Note that these equations make sense only for $N \geq 5$; players don't get refunded if they have less than 5 friends. In figure 2 we have graphed this relationship:

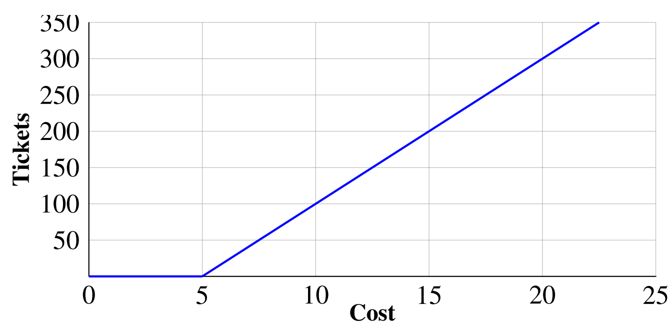


Figure 2

For any table where rate of change of one variable with respect to the other is **not** constant, the graph of these data will **not** be a straight line. Students will begin to explore these types of relationships more in Secondary 1. In 8th grade students simply need to recognize if a relationship is linear or not.

Example 9.

I have a 120 gallon steel drum full of water to keep my garden thriving through a long dry spell. Each day I use four and a half gallons watering my plants. How much water do I have in the drum after 10 consecutive dry days? After d consecutive dry days? How long can I last without rain or refilling my drum?

Solution. If I use 4.5 gallons of water each day, in 10 days, I use $4.5(10) = 45$ gallons of water, so there are 75 gallons of water still in the drum. After d days there are still $120 - 4.5d$ gallons in the drum. Using the symbol w to indicate the amount of water in the drum, this gives me the relation $w = 120 - 4.5d$. Figure 3 is the graph of that relation.

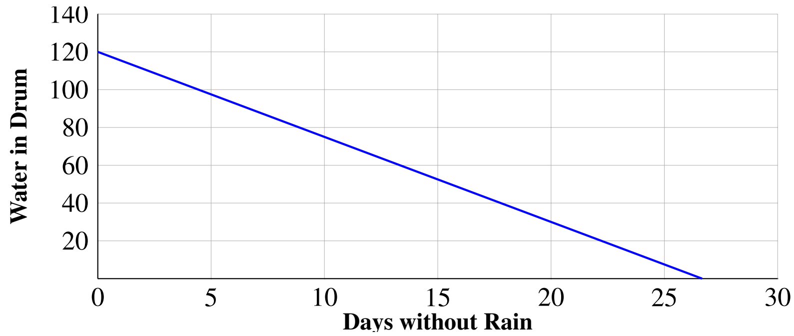


Figure 3

Notice that this time the line is pointing downward, that is because the amount of water in the drum decreases as the number of days increases. If we calculate (change in w)/(change in d) for any two points on the graph the result will be -4.5, indicating that each day we have 4.5 gallons less in the drum. It is important to note that language plays a role here: the word *less* accounts for the negative sign. We should be wary of expressing this twice. it would be wrong to say that "each day we have -4.5 gallons less in the drum." What is correct is "the ratio of change in water to the change in day is -4.5 gallons to 1 day," or the unit rate of change in water in one day is -4.5 gallons.

Example 10.

Consider the image in Figure 4:

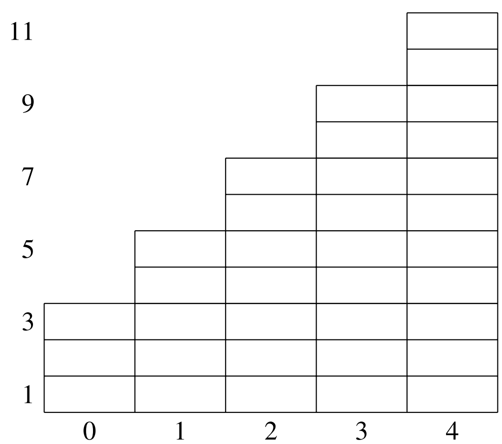


Figure 4

There is a pattern here: each time we move to the right by one unit, the height of the stack increases by 2. We have labeled the axes in figure 4 with x representing the number of moves to the right, and y the height of the stack. So, the first stack has the value 0, indicating that there are no moves to the right yet, and the last stack is 4 moves to the right. The height of the stack starts at 3, and with each move to the right, increases by 2. This tells us that the algebraic relationship is $y = 3 + 2x$.

Extension

The following example introduces a context (mountain climbing) where the rate of change may change several times during the experience, but is otherwise constant (we might say "locally constant." This is something worthwhile to look at for a while, without making too much of it. Students should come to understand that at any stage of their learning they are not at the end of that process.

