Chapter 8
Integer Exponents, Scientific Notation and Volume

We have already introduced the notation x^2 for $x \times x$ and x^3 for $x \times x \times x$, and it is easy to see how to extend this to all positive integers: x^n just means that we multiply x by itself n times. So, $2^6 = 64$, $3^4 = 81$ and so forth. Since we are using whole numbers just to count factors, clearly $x^{m+n} = x^m \times x^n$: multiplying x to itself m times, and then n more times is the same as multiplying x by itself $m + n$ times. We then ask the question: can we make sense of x^p for *all* integers p, so that the usual rules of arithmetic on the exponents apply? The answer is "yes," and the exploration of this is the content of the first section of this chapter. We put particular emphasis on the assertion that $x^0 = 1$ for all numbers $x \neq 0$. There are several ways to to see that this is the right definition, but the fact is that it the only possibility that is consistent with the rules of arithmetic, as we shall show.

In the next section we revisit place value, recalling that when a number is exhibited in place 10 notation, each place represents a power of ten, we move on to a shorthand for representing numbers, using exponential notation. Scientific notation is important, not just as a convenience for dealing with very small or very large numbers, but as a way of understanding "orders of magnitude." When it is said that phenomenon A is two orders of magnitude more likely than phenomenon B (as in the scale for Hurricane intensity) we do not mean that A is twice as likely as B; we mean that A is $100 = 10^2$) times more likely than B. We pose many problems illustrating the meaning of "orders of magnitude" that should convince students that this is not just a shorthand, but conveys a rich meaning that other wise could be missed.

Finally, in the last section we introduce certain volume calculations (for a cylinder, cone and sphere), the purpose of which is to work with the relations among these solid figures, and secondly, to apply the mathematics of the preceding sections. As an example, Farmer Brown has fields that can produce grain, and silos that can store them. Given the correspondence between square feet of farmland and cubic feet of grain, we ask these questions: a) for a certain size of field, what storage capacity is needed? b) Given the size of the silo, how many square feet need to be planted in wheat so as to fill the silo?

Section 8.1: Integer exponents

Know and apply the properties of integer exponents to generate equivalent numerical expressions. For example, $3^2 \times 3^{(-5)} = 3^{(-3)} = 1/(3^3) = 1/27$. *8.EE.1*

In previous discussions about area and volume we already have introduced the notation x^2 and x^3: the first is the product of two x's, the second the product of three x's. We can now introduce the same notation for all counting numbers (positive integers): x^n is the product of n x's for any positive integer n. Notice that multiplication of such objects amounts to addition in the exponent: $x^3 \times x^8 = x^{(3+8)} = x^{11}$, because multiplying a number to itself 3 times, and then multiplying that by the product of 8 of the same numbers is the same as multiplying that number by itself

11 times. In general, we can say that for positive integers, we have

$$x^p \times x^q = x^{p+q}$$

for all positive integers p and q.

Can we extend this notation to all integers, positive, negative and zero? If we want to be able to write

$$x^5 \times x^{-3} = x^{5-3} = x^2 \ ,$$

we have to understand multiplication by x^{-3} as an operation going from x^5 to x^2, that is. removing 3 of the multiplications by x. But we know such an operation: it is that of dividing by x three times, or - what is the same - dividing by x^3. So, we take this as the definition of negative exponents:

$$x^{-p} = \frac{1}{x^p}$$

The addition rule above now holds for all nonzero integers.

Since x^2 means: multiply the expression x by itself, if we replace x by x^3, then $(x^3)^2 = x^3 \times x^3 = x^6$. But x^6 is the same as $x^{3\cdot 2}$, so we conclude that $(x^3)^2 = x^{3\cdot 2}$. This of course is true for all integers, not just 2 and 3, so we have this understanding of the product rule for exponents:

$$x^{p\cdot q} = (x^p)^q = (x^q)^p, \quad \text{for all positive integers } \ p \ \text{ and } \ q.$$

What meaning do we attach to the expression x^0? We follow the logic of the rules of arithmetic: If we start with the expression x^5, and cancel all the x, we get 1. That is, we have:

$$x^5 \times x^{-5} = x^{5+(-5)} = 1 \ .$$

But $5 + (-5) = 0$, giving us the rule $x^0 = 1$.

Another way to look at it is this: we are studying the multiplicative structure of expressions. When we work with the additive structure of expressions, the simplest expression is 0. So, in the case of multiplicative structure, the simplest expression should be 1. It follows that anything multiplied by itself no times is 1. Finally, we have to understand that these rules apply to all values for x except 0, because the logic doesn't apply for $x = 0$.. We can't divide by zero, so 0 to a negative exponent doesn't make sense.

Let's pay particular attention to raising a negative number to a power. Since $-a$ and $(-1)(a)$ are the same thing, we can calculate using the commutative property. For simplicity, suppose $a > 0$, so we already feel comfortable with a^n. How about $(-a)^n$? Write this as$(-1)^n \times a^n$, and so we just have to know what $(-1)^n$. If n is a positive integer, this is the product of n (-1)'s. Multiplication by -1 is the same as reflection in the origin, so $(-1)^2$ is the reflection of -1 in the origin, and is 1; $(-1)^3$) reflects -1 in the origin and then reflects back again, so is -1. Continuing with this representation as reflecting back and forth around the origin, we see that $(-1)^n$ is 1 if n is even, and is -1 if n is odd. This is true also for negative exponents, since $(-1)^{-1} = -1$ (-1 is its own multiplicative inverse).

Let us summarize the operational techniques with exponents: vskip 0.2in

In the following, a and b can be any number, and p , q and n are integers (positive or negative):

- $a^{p+q} = a^p \times a^q$
- $(\frac{a}{b})^n = a^n b^{-n}$
- $(a \cdot b)^n = a^n \times b^n$
- $(a^p)^n = a^{p+n}$
- $a^{-p} = \frac{1}{a^p}$.
- $(-a)^n$ is equal to a^n if n is even, and is equal to $-(a^n)$ if n is odd.

Keep in mind that the use of exponents is in reference to the multiplication of positive numbers, and that, for $a > 1$, the numbers a^n, for $n > 0$, are to right of 1, and for $n < 0$ between 0 and 1. Figure 1 illustrates this for $a = 2$.

