Chapter 3
Representations of a Linear Relation

The purpose of this chapter is to develop fluency in the ways of representing a linear relation, and in extracting information from these representations. In the first section we shall study linear relations among quantities in detail in each of the realizations: formulas, tables, graphs and context, and develop fluidity in moving among them. Although the Core Standards refers to the concept of *function*, in this chapter we continue this as a discussion of linear relations, which, when of the form $y = mx + b$ is called a function. The transition in thinking from "equations" to "relations" to "functions" is - as was that from "unknowns" to "variables" - subtle but significant. The outcome for the student is to develop a way of seeing functions dynamically, and as expressions of the behavior of two variables relative to each other. For these reasons, we move ahead slowly; in this chapter developing technique in studying linear relations, in chapter 4 concentrating on the simultaneous solution of two equations, and then returning to an in-depth study of functions in chapter 5.

In the second section we go more deeply into the relationship of the geometry and algebra of lines; giving the slope conditions for two lines to be parallel or perpendicular. There are two advantages in introducing this topic now. First, it provides an opportunity to introduce translations and rotations and use their basic properties, and second it gives an application of the idea of slope in comparing two lines.

Section 3.1: Linear relations: creating graphs, tables, equations of lines

Interpret the equation $y = mx + b$ as defining a linear function whose graph is a straight line. 8.F.3.

Determine the rate of change and initial value of the function from a description of a relationship, or from two (x, y) values, including reading from a table or a graph. 8.F.4.

In example 13 of Chapter 2, we considered three points: $(0, 5)$, $(2, 9)$ and $(-1, 3)$, and calculated the rise/run for each pair of points always arriving at the answer: $slope = 2$. So, $(2, 9)$ and $(-1, 3)$ are on a line through $(0, 5)$ of slope 2. But there is only one line through $(0, 5)$ of slope 2, so it must be that all three points lie on the same line.

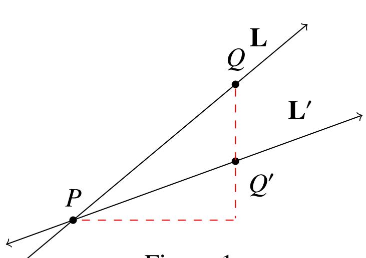


Figure 1

It is intuitively clear that there is only one line through a given point and of given slope, and figure 1 shows us why. The lines L and L' intersect at the point P; we have drawn the slope triangle with a run of one unit for both lines. If the lines L and L' are different, then the rise for these lines is different (otherwise Q and Q' would be the same point), and so the slope is different.

We can also give an algebraic argument. Start with two lines L and L' and let the equation for L be $y = mx + b$, and that for L' be $y = m'x + b'$. If the lines intersect that tells us that there is a value for x at which the expressions $mx+b$ and $mx' + b'$ have the same value (namely the y coordinate of the intersection point). Let x_0 be that value of x, so that $mx_0 + b = m'x_0 + b'$. Now, if the lines have the same slope, $m = m'$, and now subtracting mx_0 from both sides, we get $b = b'$; that is, the lines L and L' have the same equation, and therefore are the same line.

Let's pick up where we left off in Chapter 2, and recall the definition of *the equation of a line*. We start with a line L, and a point P on L with coordinates (a, b). If (x, y) is any other point on the line we must have

$$\frac{y-b}{x-a} = \text{slope of } L.$$

Because there is only one line through P of that slope, we also know that if (x, y) is *not* on the line L, this computation cannot give the slope of L. Thus the above equation is a test for a point to be on the line L, and thus is called *the equation of the line*. Returning to the points (0,5), (2,9) and $(-1, 3)$, we can use any two to calculate the slope, 2, and then use any of the three points to test for membership in the line:

$$\frac{y-5}{x-0} = 2 \quad \frac{y-9}{x-2} = 2 \quad \frac{y-3}{x-(-1)} = 2$$

Thus, for example the point (3,11) is on the line because it passes the test: the slope calculation with any of the given points always gives 2. On the other hand, (6.2) is not on the line, for each of the computations gives a number different from 2. Of course, we don't have to test the slope equation with every point on the line, but just with one point (and maybe another to check the calculation).