Example 11.

The Timpanooke trail (see Figure 5) is a 7 mile trail from the foot of Mt. Timpanogos (at 7200 feet) to the peak (at 11900 feet). The trail has three different segments: the first is a three and a half mile horse trail with a steady altitude gain; the second is a two and a half mile traverse across a nearly level basin, and the last is a one mile steep climb to the peak. The accompanying table shows the altitudes at each of these transition points, and the time it takes an average hiker to cover each leg. Make two graphs; on both the horizontal axis is "miles" and on one, put "altitude" on the vertical axis, and on the other, "hours." Calculate, for each leg of the trek, the rate of change of altitude with respect to miles, and of hours with respect to miles. Compare and contrast the two graphical representations. Can you explain the similarities in the two graphs?

Figure 5: `http://farm3.static.flickr.com/2659/3692964206215e54c7d7.jpg`

Timpanooke Trail				
Altitude	7200	8700	10,700	11,900
Miles	0	3.5	6	7
Hours	0	2	3.5	5

End Extension

Summary. In the preceding examples of linear relations (except Example 11 where the process is more complex), we have seen from the tables of values that the rate of change in y with respect to x is constant. That constant is positive if the graph points upwards as we move from left to right, and negative if the graph points downward. If the graph is horizontal, there is no change in y, so the rate of change is 0. It seems to always turn out that the graph of a linear relation is a straight line, but this is something we cannot yet explain. It is important to always keep in mind that the subject of mathematics - indeed every science - is not to just record observations, but to study them enough to be able to explain them. That is the only way that the working scientist can make progress, be it in mathematics, medicine or space travel. In classroom discussions, it is important, where relevant, to distinguish between *observation* vs *understanding*. There is nothing wrong with drawing conclusions from observation and holding them dearly, but real progress comes from finding out *why* observations are indeed factual. Here we're observing that the relationship is linear, and in the next section we shall develop the mathematics necessary to *understand* why the graph of a relation of constant change is a straight line.

To begin this understanding, we ask: what is the property of a line that guarantees that it will be described by a relation of the form $y = mx + b$? What we know about a line is that it is determined by two points: place a straight edge against the two points and draw the line. A line is also determined by a point and a direction: lay the straightedge against the point, and set it in the intended direction and now draw the line. To relate these to a condition on linear relations, we need to find an algebraic way of expressing this geometric, constructive criterion. This is done with the concept of *slope* of a line and its relation to rate of change.

Section 2.2: Slope of a Line

Describe the effect of dilations ... on two dimensional figures using coordinates. 8.G.3: that the image of a line is a line parallel to it; that, under a dilation a line segment goes to a line segment whose length is the length of the original segment multiplied by the factor.

Use similar triangles to explain why the slope m is the same between any two distinct points on a non-vertical line in the coordinate plane; derive the equation y = mx for a line through the origin and the equation y = mx + b for a line intercepting the vertical axis at b. 8.EE.6.

In order to respond to this last standard in the way it is stated, a chapter on transformational geometry up to similarity would have to precede this chapter. We felt that it is important in eighth grade to begin the year by completing the set of ideas around linearity, and that an initial chapter on geometry would be a diversion from this main point of 8th grade mathematics. Since all that is needed to understand the main fact about slope are the two properties of dilations cited above, we decided to minimize the geometry to these facts, and then return to the relation of the rate of change of a linear function and the slope of the graph of that function: they are the same. Dilations are connected to scaling, so it could be useful to recall at this time that discussion in 7th grade.

A *dilation* is given by a point C, the center of the dilation, and a positive number r, the factor of the dilation. The dilation with center C and factor r moves each point P to a point P' on the ray CP so that the ratio of the length of image to the length of original is r: $|CP'|/|CP| = r$.

Figure 5 illustrates a dilation. In the figure, the center of the dilation is C, and its factor is r. We have exhibited 3 original points, P, Q, R and their images under the dilation P', Q', R. The letters a, b, c are the distances of P, Q, R from C.

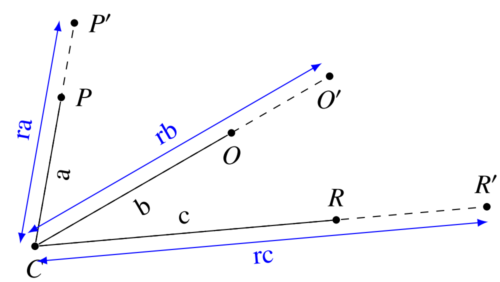


Figure 5

EXAMPLE 12.