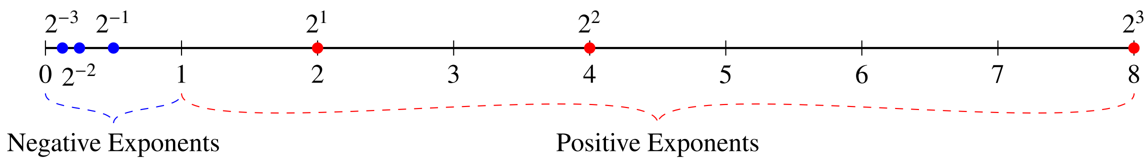


Figure 1.

It is good to have as a reference the values of the powers of small digits.

n^1	n^2	n^3	n^4	n^5	n^6	n^7	n^8	n^9	n^{10}
1	1	1	1	1	1	1	1	1	1
2	4	8	16	32	64	128	256	512	1028
3	9	27	81	243	729	$\cdots$			
4	16	64	256	1028	$\cdots$				
5	25	125	625	$\cdots$					
10	100	1000	10000	$\cdots$					

The powers of 10 are particularly easy: for p positive, 10^p is a 1 followed by p zeros, and 10^{-p} is a decimal point followed by zeros and ending in the pth position with a 1:

$$10^2 = 100 \quad 10^5 = 100,000 \qquad 10^{-2} = .01 \quad 10^{-5} = .00001 \ .$$

One of the values of exponents is that their use makes the understanding of our *place value* notation and computation more clear. The number 5283.7 is expressed in "long form" as

$$5 \times 1000 + 2 \times 100 + 8 \times 10 + 3 + 7 \times \frac{1}{10} \ .$$

Using exponents this becomes

$$5 \times 10^3 + 2 \times 10^2 + 8 \times 10^1 + 3 \times 10^0 + 7 \times 10^{-1} \ .$$

Example 1. Place Value Operations

- Double 63. To double a number we could multiply the number by 2. Or, we could double every digit. This way, double 63 is 126. But what if we double 67, is the answer 1214? No, because the "place" value of the 2 in 12 and the one in 14 are the same and so must be added, and the answer is 134. Although this looks like a magical trick - it is not. Writing out the place value in long form, doubling the digits looks like this:

$$2\times67 = 2\times(6\times10+7) = 2\times(6\times10)+2\times7 = 12\times10+14 = 12\times10+1\times10+4 = 13\times10+4 = 134$$

- Multiply 102 by 54. We put these numbers in long form and use the rules of arithmetic:

$$102 \cdot 54 = (10^2 + 2)(5 \times 10 + 4) = 5 \times 10^3 + 4 \times 10^2 + (2 \cdot 5) \times 10 + 2 \cdot 4$$

$$= 5 \times 10^3 + 4 \times 10^2 + 10^2 + 8 = 5000 + 500 + 8 = 5508 .$$

- Take 3 percent of 5000. In exponential notation, 3 percent is 3×10^{-2} and 5000 is 5×10^3. So, we use the rules of exponents to solve:

$$(3 \times 10^{-2}) \cdot (5 \times 10^3) = 15 \times 10^{-2+3} = 15 \times 10^1 = 150 .$$

Example 2. Operations with exponents

- $8 \cdot 16 = 2^3 \cdot 2^4 = 2^7 = 128$.
- $64 \cdot 18 = 2^6 \cdot 2 \cdot 3^2 = 2^7 \cdot 3^2 = 128 \cdot 9 = 128 \cdot (10 - 1) = 1280 - 128 = 1152$.
- Approximate $\sqrt{6340}$. This is a little less that 64×100, so its square root is a little less than $8 \times 10 = 80$.

Example 3. March Madness (from the Chicago Maroon, March, 2012)

March Madness - the NCAA final basketball tournament - has the form of a *single-elimination* tournament. In such a tournament, we start with a certain number of teams, and we pair them off into games: each team plays a game. This is called the *first round*. All the losers in the first round are eliminated; in the second round all the winning teams are paired off into games, and all the second round losers are eliminated. This process continues until only two teams remain: this is the *final round* and the winner is the champion of the tournament. For a graphic of the 2012 women's basketball *brackets* see the figure on the next page.

Since there are two teams in the final round, there had to be four teams in the semi-final round, and thus eight teams in the preceding round and so forth. So, it is necessary, for a single elimination tournament to work, with no teams ever idle, that we start with a number of teams that is a power of two, and that exponent is the number of rounds. So, for example, if we start with 16 teams, since $16 = 2^4$, there are 4 rounds and $8 + 4 + 2 + 1 = 15$ games.

In March Madness we start with 64 teams. How many rounds are there? How many teams are in the second round? in any round? How many games total are played?

Solution. Since $64 = 2^6$, there are six rounds. Each round eliminates half the remaining teams, so there are 32 teams in the second round, 16 in the third, and so forth. There are $32 + 15 + 8 + 4 + 2 + 1 = 63$ games. Another way of counting is that there are 63 teams that are NOT champions, and each game produces one non-champion.

Section 8.2: Scientific Notation

Scientific Notation and Place Value

Use numbers expressed in the form of a single digit times an integer power of 10 to estimate very large or very small quantities, and to express how many times as much one is than the other. For example, estimate the population of the United States as 3×10^8 *and the population of the world as* 7×10^9*, and determine that the world population is more than 20 times larger. 8.EE.3*

In today's world we work with very big numbers (astronomical distances) and very small numbers (microscopic distances), so a shorthand has been invented to make it easier to handle such numbers. To illustrate, suppose we want the product of 300,000,000 and 7000. We know that this is going to be 21 followed by a certain number of zeros, but how many? We calculate this way:

$$300,000,000 \times 7000 = (3 \times 100,000,000)(7 \times 1000) = (3 \times 7)(100,000,000 \times 1000)$$

$$= 21 \times 100,000,000,000 = 2,100,000,000,000 \ .$$

The second multiplication $(100,000,000 \times 1000)$ amounts to putting the 0's at the end of the second factor behind the zeros of the first factor; in other words, the number of zeros in the product is the sum of the numbers of zeros in the factors. In exponential notation: $10^8 \times 10^3 = 10^{11}$. Using this notation, the above calculation now looks like this:

$$(3 \times 10^8)(7 \times 10^3) = 21 \times 10^{11} \ .$$

Whenever we write numbers using the symbol: $\times 10^n$, we say are using *scientific notation*. As this example shows, this notation makes calculations easier to read. It also makes it easier to make comparisons; for example the statement "Earth is 93 million miles away from the Sun, and Mars is 143 million miles away from the Sun," we are using scientific notation (simply replace the word "million" with $\times 10^6$. This is easier to understand than the statement "Earth is 93000000 miles away from the Sun, and Mars is 143000000 miles away from the Sun."