The equations above are not in simplest form, and if we clear of fractions and simplify to the form $y = mx + b$ we do get the same values of m and b. This could not be otherwise, for we can identify m and b as characteristics of the line: m is its slope and $(0, b)$ is the intersection of the line with the y-axis. So, when put in simplest form ($y = mx + b$), there is only one equation of the line.

Let's follow this through for each of the above equations, first clearing of fractions, and then isolating y on the left hand side of the equation

$$\begin{array}{ccc}
\dfrac{y-5}{x-0} = 2 & \dfrac{y-9}{x-2} = 2 & \dfrac{y-3}{x-(-1)} = 2 \\
y - 5 = 2x & y - 9 = 2(x-2) & y - 3 = 2(x+1) \\
y = 2x + 5 & y = 2x - 4 + 9 & y = 2x + 2 + 3 \\
 & y = 2x + 5 & y = 2x + 5
\end{array}$$

Is (3,10) on the line? We calculate the slope of the line segment between (3,10) and (2,9), and get 1. Thus (3,10) is not on the line. But (3,11) is a point on the line, since $(11 - 9)/(3 - 2) = 2$. More importantly, note that every slope calculation (as those just executed) always simplifies to a unique equation $y = mx + b$.

Once we know the slope of the line, we can use any point on the line to calculate the equation of the line. If we know two points on the line, we can use those points to compute the slope. Then, using one of the points and the slope, calculate the equation of the line. Restating this: if we know a point on a line and the slope of the line, we can calculate the equation of the line. This corresponds to the geometric fact that a point and a direction determine a line. Next, if we know two points on a line, we can calculate the equation of the line; corresponding to the the geometric fact that two points determine a line.

To sum up: The equation of the line (the test for a point (x, y) to be on the line) can always be written in the form $y = mx + b$, called the *slope-intercept form of the equation of a line* because m is the slope, and $(0, b)$, the y intercept is on the line. No matter what points on the line we choose for the calculations, the slope-intercept form of the equation will always be the same.

One last important point, to which we will return in the next section and again in Chapter 5. An equation of the form $y = mx + b$ describes a process: As the value of x changes, the value of y changes along with it. And, the slope is calculated as the quotient of the change in y by the change in x between any two points on the line:

$$m = \frac{y_1 - y_0}{x_1 - x_0}$$

Thisis the *rate of change* of y with respect to x, and the fact that the graph is a line tells us that the rate of change is constant. Since b is the value of y when $x = 0$, we also refer to b as the *initial value*.

Example 1.

Masatake runs at a constant rate. At a recent marathon, his friend Jaime positions himself at the 5 mile marker, and Toby is at the 8 mile marker. Masatake passes Jaime at 8:40 AM, and then passes Toby at 9:01 AM. If Masatake can keep up that rate, at what time will he finish the race? A marathon is 26.2 miles long.

Solution. The problem states (twice) that we assume that Masatake runs at a constant rate. Therefore the relation between minutes and miles is linear and the rate of change is constant. Note that the order of the variables is not specified, so that we can talk about the rate as minutes per mile or miles per minute. As we'll see, the context often shows us which to choose: in this case since the information desired has to do with time, we probably should describe the rate as minutes per mile. In any case, it is desirable to be flexible in the move from one to the other. Using the two measurements we can find that rate: between the two sightings he runs 3 miles in 21 minutes, so is running at a rate of 21/3 = 7 minutes per mile. When Toby saw Masatake, he still had 18.2 miles to run. At 7 min/mi, remembering that

$$\text{Minutes} = \frac{\text{Minutes}}{\text{Miles}} \times \text{Miles} ,$$

it will take him (7)(18.2) = 127.4 minutes, or about 2 hours and 7 minutes more. So Toby expects him to finish the race at 11:08 AM. If we also ask, at what time did Masatake start the race? - we know that he started 5 miles before he passed Jaime, and that is $5 \cdot 7 = 35$ minutes. Masatake started the race at 8:05, and he will have run the whole marathon in 3 hours and 3 minutes.