> In Figure 6 we illustrate the effect of a dilation with center $C = (0, 0)$, and factor $r = 2.5$ on a triangle in the first quadrant of a coordinate plane.

Observe the connection of this image with those in the 7th grade discussion of scale drawings. Note also that a point $(x, 0)$ is moved to the point $(2.5x, 0)$, and a point $(0, y)$ is moved to $(0, 2.5y)$. In fact, any point (x, y) is moved out to the point on the lines through the origin and that point whose distance from the origin is 2.5 times that of (x, y). That the coordinates of this point are $(2.5x, 2.5y)$ is easily observed, and gives the coordinate description of a dilation with center the origin. Students should work many examples of this type to conclude that

> In a coordinate plane, the dilation with center the origin and factor r is given by the coordinate rule $(x, y) \rightarrow (rx, ry)$.

Expressing a transformation of the plane in terms of coordinates provides an algebraic tool to help work with calculations, but it is not as important at this stage as being able to understand the properties of dilations. Students

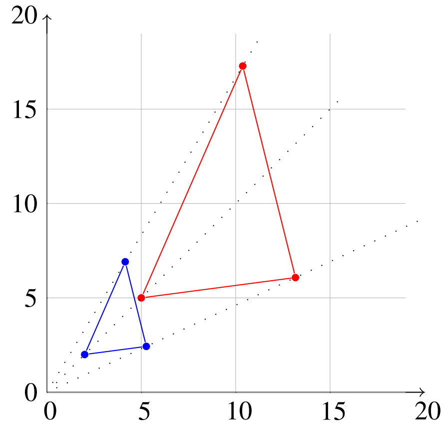


Figure 6

should experiment with specific examples of dilations (with factors less than 1 as well as greater than 1); enough to see that these properties are true for any dilation:

Properties of the dilation with center C and factor r:

a. If P is moved to P', then $|CP'|/|CP| = r$. That is, the distance of P' from C is r times the distance of P from C

b. If P is moved to P' and Q is moved to Q', then $|Q'P'|/|QP| = r$. That is, under a dilation, the length of any line segment is multiplied by the factor of the dilation.

c. The dilation takes parallel lines to parallel lines.

d. A line and its image are parallel.

e. A line through the center is its own image, even though the points on the line move by the factor r.

The first is part of the definition of a dilation. The second tells us that *every* length, not just those on lines through the center, is multiplied by the factor of the dilation. The last two about parallelism are central properties of dilations. They are easy to observe through examples, and they are intuitively plausible. At this stage it could be desirable to see why these last two are true, starting with the fact that parallel lines have no point of intersection. We will return to these geometric facts about dilations in Chapter 9; for now what is important is for the student to understand that the characteristic property of a straight line follows from the properties of dilations.

In the preceding section we observed that the graph of a proportional relation is a straight line through the origin, we now turn to understanding why this statement and its converse is true. The key here is the above set of properties of dilations. Let's start with the statements that we want to understand:

- A non-vertical straight line through the origin is the graph of a proportional relation $y = mx$.
- The graph of the proportional relation $y = mx$ is a non-vertical straight line through the origin.

We start with the first statement, and then show that it implies the second. In Figure 7 we have drawn a typical line L through the origin. P is the point whose first coordinate is 1 and whose second coordinate is m. Q is any other point on the line with coordinates (x, y). We introduce the dilation with center the origin that takes the point P' to Q'. Since the length 1 goes to the length x, the factor of the dilation is x. Now, the dilation takes the vertical line PP' to a parallel, and therefore also vertical, line through Q'. That line has to intersect L at Q, since the line L is not changed in the dilation. Now, the dilation multiplies the length of PP' by x, so the length of QQ' is mx. But that is y, so we can conclude that $y = mx$. Since Q was any other point on L, we have shown that L is the graph of the proportional relationship $y = mx$.

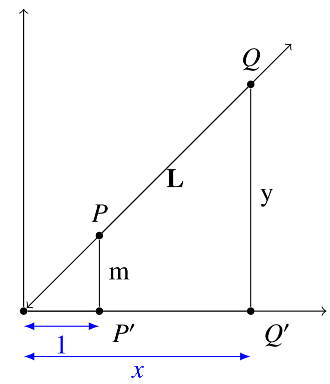


Figure 7

As for the converse, we start with a proportional relationship $y = mx$. Draw the line through the origin and the point $(1, m)$. By the argument above, this line is the graph of the proportional relationship $y = mx$.