A number is said to be written in *normalized scientific notation* if it is given in the form $a \times 10^n$, where a is a number whose absolute value is greater than or equal to 1 and strictly less than 10 and n is an integer. a is called the *significant figure* of the number, and n its *order of magnitude*. To illustrate: Earth is 9.3×10^7 miles from the Sun, and Mars is 1.43×10^8 miles from the Sun and $(3 \times 10^8)(7 \times 10^3) = 2.1 \times 10^{12}$.

Example 4.

- Express 35,000,000 in normalized scientific notation. This is 35 followed by 6 zeros, so is 35×10^6. To put this in normalized scientific notation, we have to move the decimal point one place to the left, and raise the exponent by 1, giving us the answer: 3.5×10^7. Note that we could also write 35000000 as 350×10^5 or 0.35, depending upon what it is that we want to emphasize. Generally speaking, as every three places is denoted by a comma, it is best to go by multiples of 3.

- Express 3,650,000 in scientific notation. This could be 365×10^4 or 3.65×10^6; it is the second that is in normalized scientific notation.

- Express 3,651,284 in scientific notation with 3 significant figures. This means that, for purposes of estimation, we care only about the first three digits, and so the answer is 3.65×10^6.

Computers and calculators use a different notation for scientific notation: 36400000 appears as $3.64E7$, meaning 3.64×10^7. Let us take a moment to introduce the vocabulary for large numbers.

Name	Number	Scientific Notation
One	1	10^0
Ten	10	10^1
Hundred	100	10^2
Thousand	1000	10^3
Ten Thousand	10,000	10^4
Hundred Thousand	100,000	10^5
Million	1,000,000	10^6
Billion	1,000,000,000	10^9
Trillion	1,000,000,000,000	10^{12}
Quadrillion	1,000,000,000,000,000	10^{15}

Table 1

and so forth. Notice one value of scientific notation: the middle column grows more and more unreadable, while the last column can be grasped. So, for example, a septillion will be 10^{24}, or 1 followed by 24 boring zeros.

Fractional decimals can be similarly written, using scientific notation, but this time the powers of 10 are negative:

$$0.1 = 10^{-1} \quad 0.001 = 10^{-3} \quad 0.367 = 3.67 \times 10^{-1} .$$

As far as names go, we speak of powers of 10 in the denominator by adding "th" to the end of the word, so 0.3 is 3 tenths, or 3×10^{-1}, 0.000005 is 5 millionths, or 5×10^{-6}, and so forth. You might want to notice that there is a little anomaly in scientific notation: $10,000 = 1 \times 10^4$ but $.00001 = 1 \times 10^{-5}$; that is, the exponent of 10 is not always the number of zeros from the decimal point. This is because the first place to the left is the 0th place, while the first place to the right is the (-1)st place. Finally, there do not exist special names for the negative powers of ten, but there are such names in the metric system (for grams, meters, etc.), as in this table:

Name	Number	Scientific Notation
meter	1 meter	10^0 meter
decameter	tenth of meter	10^{-1} meter
centimeter	hundredth meter	10^{-2} meter
millimeter	thousandth meter	10^{-3} meter
micrometer	millionth of a meter	10^{-6} meter
nanometer	billionth of a meter	10^{-9} meter
angstrom	tenth of a nanometer	10^{-10} meter

Table 2

An *angstrom* is the unit of measurement used to measure lengths at the atomic level.

It now makes sense to inquire: how do we calculate arithmetic operations in scientific notation? First, as for addition, the issue shouldn't come up: if the numbers are not of the same order of magnitude, the question won't come up. So, for example, $3.7 \times 10^4 + 6.1 \times 10^4 = 9.8 \times 10^4$, by the distributive property of arithmetic. But we won't be asked to calculate

$$7.104 \times 10^7 + 2.100 \times 10^2 = ?$$

because it just doesn't make sense: the first term is 5 orders of magnitude large than the second. It is like asking: if I weigh myself on a scale, and then directly after a fly lands on my head, will the scale show a difference? The answer is clearly "No." So, in the displayed question, the first number has 3 decimal points of accuracy (so is 7104×10^4), and thus the second number, 2.100×10^2, is under the radar.

However, orders of magnitude in scientific notation play a major role when the problem involves multiplication and division. For example, 120 million divided by 30 gives 4 million. The fact that "120 million" and "30" have different order of magnitude is very relevant to answering the question. Writing "120 million" as 1.2×10^8 and "30" as 3×10, this calculation becomes:

$$\frac{1.2 \times 10^8}{3 \times 10} = \frac{1.2}{3} \times \frac{10^8}{10} = 0.4 \times 10^7 = 4 \times 10^6 = 4,000,000 .$$

Summary:

- $(a \times 10^n)(b \times 10^m) = (a \times b)(10^{n+m})$
- $\frac{(a \times 10^n)}{(b \times 10^m)} = \frac{a}{b} \times (10^{n-m})$

Example 5.

a. Multiply 3.2×10^4 by 6×10^{-1}.

b. Divide 3.3×10^6 by 1.1×10^5.

SOLUTION.

a. First, let's remind ourselves what we are being asked without scientific notation: Multiply 32,000 by 0.6. Since that is what is being asked, let's go back to scientific notation:

$$(3.2 \times 10^4) \times 6 \times 10^{-1} = ((3.2) \times 6) \times (10^4 \times 10^{-1}) = 19.2 \times 10^3 = 1.92 \times 10^4$$

which is, in standard notation 19,200.

b. Here we are asked to divide 3.3 million into 110,000 parts.

$$\frac{3.3 \times 10^6}{1.1 \times 10^5} = \frac{3.3}{1.1} \times \frac{10^6}{10^5} = 3 \times 10$$

or 30.

EXAMPLE 6.