Notice that in working this problem we did not seek an equation to solve, but instead thought about the problem algebraically, using the basic rate equation. The basic fact used here is that the time running between two points is proportional to the distance between the points, and we find the constant of proportionality, 7, using the two given points. In the end, since we now know the initial value of time, we can write the equation for the line:

$$Time = 8:05 + 7\ (Miles) ,$$

however, be careful in computing with these numbers, since there are 60 minutes (not 100, as the notation might suggest) in an hour.

Example 2.

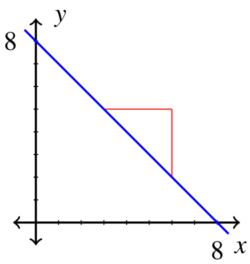
Given the point $P : (3, 5)$ and the number $m = -1$, find the equation of the line through through P with slope m.

Example 3.

Given the points $(2, 1), (-1, 10)$, find the equation of the line through those points. First we calculate the slope using the given points:

$$\frac{10 - 1}{-1 - 2} = \frac{9}{-3} = -3$$

SOLUTION. Following the above, the point $X : (x, y)$ is on the line if the slope calculation with the points X and P gives -1:



$$\frac{y-5}{x-3} = -1$$

This simplifies to $y = -x + 8$.

Now, the equation of the line is given by the slope calculation using the generic point (x, y) and one of the given points (say (2,1)):

$$\frac{y-1}{x-2} = -3$$

or, $y - 1 = (-3)(x - 2)$, which simplifies to $y = -3x + 7$.

EXAMPLE 4.

Given a straight line on the coordinate plane, such as that in figure 2, find its equation. One way to do this is to discover the values of b and m by locating the y intercept and drawing a slope triangle. In Figure 2, the y-intercept is 7, and if we go across by 1 unit, the graph goes up by 3 (be careful to note the different scales on the coordinate axes). This gives $y = 3x + 7$ as the graph. Another way is to locate two convenient points (such as (1,10) and (6,25) since the line goes through these intersections of gridlines), and use them to calculate the slope.

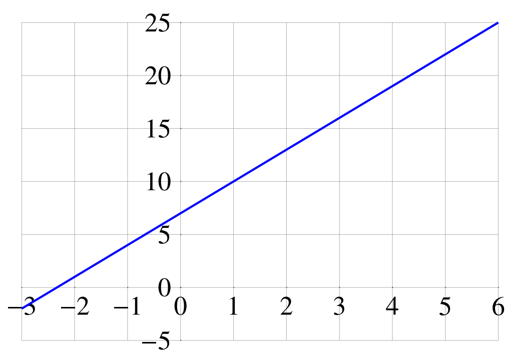


Figure 2

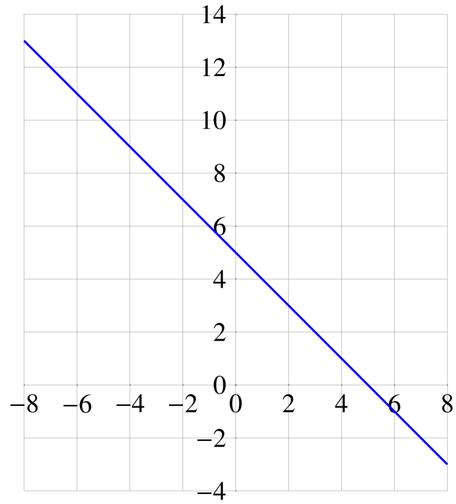
Typically, the first thing to do in trying to understand a relation is to make a table of solutions to see what information we can gather. Then, plot the data points on a graph, and connect them.

EXAMPLE 5.

Reconsider the relation $x + y = 5$. Make a table of representative values and graph.

x	−8	0	1	3	4	5	8
y	13	5	4	2	1	0	−3

From the table we see that as x increases, y decreases. In fact, whenever x increases by 1, y decreases by 1, confirming that the slope is -1. Here is the graph:



> Connecting the points on the graph, we get a straight line. Every point on the graph is a solution, even if it wasn't included in our list of solutions. For any point (x, y) on the graph, if we add x and y we get 5.

