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While this chapter was written in a way that it can all be done using paper and pencil (and the tools suggested below) you may choose to use geometry software such as Geometer's Sketchpad or GeoGebra for students to explore these concepts. There are also many applets online that will show animations of the transformations.

The following tools should be available to students to use throughout the chapter: straight edge, patty (or tracing) paper, compass, protractor, additional graph paper, colored pencils.
Large table-sized grids could be kept in the classroom and used daily for both notes and homework. Scaled, cut-out shapes for each corresponding section would also need to be made.



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# Chapter 9 Geometry: Transformations, Congruence, and Similarity

**Utah Core Standard(s):**

- Verify experimentally the properties of rotations, reflections, and translations: (8.G.1)
  a) Lines are taken to lines, and line segments to line segments of the same length.
  b) Angles are taken to angles of the same measure.
  c) Parallel lines are taken to parallel lines.
- Understand that a two-dimensional figure is congruent to another if the second can be obtained from the first by a sequence of rotations, reflections, and translations; given two congruent figures, describe a sequence that exhibits the congruence between them. (8.G.2)
- Describe the effect of dilations, translations, rotations, and reflections on two-dimensional figures using coordinates. (8.G.3)
- Understand that a two-dimensional figure is similar to another if the second can be obtained from the first by a sequence of rotations, reflections, translations, and dilations; given two similar two-dimensional figures, describe a sequence that exhibits the similarity between them. (8.G.4)

**Academic Vocabulary:** transformation, translation, reflection, rotation, rigid motion, image, pre-image, corresponding vertices, corresponding segments, corresponding angles, corresponding parts, coordinate rule, perpendicular bisector, line of reflection, slope, horizontal line, vertical line, clockwise, counterclockwise, center of rotation, angle of rotation, origin, concentric circles, congruent, dilation, center of dilation, scale factor, similar

**Chapter Overview:**
In this chapter, students explore and verify the properties of translations, reflections, rotations, and dilations. Students learn about the different types of rigid motion (translations, reflections, and rotations), execute them, and write coordinate rules to describe them. They describe the effects of these rigid motions on two-dimensional figures. Students then use this knowledge to determine whether one figure is congruent to another, understanding that two figures are congruent if there is a sequence of rigid motions that takes one figure onto the other. Then, students study dilations, again exploring and verifying the properties of dilations experimentally. They describe and execute dilations. They use this knowledge to determine whether one figure is similar to another, understanding that a two-dimensional figure is similar to another if there is a sequence of rigid motions and dilations that takes one figure onto the other.

**Connections to Content:**
Prior Knowledge: Up to this point, students have worked with two-dimensional geometric figures, solving real-world and mathematical problems involving perimeter and area. They have classified two-dimensional figures based on their properties. In 7$^{th}$ grade, students scaled figures. Students will rely on work with function and slope from previous chapters in this text in order to investigate the properties of the different transformations and to write coordinate rules to describe transformations. Students also use the skill of writing the equation of a line in order to write the equation for a line of reflection. Students have also been exposed to dilations in Chapter 2 using the properties of dilations to prove that the slope of a line is the same between any two distinct points on a non-vertical line and to derive the equation of a line.
Future Knowledge: In subsequent courses, students will expand on this knowledge, explaining how the criteria for triangle congruence (ASA, SAS, and SSS) follow from the definition of congruence in terms of rigid motions. Similarly, they will use the properties of similarity transformations to establish the AA criterion for two triangles to be similar.


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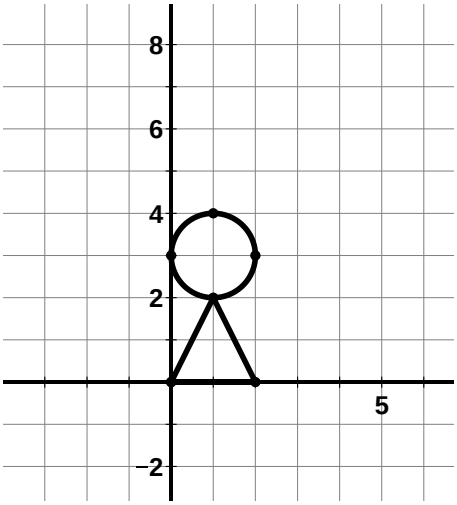
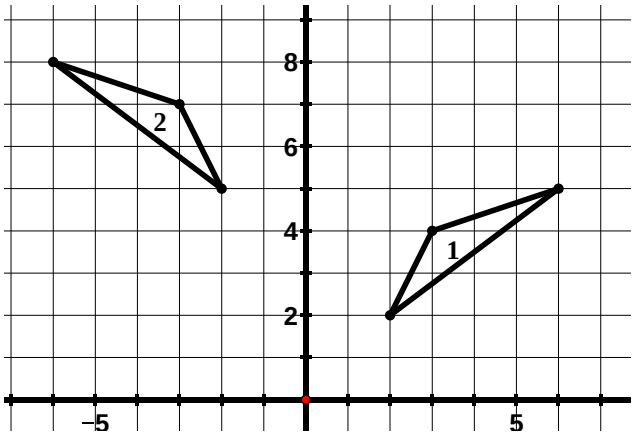
**MATHEMATICAL PRACTICE STANDARDS:**

| | | |
|---|---|---|
| | **Make sense of problems and persevere in solving them.** | Where would you put your center of dilation to enlarge the picture below by a factor of 2 and put the image entirely in the second quadrant? Mark your center of dilation with an *O* and then do the dilation.<br>  <br><br>Describe a sequence of rigid motions that would carry triangle 1 onto triangle 2.<br>  <br><br>*Throughout this chapter, students will see problems with multiple correct answers. For example, in the first problem above, there are many different places to put the center of dilation in order to meet the constraints specified in the problem. Students must use their knowledge of the properties of dilations as an entry point to solving the problem. In the second problem, there are many different sequences of rigid motion that will carry triangle 1 onto triangle 2. Students will have the opportunity to consider the different approaches taken by others, compare the approaches, and identify correspondences between the different approaches.* |