Suppose that we want to multiply 3 billionths by 7 ten-thousandths. We might write:

$$\frac{3}{1000000000} \times \frac{7}{10000} = \frac{3 \cdot 7}{1000000000 \cdot 10000} = \frac{21}{10000000000},$$

but the following is much easier to understand:

$$(3 \times 10^{-6})(7 \times 10^{-4}) = 21 \times 10^{-10}.$$

Solve Problems and Apply Scientific Notation

Perform operations with numbers expressed in scientific notation, including problems where both decimal and scientific notation are used. Use scientific notation and choose units of appropriate size for measurements of very large or very small quantities (e.g., use millimeters per year for seafloor spreading). Interpret scientific notation that has been generated by technology.8.EE.4

EXAMPLE 7.

- How many millions are there in a trillion? We write a million as 10^6 and a trillion as 10^{12}, and the question is: evaluate $\frac{10^{12}}{10^6}$. The answer is $10^{12-6} = 10^6$, or a million. A trillion is a million million.

- 0.0031×562.1 =? The easiest way to get the answer is to use a calculator. However, we may just want an estimate, in which case, moving to scientific notation is best. Rewrite the problem as $(3.1 \times 10^{-3})(5.621 \times 10^2)$. Now estimate the significant figures: this is about 3 times 5.5, which is 16.5. Next, add the exponents, and write the (estimate of) the answer as 16.5×10^{-1}, or 1.65. The calculator turns up the accurate answer: 1.7425.

- About how much is 40% of 140 million? Rewrite this as the product

$$(40 \times 10^{-2})(140 \times 10^6) = (40 \times 140)(10^{-2} \times 10^6) = (4 \times 14)(10^2 \times 10^{-2} \times 10^6) = 56 \times 10^6$$

or 56 million.

Here are some illustrations of the value of scientific notation in applications, particularly to problems that give meaning to the concept "order of magnitude."

Example 8.

In a class action suit, 4000 claimants were offered a $800 million settlement. How much is that per claimant?

In scientific notation, the question is to evaluate $(8 \times 10^8) \div (4 \times 10^3)$, which is $(8 \div 4)(10^{(8-3)})$ which simplifies to 2×10^5. Thus each claimant would receive $200,000.

Example 9.

We read in the paper that the United States has a 15 trillion dollar debt. Let's say that there are 300 million working people in the United States. How much is the debt per worker?

In scientific notation this is 15×10^{12} divided by 3×10^8, which is 5×10^4, or about $50,000 for each tax-paying citizen.

Example 10.

Tameka has a job at which she earns $10 hour. Her tax rate is 18%. Let's assume that *all* of Tameka's taxes go toward paying off the $50,000 debt of the preceding problem. How many hours will she have to work to pay off her share of the debt? If she works 2×10^3 hours a year, how many years is that?

Solution. Given the assumptions of this problem, Tameka's hourly contribution to paying the debt is 18% of $10, or $1.80. Let h represent the number of hours it takes until she pays off her $50,000. This gives us the equation $1.8h = 5 \times 10^4$, and thus $h = (5/1.8) \times 10^4$, which is 27,778. If she works 2000 hours in a year, that comes to $27.78 \div 2 = 13.89$ years. (Track the implicit use of scientific notation).

The following two examples are taken from Grade 7, but are repeated here to demonstrate the value of scientific notation.

Example 11.

Figure 2

The National Press Building on Fourteenth Street and Avenue F is 14 stories high, with 12 feet to each story. It has 150 feet of frontage on 14th St, and 200 feet on Ave F. The building has the shape of a rectangular prism. What is its volume?

We view the building as constructed by drawing upwards a 150×200 rectangle for 14 stories. Now the area of the base is $1.5 \times 10^2 \times 2 \times 10^2 = 3 \times 10^4$ sq. ft. Since each story is 12 feet high, the volume of each story is $12 \times 3 \times 10^4 = 3.6 \times 10^5$ cu. ft., and as the building is made up of a stack of 14 stories identical to the first one, the total volume is $14 \times 3.6 \times 10^5 = 5.04 \times 10^6$ cu. ft; approximately 5 million cubic feet.

As it turns out, the building sold in 2011 for $167.5 million dollars, which comes to about a $33.5 cost per cu. ft. However, the value of a building is not measured by its volume, but by the square footage of its floors. Since there are 14 floors , each a copy of the first floor, so each of 30,000 square feet, the total floor area of the building is $14 \times 3 \times 10^4 = 4.2 \times 10^5$ square feet, and the cost per square foot is $(167.5 \times 10^6) \div (4.2 \times 10^5) = 39.88 \times 10$, that is $398.80 per square foot.

Example 12.

The Pentagon, the headquarters of the U.S. Department of Defense is a regular five-sided figure with 6.5 million square feet of floor space on seven levels, two of which are underground. The side length of the interior central plaza is about one-third the side length of the building.

Figure 3

a. What is the *footprint* of the Pentagon? The footprint is the total area occupied by the building together with the central plaza.

b. What is the area of the central plaza?

c. There are 11 feet of elevation between floors of the Pentagon. What is the total volume of the above-ground building?

Solution.

a. The image shows the Pentagon to be a prism - in the sense that all floors are of the same shape and size indeed all sections by planes parallel to the ground are of the same shape and size. Thus each floor of the building comprises 1/7 of 6.5×10^6 sq. ft, or 928×10^3 sq. ft. But this is the area of the base floor of the building, not the footprint, which includes the central plaza, We are told that the the length of a side of the plaza is one-third the side length of the building. Since the plaza and the building have the the same shape, that tells us that the footprint of the plaza is a downscaling of the footprint of the entire Pentagon by a linear scale factor of 1/3. Since area scales by the square of the linear scale factor, we conclude that the area of the plaza is 1/9th of the are of the footprint. Thus the area of the floor of the building, 928,000 square feet is 8/9 of the area of the footprint. The answer then, to a) is that the area of the footprint is $\frac{9}{8}(928 \times 10^3) = 1.044 \times 10^6$ sq. ft.

b. The plaza is 1/9 of the footprint, so its area is $\frac{1}{9}(1.044 \times 10^6) = 116 \times 10^3$ sq.ft.

c. The reason this figure (the volume of the building) is interesting is to estimate the cost of heating the building in winter, and air-conditioning it in summer. So, now we are interested only in the volume of the building that is above ground. Since there are 5 stories above ground, each of height 11 feet, the building stands 55 feet high. The area of the base is 928,000 sq. ft., so the volume of the building above ground is $55 \times 928,000 = 51,040,000$ cu. ft.