Notice that if we shift the line down by 5 units (or to the left by 5 units), the slope is still -1, but the new line goes through the origin. This new line then is the graph of the equation $y = -x$, which can be rewritten as the relation $x + y = 0$. Another way to see this is to notice that moving a point downward by 5 units is the same as subtracting 5 from the y coordinate of a point (with no change in the x coordinate). We can write this as $(x, y) \rightarrow (x, y - 5)$, and say: under the downward shift by 5, the point (x, y) goes to $(x, y-5)$. We can write this $x_{\text{new}} = x$, $y_{\text{new}} = y-5$, so the relation $x + y = 5$, rewritten as $x + (y - 5) = 0$ becomes $x_{\text{new}} + y_{\text{new}} = 0$.

We will return to the subject of shifts later in this chapter; for now it suffices to note that a the geometric act of shifting downward by 5 units amounts to the algebraic act of replacing $y - 5$ by y. Similarly, a shift to the left by 5 units is realized algebraically by replacing $x - 5$ by x.

Let's go back to the signal characteristic of a linear relation between the variables x and y: the rate of change of y with respect to x is constant. This constant is the slope of the graph of the relation. In particular, if a situation is given in which two variables are related and there is a constant rate of change, then the relation is linear. What we have wanted to observe in the last example is that if we shift the graph of a linear function so that the new graph goes through the origin, then the new graph is a graph of a proportional relation. Furthermore, the rate of change of y with respect to x along both graphs is the same.

Compare properties of functions (linear) presented in a different way (algebraically, graphically, numerically in tables or by verbal descriptions). 8.F.2.

Interpret the rate of change and initial value of a linear function in terms of the situation it models, and in terms of its graph or table of values. 8.F.4.

Up until now we have been speaking informally about a "relation" between the variables x and y rather than "function" and now is a good time to put these words on a firmer footing. Technically speaking, a *relation* between the variables x and y is a region in the plane. Examples:

- **a.** $x < y$, the relation x is less than y, is the region in the plane below the line through the origin that makes a 45° angle with the x-axis;
- **b.** x and y are both less than 1 and greater than 0 is represented by the unit square in the first quadrant;
- **c.** a line in the plane expresses the relation "(x, y) is a point on the line."
- **d.** $5x - 3y + 7 = 0$ is a relation expressed algebraically: it consists of all pairs (x, y) that satisfy this relation.

In this chapter, we focus on *linear relations* as described in example **c.** above. A *linear relation* is represented by a line in the plane: (x, y) are in this relation precisely when (x, y) is on the line. As we learned in the preceding section, if the line is non-vertical, we can describe it by an equation of the form $y = mx + b$. If the line is vertical, there is no relation in the colloquial sense, because x is always the same number, while y can be anything. Similarly, if the line is horizontal ($y = 0x + b$), there is no discernible relation since x can be anything, and y always remains equal to b.

When the relation is given as a recipe for going from a value of x to its related value y, then we say that y is a *function* of x. We look for functions when we have a situation involving two variables x and y, and we have a strong suspicion that the value of x somehow determines the value of y. Then we look for the recipe that makes the "somehow" explicit. For example:

a. Our car salesman's monthly salary is determined by the number of cars sold and indeed we were given the recipe: the monthly salary is \$1000 plus \$250 for each sale in that month. Then we translated this to the algebraic expression: $C = 1000 + 250N$, where C is the number of income for N cars sold. Yes, C, compensation, is a function of the number N of cars sold.

b. The equation $y = mx + b$ expresses a function. The recipe is this: First, pick a number x. Second multiply it by m. Third, add b. The game described in Example 12 of chapter 1: "Pick a number from 1 to 20. Add three and double the result. Add 8. Take half of that number. Subtract your original number." It is a complicated way of describing the function: for any x, go to 7, which corresponds to the line $y = 7$, and that is the basis of the "trick:" to every number in the domain, the function assigns the number 7.

In Chapter 5 we will return to a full study of the concept of *function*. Here we want to explore a little further the idea of *relation* and how it leads to the concept of function.

Example 6.

We return to the relation: the sum of two numbers is 5.