Now, we shall repeat this argument for a non-vertical straight line that does *not* go through the origin. . The result is just as above, but with "proportional" replaced by "linear," and "unit rate" replaced by "rate of change." The statements are:

- A non-vertical straight line is the graph of a linear relation $y = mx + b$.
- The graph of the linear relation $y = mx + b$ is a non-vertical straight line through the point $(0, b)$ (called the y-intercept).

To see why these are true, start with a linear relation $y = mx + b$. Of course, when b = 0, the graph goes through the origin and is a proportional relationship, so, by the above argument the graph is a straight line. Now we could argue as follows: Given the equation $y = mx + b$, first look at the graph of the proportional relation $y = mx$. We now know that that is a straight line L. If we shift that graph in the vertical direction a distance of b units, we still have a straight line L'. We also know that if (x, y) are the coordinates of a point on L', then $(x, y - b)$ are the coordinates of a point on L. So we must have $y - b = mx$; that is: this equation is satisfied by the coordinates of any point on L'. But this is the same as $y = mx + b$, so L' has to be the graph of the linear relation $y = mx + b$.

For the converse, start with a non-vertical line L. Since it is non-vertical it intersects the y-axis in a point $(0, b)$. If we shift this point to the origin, we get a new line L' through the origin which is, therefore, the graph of a proportional relationship $y = mx$. But if (x, y) is on L, $(x, y - b)$ is on L' and so we again have $y - b = mx$ as a relation defining the line L, or what is the same $y = mx + b$.

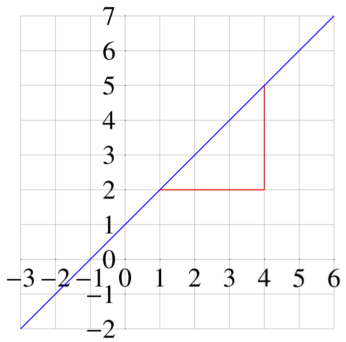


Figure 8

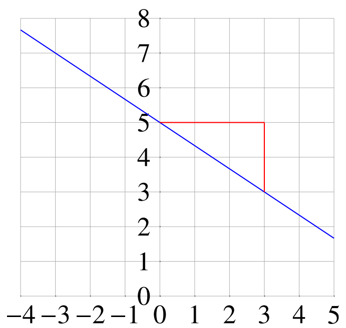


Figure 9

This argument gives a geometric meaning to the number b: it is the y-coordinate of the point of intersection of the line with the y-axis (the y-intercept). But what is the geometric meaning of the number m?

Let us start again with a non-vertical straight line in the coordinate plane, that doesn't go through the origin, but through some point $(0, b)$ on the y-axis. We know that two points on a line determine the line: just put a straight edge against both points, and draw the pencil along the straightedge. We now want to see how to describe this in terms of coordinates: how do the coordinates of two points on a line determine the relation between the coordinates of *a*ny point on the line? This is where slope comes in.

Given two points in the coordinate plane, P and Q, we define the rise to be the difference of the y values from P to Q, and the run to be the difference in the x values from P to Q. The slope of the line segment is the quotient of these two differences:

$$slope = \frac{rise}{run}$$

If P has the coordinates (x_0, y_0) and Q has the coordinates (x_1, y_1) this is

$$slope = \frac{y_1 - y_0}{x_1 - x_0}$$

Geometrically, if we draw the triangle with hypotenuse the line segment from P to Q and legs horizontal and vertical - this is the *slope triangle* - the slope is the signed quotient of the length of the vertical leg by the length of the horizontal leg. By *signed*, we mean that the slope is positive if the line points upward as we go to the right, and negative if the line points downward. (see figures 8 and 9).

Note that in the slope computation the differences have to be taken in the same order: if we subtract the y value of P from that of Q, we must subtract the x value of P from that of Q. However, if we interchange the points P and Q, we get the same number. For a vertical line, the denominator in the quotient is zero, so the slope is not defined.

For a horizontal line, the numerator is zero, so, the slope is zero. Since the equation of a horizontal line is of the form $y = b$, this corresponds to the fact that y does not change as we move along the line. What we want to show is this: for a line L, this slope calculation is the same for any two points P and Q on L and is called the *slope of the line*.