Suppose that the cost of a building in the Washington DC area is about $398.80 per sq. ft. At 6.5 million square feet, the cost of the Pentagon today would be approximately $4 \times 10^2 \times 6.5 \times 10^6 = 26 \times 10^8$, or 2.6 billion dollars.

Example 13.

On the computer a *byte* is a unit of information. A typical document contains many tens of thousands of bytes, and so it is customary to use these words: 1 kilobyte = 1000 bytes; 1 megabyte = 1000 kilobytes, 1 gigabyte = 1000 megabytes, 1 terabyte = 1000 gigabytes.

a. Rewrite this vocabulary in scientific notation. How many bytes are there in each of these terms?

1 kb = 10^3 b; 1 mb = 10^3 kb = 10^6 b; 1 gb = 10^3 mb = 10^9 b; 1 tera = 10^3 gb =10^{12} b.

b. My computer has a memory (storage capacity) of 16 gigabytes. How many such computers do I need to have, when all are combined, a terabyte of memory?

The question is: how many times does 16 gb go into 10^3 gb?

$$\frac{10^3}{16} = \frac{100}{16} \times 10 = 6.25 \times 10 = 62.5$$

so I'll need 63 such computers.

c. An online novel consists of about 250 megabytes. How many novels can I store on my 16 gigabyte computer?

16 gb is 16×10^3 mb, so we want to know how many times 250 goes into 16×10^3. Since we will be dividing by 250, it is worthwhile noting that

d.

$$\frac{1}{250} = \frac{4}{1000} = 4 \times 10^{-3} \ .$$

Thus

$$\frac{16 \times 10^3}{250} = 16 \times 10^3 \times 4 \times 10^{-3} = 64 \ .$$

Example 14.

Many chemical and physical phenomena happen in extremely small periods of time. For that reason, the following vocabulary is used: 1 second = 1000 milliseconds, 1 millisecond = 1000 microseconds, 1 microsecond = 1000 nanoseconds.

a. Rewrite this vocabulary in scientific notation. How many nanoseconds are in a millisecond? in a second? in an hour?

b. My computer can download a byte of information in a millisecond. How long will it take to download a typical book (250 megabytes)? How long will it take to download the Library of Congress (containing 36 million books). Express your answer conveniently in terms of time.

Solution.

a. 1 second = 10^3 milliseconds = 10^6 microseconds = 10^9 nanoseconds. Otherwise put, a nanosecond is one billionth of a second, a microsecond is one millionth of a second, and a millisecond is one thousandth of a second.

b. It takes 10^{-3} seconds to download one byte of information, so the rate is 10^{-3} seconds per byte. Thus to download 250 megabytes, which is 2.5×10^5 bytes, so it takes $2.5 \times 10^5 \times 10^{-3} = 2.5 \times 10^2 = 250$ seconds, or 4 minutes and 10 seconds. To download 35×10^6 such books will take $36 \times 10^6 \times 2.5 \times 10^2 = 9 \times 10^9$ seconds. Since there are 3600 seconds in an hour and 24 hours in a day, in days this is:

$$\frac{9 \times 10^9}{36 \times 10^2 \times 24} = \frac{25 \times 10^7}{24} = 1.04 \times 10^7$$

days; far too long to wait. My computer is too slow. If I get a new computer that is one million times as fast (one millionth is 10^{-6}), it will still take 10.4 days to download the library of Congress.

Section 8.3: Volume

Know the formulas for the volumes of cones, cylinders and spheres and use them to solve real-world and mathematical problems. 8.G.9

In this section we start by reviewing some of the terminology and ideas of 7th grade used in volume computations. There we talked of prisms and cones based on polygonal figures in the plane; here we move to the same concepts, based on circular figures.

Prisms and Cylinders

We conceive of measurements in the various dimensions (length, area, volume) as a natural progression through the dimensions. To begin: we start with a point on the line that we draw out along the line for a certain distance, creating a line segment, and the measurement of that line segment is its *length*. Suppose that we start with a line segment in the plane of length l and draw it out for a distance w perpendicular to the line segment: we obtain a rectangle of side lengths l and w. The *area* of that rectangle is the product: $A = l \cdot w$ Take a figure F on a plane in three dimensions of area A; dragging it out in a direction perpendicular to the plane for a distance h, we get the *prism* with base F. Following the preceding logic, the measure of this solid figure (called its *volume*) is the product of h with A: $V = A \cdot h$. In particular, if the figure in the plane is the rectangle of side lengths l and w, the then solid figure (called a *rectangular prism*) is of volume $V = l \cdot w \cdot h$. If the figure we started with was a triangle of base b and altitude a, then the solid figure (called a *triangular prism* or *wedge*) has volume $V = \frac{1}{2}abh$.

In 7th grade we solved problems for prisms and cones based on polygonal figures. Suppose we start with a circle in the plane of radius r, and therefore of area πr^2. Drawing it out, the solid figure we get is a *circular cylinder* and its volume is $V = \pi r^2 h$: that is, the product of the area of the base with the height. It is customary to drop the adjective "circular", and call the this the *cylinder* (see Figure 4).

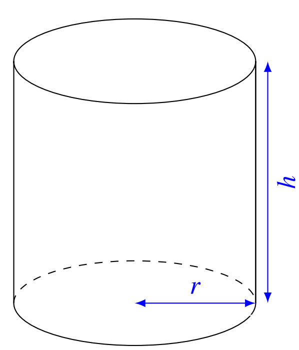


Figure 4

Example 15. Volume computations

First, as a reminder, let's begin with a review of seventh grade problems.

a. Find the volume of a prism built on a rectangle of side lengths 50 feet and 18 feet, and of height 25 feet.

Solution. The volume is the product of these lengths, so is $50 \times 18 \times 25 = 22500$ cubic feet. If you remember that 50 is half a hundred and 25 is one-fourth a hundred, this makes the computation easier:

$$50 \times 18 \times 25 = \frac{1}{2}10^2 \times 18 \times \frac{1}{4}10^2 = \frac{1}{8} \times 18 \times 10^4 = 2.25 \times 10^4 = 22500 .$$

b. A *wedge* is a triangular prism whose base is a right triangle of side lengths 2 in by 5 in, and whose height is 8 in. What is the volume of the wedge (as shown in Figure 5)?