Let x and y represent the two numbers. The sum of x and y is $x + y$; the assertion is that this is 5. This relation can be expressed by the equation

$$x + y = 5$$

The recipe: "Pick a number for x and solve for y" describes a function since we know how to solve linear equations, so this does give us a y for every x. If we write the solution symbolically we get: $y = 5 - x$, whose set of rules are: given the input x, subtract it from 5: that is the output y. This is one reason to think of the function as a black box rather than as a set of instructions: for there could be many different sets of instructions that give rise to the same function (and that is the basis for this kind of "math trick.")

Example 7.

Let's return to Masatake's marathon. His friend Jack also ran the race, but because of the crowd he didn't start running until 8:10. Jack however runs faster than Masatake, at 6.8 minutes per mile. Does Jack finish before Masatake?

Solution. At 6.8 minutes per mile, Jack runs the marathon in $(26.2)(6.8) = 178.16$ minutes, or 2 hours and about 58 minutes, and arrives at the finish line at 11:03 - a photo finish with Masatake!

To summarize: to find the equation of the line through the two points (x_0, y_0), (x_1, y_1), first calculate the slope of the line using the given points:

$$m = \frac{y_1 - y_0}{x_1 - x_0}$$

If (x, y) is any point in the plane then it is on the line L if and only if the slope calculation gives m:

$$\frac{y - y_0}{x - x_0} = m$$

If the variables x, y are in a linear relation (that is, the graph of the relation is a line), the relation can be expressed in the form of the function $y = mx + b$, where m is the slope and b is the y-intercept.

A *linear relation*, as we have been using it, is determined by an equation of the form $Ax + By = C$. If both A and B are zero, we just have the statement $0 = C$, which does not describie any relation between x and y. If just $B = 0$, we get the vertical line $x = C/A$ which does not describe a function of x, since x doesn't even change. If just $A = 0$, we get the line $y = C/B$ which does describe a function: for any x, let $y = C/B$. This is called a *constant* function, because the input x has no effect on the output $y = C/B$. If A and B are both nonzero, we can write the equation in the form $y = mx + b$, exhibiting y as a function of x: $y = -(A/B)x + (C/B)$. The graph of this relation is a non-vertical, non-horizontal line of slope $-(A/B)$ and y-intercept C/B.

Example 8.

Find the slope of the line given by the relation $3x - 7y = 11$, and write the equation in slope-intercept form.

Solution. By the above, the slope of the line is $-(3)/(-7) = 3/7$. Set $y = 1$ and solve the equation for x. We find $x = 6$, so (6,1) is a point on the line. We could also use the laws of arithmetic to write y in terms of x, getting

$$y = \frac{3}{7}x - \frac{11}{7}$$

from which we conclude that the slope of the line is 3/7, and the point $(0, -11/7)$ is on the line.

Section 3.2: Parallel and Perpendicular lines

Let's return to example 3 of Chapter 2, describing two different ways of measuring an amount of water: by the height of the column of water in a specific equation, and by the weight of the water. The point there was that the measured weight was *not* proportional to the height of the water because we had to account for the weight of the container. Subtracting that gave us the true weight of the water which is indeed proportional to the height of the column of water. In Figure 4 we have graphed the data of that example: the horizontal axis is the *height* of the column of water, and *weight* is on the vertical axis. The blue line is that of the measured weights (including the container), and the red line is the weight of the water. Here we want to note that the two lines are parallel, and that the difference between the measured weight and the actual weight is always the same, no matter what the height.