Let L be a non-vertical line, P ,Q and P', Q' two different pairs of points on the line, and T and T' the right triangles whose hypotenuses are the given line segments, and whose legs are horizontal and vertical. Label the vertices at the right angles as V and V' (see figure 10). These two triangles appear to be related by a dilation; we want to show that they are. First, if there is a dilation that takes T to T', it must be the case that P goes to P', so the line L is a line through the center of the dilation. Also, V goes to V', so, by the same reasoning V and V' also lie on a line through the center of the dilation. Let L' be the line through V and V'. The point of intersection C of L and L' has to be the center of the dilation (figure 11), and its factor r has to be the ratio of the length of CP' to that of CP. Let's verify that this dilation does take T to T'. First of all, it takes P to P', since that is how r was chosen. Since the dilation preserves "horizontal," and preserves the line L', it takes the segment PV to $P'V'$, and so the ratio of those lengths is also r. Since the dilation preserves "vertical," and preserves the line L, it takes the segment QV to $Q'V'$, so the ratio of those lengths is also r. Thus, in moving from T to T', the length of every side is multiplied by the same factor r, so when we calculate the $rise/run$, the r's cancel, and the quotient is the same for both triangles.

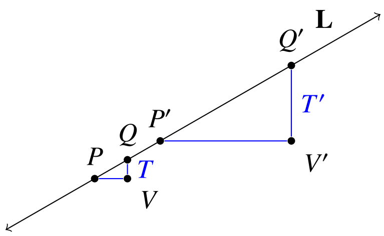


Figure 10

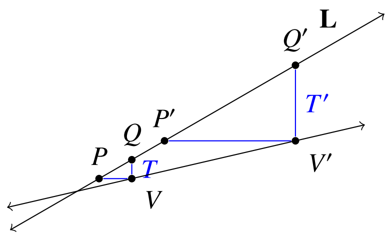


Figure 11

There is one case not covered: the lines L and L' may not intersect; that is, the are parallel. In this case, (see Figure 12) under the shift of P to P', the triangle T slides along these tracks to T' without changing the lengths of the sides.

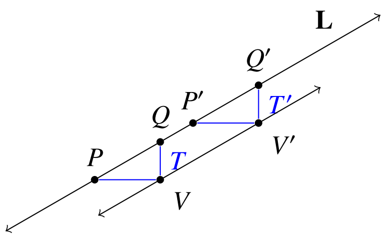


Figure 12

Section 2.3: The Equation y = mx + b.

To wrap up this chapter, we bring together all the preceding material, not simply to summarize it, but also to lead in to the next chapter, a study of linear functions and lines, the purpose of which is to develop flexibility in moving among the representations of linear relations.

- For a line L, for any two points P, Q on the line, the quotient

$$\frac{rise}{run} = \frac{\text{change in } y \text{ from } P \text{ to } Q}{\text{change in } x \text{ from } P \text{ to } Q}$$

  is constant, and that constant is the *slope of the line*.

Example 13.

$(0, 5)$, $(2, 9)$, $(-1, 3)$ are three points on a line. Calculate rise/run for each pair of points.

Solution. First we should verify that indeed the three points lie on a line, using the slope calculation In each case that calculation produces 2 as the slope of the line, for example, taking the third and first points, we have:

$$\frac{3-5}{-1-0} = \frac{-2}{-1} = 2$$

Let us see what this example tells us. Start with two pairs of points; suppose that we find that the slope calculation produces the same number. This does not mean that the two pairs of points are on the same line (what does it mean?). However if there is a point in common to the two pairs, then all three do lie on the same line. This tells us something important. Given a line, pick two points P, Q on the line. Calculate the slope m using these two points. Now take any point X on the plane, and calculate the slope of the slope triangle using the points X and P (or Q). If the result is m, then, by this observation, X is on the line; if it is not m, then X is on the line. We have generated a protocol for deciding whether or not a point X is on the line through P and Q.

Going back to example 12, the line through any pair of these points has slope 2. So, for any point (x, y), if any of the calculations

$$\frac{y-5}{x-0} \qquad \frac{y-9}{x-3}, \qquad \frac{y-3}{x-(-1)}$$

gives the value 2, then they all do, and (x, y) is a point on the line. If any of these computations do not give 2, then none do, and (x, y) is not on the line. So, we have this test for a point (x, y) to be on the line:

$$\frac{y-5}{x-0} = 2$$

By multiplying both sides by x we get $-5 = 2x$, or $y = 2x + 5$. This is called the equation of the line. Instead of choosing the first point, we could have chosen one of the other two getting the test:

$$\frac{y-9}{x-2} = 2, \qquad \frac{y-3}{x-(-1)} = 2$$

No matter what point we choose for the test, after simplification we will always get the equation $y = 2x + 5$.

Chapter 3 starts with an examination of techniques to find the *equation of a line*, beginning with this example.