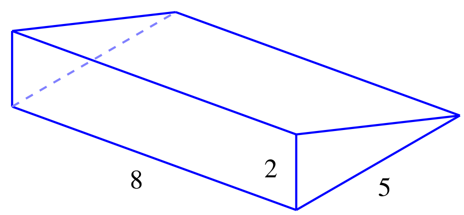


Figure 5

Solution. We see this as a 2×5 right triangle dragged out 8 inches, so the area is

$$(\frac{1}{2}2 \cdot 5) \cdot 8 = 40 \text{ cubic inches} .$$

c. A construction company wants to build a small shed covering a rectangular plot, that is 10 feet high, 18 feet long and has 3600 cubic feet of volume. What should its width be?

Solution. The shed is a rectangular prism of 3600 cubic feet, its height is 10 feet and its length is 18 feet. If we let w represent the width, we must have $18 \times w \times 10 = 3600$, or $180w = 3600$. That does it! So $w = 3600/180 = 20$ feet.

d. An ice cream company wants to package a pint of ice cream in a circular cylinder that is 4 inches high. What does the radius of the base circle have to be?

Solution. A pint is 16 fluid ounces, but we need this in cubic inches. We search and find that 1 fl oz is 1.8 cu. in. Now we can proceed. A pint is 16 fl oz, so is $16 \times 1.8 = 28.8$ cu. in. We want to put this in a cylinder of height 4 in, and of base radius r, and we want to find the value of r. What we know is that $\pi r^2 h = V$, and we know $V - 28.8$, $h = 4$ and let's take $\pi = 3.14$. We have to solve the equation

$$(3.14)r^2(4) = 28.8 \quad \text{or} \quad r^2 = \frac{28.8}{4(3.14)} = 2.29 .$$

Since $1.5^2 = 2.25$, we can conclude that the radius is slightly more the one and a half inches.

In seventh grade we introduced Cavalieri's principle in order to explain facts about volume calculations, on the grounds that this principle is intuitively plausible, although not technically part of the middle school curriculum. We shall once again base our understanding on this mathematical tool.

> **Cavalieri's principle**: Suppose that we stand two figures side by side. Suppose that every horizontal slice through the two figures gives two planar figures of the same area. Then the volume of the two solid figures is the same.

Notice that we do not require that the figures in the sections have the same size and shape, only that they have the same planar area. This more general interpretation will be useful to us in studying curved solids in three dimensions. We also want to notice that Cavalieri's principle applies in two dimensions (where the section is made by a line parallel to the base, and the hypothesis is that the lengths of the line segments are the same. This gives us another explanation of the formula $V = \frac{1}{2}bh$ for the area of a triangle, where b is the length of the base of the triangle, and h is its height (see Figure 6).

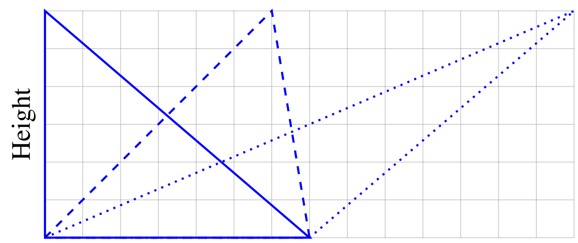


Figure 6

Figure 7 illustrates the reasoning behind Cavalieri's principle, where the individual figures could represent rectangles in the plane, or rectangular prisms or cylinders in three dimensions.

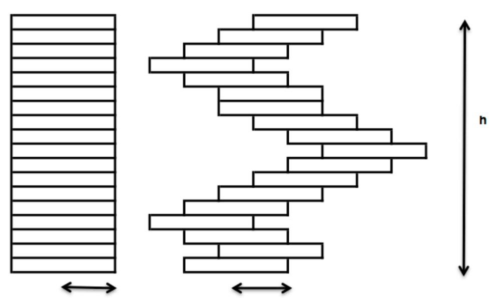


Figure 7

Example 16.

A buttress to a wall is a support structure that extends out from the wall to the ground, as in Figure 8. That buttress is made of blocks, all with base area of 3 sq. ft. The top of the buttress lies against the wall 30 feet above the ground. What is the volume of the buttress?

Solution. Using Cavalieri's principle, we can consider this as a column of bricks of area 3 sq. ft. that rises 30 ft. above the ground. So, the volume is $Bh = 3 \times 30 = 90$ cu. ft.

Cones

Recall from 7th grade that a *cone* is a three dimensional shape consisting of a figure in the plane (called the *base* B), a point A not on that plane (called the *apex*) and all line segments joining A to a point on B. In 7th grade we

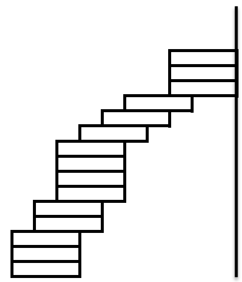


Figure 8

discussed cones whose base is a polygon in the plane, here we turn to the *right circular cone*: a cone whose base is a circle in the plane (see the image on the right in Figure 9). Last year we discussed the formula for the volume of the pyramid on the left: the base is a square of side length a and the height is h: $V = \frac{1}{3}a^2h$: the volume of the pyramid is one-third the product of the area of the base and the height. This formula is true for every cone, in particular the right circular cone:

Volume of a Cone: The volume of a right circular cone is one-third the product of the area of the base and the height. If the height is h and the radius of the base is r, then $V = \frac{1}{3}\pi r^2 h$.