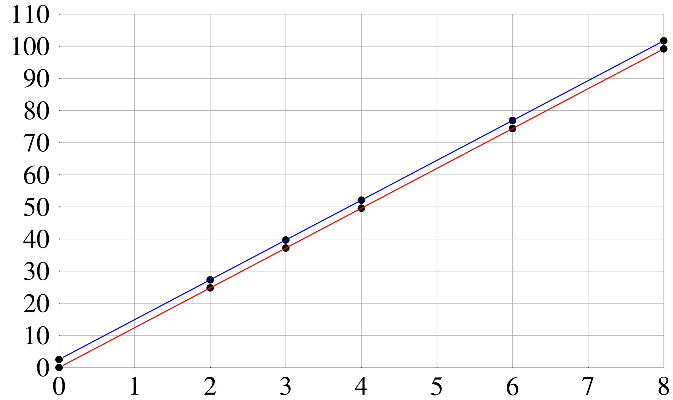


Figure 4

This may not appear to be an amazing fact, since the difference is obviously the weight of the empty container, but it does illustrate an important fact about parallel lines: we can carry one onto the other by a vertical shift of a constant amount. Indeed we can also move one line on the other by a horizontal shift of a constant amount. For this reason, we now replace the intuitive notion of parallelism with a definition in terms of shifts, or more generally, translation. A *translation* by (a, b) is a motion of the plane by a horizontally, followed by a motion of the plane by b vertically.

Example 9.

Consider the graph of $x + y = 5$, as described in Example 6. Shift the graph upward by 3 (that is by $(0, 3)$). What is the equation of the new line?

Solution. Since we have increased y by 3, $x + y$ has increased by 3, so the equation of the new line $x + y = 8$. Here is another way to see this. Let $(x_{\text{new}}, y_{\text{new}})$ be the coordinates of the point to which $(x_{\text{old}}, y_{\text{old}})$ is moved. We know that

$$x_{\text{old}} = x_{\text{new}} \quad \text{and} \quad y_{\text{old}} = y_{\text{new}} - 3$$

The equation of the old line is

$$x_{\text{old}} + y_{\text{old}} = 5$$

which is, in terms of the new coordinates:

$$x_{\text{new}} + y_{\text{new}} - 3 = 5$$

Since we are drawing the lines on the same coordinate plane, we can remove the word "new" to get

$$x + y - 3 = 5 \quad \text{or} \quad x + y = 8$$

Finally we can verify these observations with the table:

	x	-8	0	1	3	4	5	8
Old	y	13	5	4	2	1	0	-3
New	y	16	8	7	5	4	3	0

EXAMPLE 10.

Translate the line $y = 2x$ by one unit in each coordinate, so that $(x, 2x)$ goes to $(x + 1, 2x + 1)$. Find the equation of the new line.

SOLUTION. Here we have the relation

$$x_{\text{old}} = x_{\text{new}} - 1 \quad , \quad y_{\text{old}} = y_{\text{new}} - 1$$

So the equation

$$y_{\text{old}} = 2x_{\text{old}}$$

becomes

$$y_{\text{new}} - 1 = 2(x_{\text{new}} - 1)$$

removing the word "new," this becomes $y - 1 = 2(x - 1)$, which simplifies to $y = 2x - 1$

If the line $y = mx$ is translated by (a, b), then the equation of its image is

$$y - b = m(x - a)$$

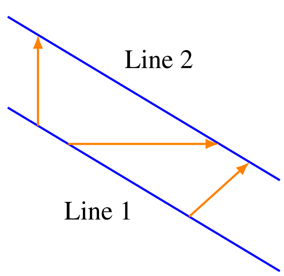


Figure 5

Two lines are said to be *parallel* if there is a translation that takes one into the other. The statement in the box tells us that if two lines are parallel, they have the same slope. It is also true that if two lines have the same slope, they are parallel, that is: there is a translation that takes one into the other. How do we find that translation? Consider the image in Figure 5 depicting two lines with the same slope:

Each orange arrow represents a translation. Observe, using two pieces of transparent graph paper, that each of those translations takes line 1 to line 2. You should conclude that, given any points P on line 1, and Q on line 2, the translation from P to Q takes line 1 to line 2. Since this diagram can be of any two lines with the same slope (we have omitted coordinates to emphasize this point); we can conclude

Two lines are parallel if and only if there is a translation of one line to the other. Parallel lines have the same slope and lines with the same slope are parallel.

In Chapter 8 where we will study translations in more detail, we will note that if two lines are parallel they never intersect, and conversely, if two lines never intersect they are parallel. This statement (a version of the *Parallel Postulate* of Euclid) cannot be verified by observation because we cannot see infinitely far away. For this reason, it has been discussed throughout history, the issue being whether or not it is a necessary part of planar geometry. It turned out, in the 19th century, that it is, for there are geometries different from planar that satisfy all of the conditions of planar geometry but the Parallel Postulate.