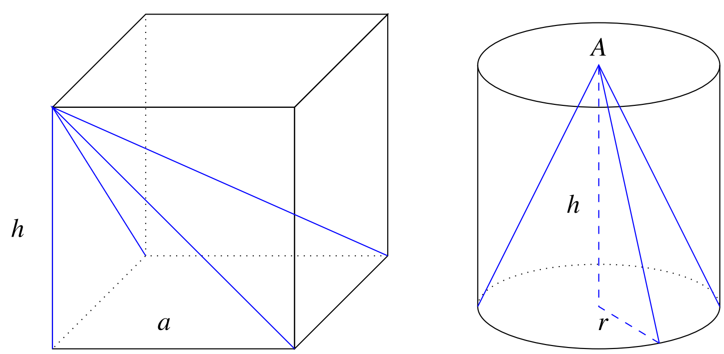


Figure 9

We can verify this fact by the following experiment. Select a cone and a cylinder of the same base radius r and height h; make sure that the cone is closed at its vertex and open at its base. Fill the cone with water and pour that water into the cylinder. The volume of the cone is the volume of the column of water in the cylinder. Depending upon how carefully the measurements are made, it will turn out that the height of that cylinder is $\frac{1}{3}h$, and so the volume of the cone with base radius r and height h is $\frac{1}{3}\pi r^2 h$.

In 7th grade we saw how to fill the cube on the left with three copies of the pyramid shown on the left in Figure 9, thus confirming this formula. The ancient Greeks took this to show that the 1/3 was the right factor for all cones, and tried in vain to find a way to fill a cylinder with three copies of the circular cone it circumscribes. It was not until the Calculus was developed that this "1/3" was understood. Actually, about a century earlier, Cavalieri showed how to effectively approximate the volume of a cone by slicing it into thin discs parallel to the base, noting that the radius of the discs decreases linearly, and then adding the volumes of the discs.

Extension

EXAMPLE 17.

Let's illustrate Cavalieri's computation, by finding the volume of a tower of Hanoi (in the background on the left in Figure 10).

Figure 10

The tower of Hanoi is the stack of cylindrical discs in the corner behind the work desk. The radii of the discs increase linearly as we go down the tower. The game using the tower of Hanoi employs another two vertical rods as well: the point being to move all the discs onto one of the other rods, one disc at a time so that no disc is ever placed on top of a smaller disc. The NCTM site, *Illuminations* has a java script to play the game (`http://illuminations.nctm.org/Activity.aspx?id=4195`). Our interest however, is just to try to calculate the volume of the entire tower. Now the game can be played with any number of discs; the tower in the photo has 32 discs. As the computation is tedious, we'll work it out for 9 discs (the most common size for playing the game).

Let us take the radius of the base of the tower of Hanoi to be r units, and the height h units. Then, since there are 9 discs, the height of each disk is $h/9$. Since the radii of the of the disc increase linearly, and there are 9 of them, the radius of the top disk will be $1/9$, of the second disc, $2/9$ and so forth. To calculate the volume of the tower, we'll work downwards from the top, adding one disc at a time. Now the first disc is a cylinder of base radius $r/9$ and height $h/9$, so its volume is

$$\text{Volume of top disc } = \pi(\frac{r}{9})^2\frac{h}{9} = \pi\frac{r^2h}{9^3}\ .$$

The radius of the second disc is $2r/9$, and its height is $h/9$ and we have:

$$\text{Volume of second disc } = \pi(\frac{2r}{9})^2\frac{h}{9} = \pi\frac{4r^2h}{9^3}\ .$$

The radius of the third disc is $3r/9$, and so The radius of the second disc is $2r/9$, and its height is $h/9$ and we have:

$$\text{Volume of third disc } = \pi(\frac{3r}{9})^2\frac{h}{9} = \pi\frac{9r^2h}{9^3}\ .$$

The pattern is clear: each time we move down a disc, the coefficient of r is the next integer over 9, and as the height is always $h/9$, we see that:

$$\text{Volume of the } k\text{th disc } = \pi(\frac{kr}{9})^2\frac{h}{9} = \pi\frac{k^2r^2h}{9^3} \ .$$

The volume of the tower of Hanoi is the sum of the volumes of the individual discs, and the volume of each disc is a factor of $\pi r^2 h$; the factor for the kth disc is $k^2/9^3$. Now, we do the calculation:

$$\frac{1^2 + 2^2 + 3^2 + \cdots 9^2}{9^3} = \frac{285}{729} = 0.391 \ ,$$

which is beginning to look a little like 1/3. What we will see, if we do what Cavalieri did, is that the more discs there are in the tower of Hanoi, the closer that factor gets to 1/3. In fact, for the 32 disc tower of Hanoi in the photo above, the factor of $\pi r^2 h$ is

$$\frac{1^2 + 2^2 + 3^2 + \cdots 32^2}{32^3} = \frac{11440}{32^3} = 0.349 \ ,$$

Were we to do this for a tower with 1000 discs, we'd find that the factor is 0.3338335 - getting really close to 1/3. Cavalieri did not just keep finding the sum; without computers even the first sum (up to 9) that we did would have been too difficult. He was smarter, he studied the algebraic form of the answer to be able to conclude that it gets closer and closer to 1/3 as the discs get thinner and the number of discs gets larger.

End Extension

The Sphere

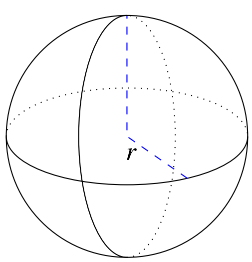


Figure 11

A *sphere* of radius r is the set of all points in space that are of a distance r from a point C, called the *center* of the sphere (see Figure 11).

We can find a formula for the volume of a sphere with physical models as we did above for the cone. Pick a hemisphere (half of a sphere) of radius r, and a cylinder of radius r and height r. Now, fill the hemisphere with water and pour the water into the cylinder. Again, depending upon the care of measuring, we will find that the water level comes to about 2/3 the way to the top. So the volume of the column of water, and therefore, that of the hemisphere is $V = \pi r^2(\frac{2}{3}r) = \frac{2}{3}\pi r^3$. Since the hemisphere is half a sphere, we get

The volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$.

This fact was known to to the Greeks by a variety of reasons; basically analogy and experimentation. At the end of this section, in an extension, we provide an argument based on Cavalieri's principle. Although Cavalieri's work was done 2000 years after the *Elements* of Euclid, the Greeks must have known something like that. A rigorous demonstration of this fact involves the basics of Calculus

Example 18.

A farmer wants to raise 250,000 sq ft of wheat, and have it watered with a rotary irrigator. Approximately what should the radius of the circle be?

Solution. Farmer Brown gave the problem to his nephew, Zack, just learning the power of approximation. Zack reasoned: the area of the circle is πr^2, where r is the radius. So, we must solve

$$(*) \qquad \pi r^2 = 250,000\ .$$

Divide both side by π and use 3 as an approximation of π, and 8 as an approximation of 25/3 to get $r^2 = 8 \times 10^4$ (approximately). Now $\sqrt{8 \times 10^4} = 2\sqrt{2} \times 10^2$, so, using 1.5 as an approximate value for $\sqrt{2}$, we get that the radius is about 300 ft.

Zack's sister Gwen expressed concern that the approximations could significantly warped the answer, so she tried a different method. Starting from (*) she concluded that:

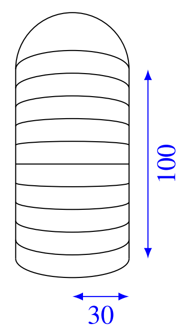
$$r = \sqrt{\frac{25 \times 10^4}{\pi}} = \frac{5 \times 10^2}{\sqrt{\pi}}\ .$$

Gwen took out her calculator, and using the approximation 3.142 for π, and calculating square roots using the methodsof the last chapter, that $r = 282.167$.

After Farmer Brown thanked his kin, which answer did he use?

Example 19.

The same farmer has a silo with a base radius of 30 feet and a storage height of 100 feet. A silo is a storage bin that is a cylinder with a hemisphere on top. The "storage height" is the part which can be filled with grain - it is just the cylinder. The weight of a cu. ft. of grain is 62 lbs.



a. A cu. ft. of grain weights 62 lbs. How many pounds of grain can the farmer store in the silo?

b. How high (including the hemispherical top) is the silo?

c. 1000 sq ft of wheat produces 250 lbs of grain. Is the silo large enough to hold the grain? By how much?

Solution.

- **a.** We are old that the volume of the maximum grain is that of the volume of the cylinder of base radius 15 ft. and of height 100 ft. So, for $r = 15$ and $h = 100$, the volume of the cylinder is

$$V = \pi r^2 h = \pi(15^2)(100) = \pi(225)(100) = (3.14)(2.25) \times 10^4) = 7.065 \times 10^4 \text{ cu. ft.}$$

which is 70,650 cu. ft. Since a cu. ft. of grain weighs 62 lbs, the maximum weight of grain that the silo can hold is

$$62 \times 7.065 \times 10^4 = 438 \times 10^4 = 4,380,000 \text{ lbs}\ , .$$

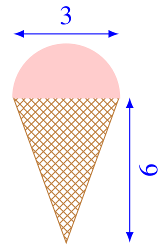
- **b.** The top cylinder is a hemisphere of radius 30 ft. Thus its height is 30 ft., and the height of the whole silo is 130 ft.

- **c.** Fron Example 18, we know that Farmer Brown devotes $250 \times 10^3 = 2.5 \times 10^5$ sq. ft. to his grain. Every thousand sq. ft. produces 250 lbs of grain. Now convert square feet to lbs.:

$$2.5 \times 10^5 \text{ sq.ft.} = 2.5 \times 10^5 \text{ sq.ft.} \frac{2.5 \times 10^2 \text{ lbs}}{10^3 \text{ sq.ft.}} = 6.25 \times 10^4\ ,$$

which is 62,500 lbs. So, Farmer Brown's silo can accommodate his production.

Example 20.



An ice cream cone consists of a cone filled with ice cream topped with a hemisphere of ice cream. If the cone is 4 inches long and the top has a diameter of 3 inches, how much ice cream (in cu in) fits in the cone. If 6 cu in of ice cream is equal to 1 fl oz, how many ounces of ice cream is that? If one fl oz of ice cream is 143 calories, how many calories is that?

Solution. We assume that the ice cream cone is completely full and is topped with a hemisphere of ice cream. Now, let us use the given data: the cone has height 4 in, and radius 1.5 in. The hemisphere has radius 1.5 in. Then the total volume is

$$\frac{1}{3}\pi(1.5)^2(4) + \frac{2}{3}\pi(1.5)^3 = \pi(\frac{1}{3}(2.25)(4) + \frac{2}{3}(3.375)) = 5.25\pi$$

cu. in. Using 3.14 as an approximation for π, this gives us 16.48 cu. in., or 2.75 fl. oz, since there are 6 cu. in. in a fl. oz. At 143 calories a fl. oz., the cone contains about 393 calories.

Extension

Example 21.

It fascinated the Greeks that, in terms of volume, a cylinder consisted of a cone and a hemisphere. They sought, without success, a constructive method to show that a cylinder can be decomposed into a cone and a hemisphere, or at least one that does not depend upon physical measurements. It was not until Calculus was invented that one found a mathematical proof of this fact; but we can use Cavalieri's principle to see why this is true. Place a cylinder, hemisphere and cone on a table as shown in Figure 12.

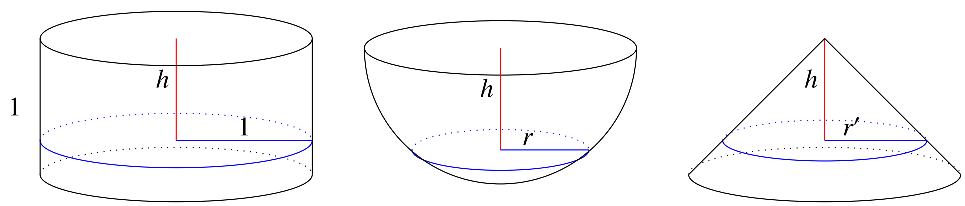


Figure 12

The height and base radius of the cylinder and the cone are both 1 unit, and the radius of the hemisphere is also 1. Now let us take a section of this setup by a plane parallel to the base and a distance h from the top of the figures.

The plane section of each figure is a circle, for the cylinder it is a circle of radius 1; let r be the radius of the circular section of the hemisphere, and r' that of the cone. If we can show that $r^2 + r'^2 = 1$, then by Cavalieri's principle we are done: the area of the section of the cylinder is equal to the sum of the area of the section of the hemisphere and the area of the section of the cone. Let's start with the hemisphere: by the Pythagorean theorem, $h^2 + r^2 = 1$, since the hypotenuse of that triangle is a radius of the sphere, of length one unit. Now for the cone: the triangle of sides labeled h and r' is isosceles, so $r' = h$. Thus $r^2 + r'^2 = 1 - h^2 + h^2 = 1$.