Two lines are *perpendicular* if they intersect, and all angles formed at the intersection are equal. This of course is the same as saying that all these angles have measure 90°.

To understand perpendicularity, we use the idea of *rotation*. A rotation is a motion of the plane around a point, called the center of the rotation. To visualize what a rotation is, take two pieces of transparent coordinate paper, put one on top of the other and stick a pin through both pieces of paper. The point where the pin intersects the paper is the *center* of the rotation. Now any motion of the top piece of paper is a visualization of a rotation. For any figure on the bottom piece of paper, copy it onto the top, then rotate the top piece of paper and copy the figure on the top to the bottom. That image is the rotated image of the original figure.

In figure 6, we see the result of rotating the red line (the line with positive slope) through a right angle (90°) with the center C. The blue line (with negative slope) is the image of the red line under the rotation.

Notice that the dark lines and the light lines correspond under the rotation, so they have the same lengths. Notice also that these are the triangles that are drawn for the slope computation except that the rise and run has been interchanged: in terms of lengths, rise(red) = run(blue), run(red) = rise(blue). However, there is one last (and important) thing to notice: the slope computation is in terms of differences between coordinates, and not lengths. In our diagram the sign of one pair of differences (represented by the black lines) has changed, while the sign of the other pair of differences has not. We can summarize this as follows:

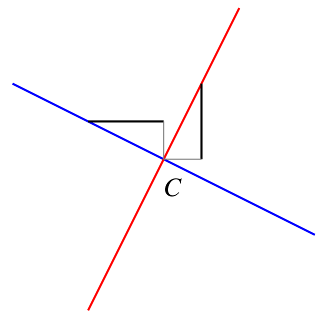


Figure 6

For the red line,

$$\text{slope(red)} = \frac{\text{length(black)}}{\text{length(gray)}}$$

and for the blue line,

$$\text{slope(blue)} = \frac{\text{length(gray)}}{-\,\text{length(black)}}$$

from which we can conclude that the product of the slopes of the blue and red lines is -1. Since we did not use any coordinates to make this argument, this statement is general, so long as neither line is horizontal or vertical.

To recapitulate: if we rotate a line L (red in figure 6), 90° about a point P on the line, getting the new line L' then the products of the slopes of L and L' is -1. The following statement follows from this assertion:

> If lines L_1 and L_2 are perpendicular at their point of intersection, then the product of their slopes is -1. If the product of the slopes of lines L_1 and L_2 is -1, then they are perpendicular at their point of intersection.

To see this, first, let us suppose two lines L_1 and L_2 intersect perpendicularly at a point P. Now rotate the line L_1 by 90° to get the line L_1'; then the product of the slopes of L_1 and L_1' is -1. But since there is only one line perpendicular to L_1 at P, L_1' and L_2 are the same line. To show the second statement: asuppose that L_1 and L_2 intersect at P but this time suppose the products of their slopes is -1. Again rotate L_1 by 90 ° to get the line L_1', and again L_1' has the same slope as L_2, so must coincide with L_2.Thus L_1 and L_2 are perpendicular.

When the lines are given by a linear relation, it is easy to write the relation of the line perpendicular to it:

> If a line L is given by the relation $Ax + By = C$, then the equation $Bx - Ay = D$ (for any D) describes a line L' perpendicular to L.

This is because line L has slope $-A/B$, and line L' has slope B/A.

Example 11.

Consider the line L given by the equation $3x + 4y = 20$. Find the equation of the line L' perpendicular to L that passes through the point $(4, 2)$.

Solution. The statement in the box tells us that all lines perpendicular to L have an equation of the form $4x - 3y = D$, where D is to be determined. So the line we are looking for has an equation of this form. Since $(4, 2)$ is on that line, we can find D simply by replacing x by 4 and y by 2 and calculating: $4(4) - 3(2) = 10$. The equation of L' is $4x - 3y = 10$.