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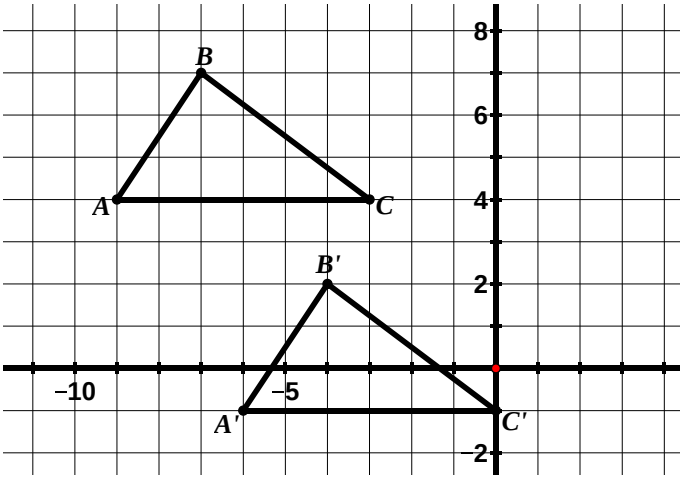
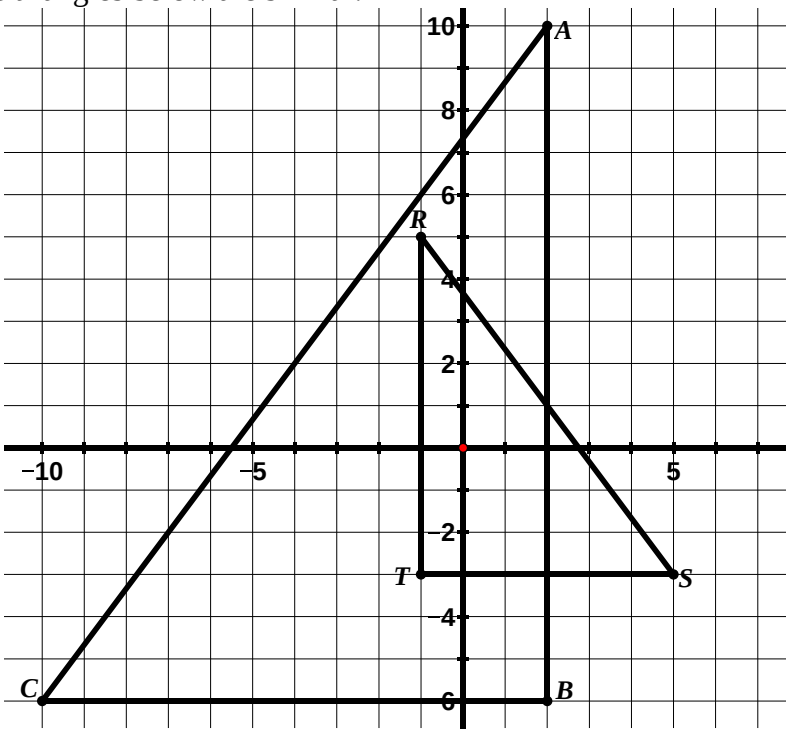
| | | |
|---|---|---|
| | **Reason abstractly and quantitatively.** | Write a coordinate rule to describe the translation below.<br>  <br><br>*Throughout the chapter, students write coordinate rules to describe transformations. They also perform transformations described by a coordinate rule.*<br><br>Determine the coordinate rule for a $90^0$ rotation clockwise about the origin. Connect this rule to the slopes of perpendicular lines.<br><br>*Students are also asked to explain why and how slopes of corresponding segments change under a given transformation, connecting the coordinate rule to the slope change. In this example, students are able to see that a rotation of a figure $90^0$ clockwise can be described by the following coordinate rule $(x, y) \rightarrow (y, -x)$ helping them to understand why the slopes of perpendicular lines are opposite reciprocals of each other.* |
| | **Construct viable arguments and critique the reasoning of others.** | The triangles below are similar.<br>  <br><br>List the sequence of transformations that verifies the similarity of the two figures. |



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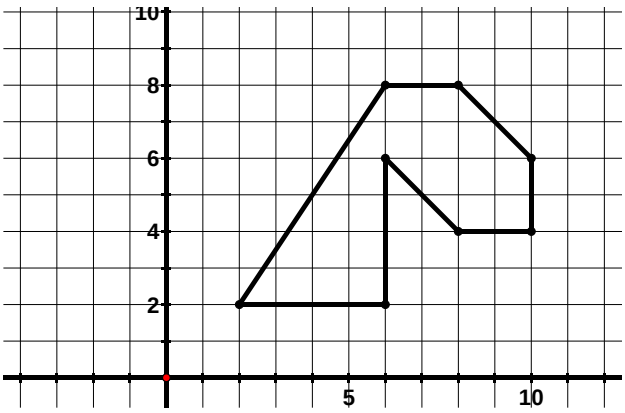
| | | |
|---|---|---|
| | | Write a similarity statement for the triangles.<br><br>*Students will construct an argument that verifies the similarity of these two figures based on their understanding of the definition of similarity in terms of transformational geometry, that is, they must identify a sequence of rigid motions and dilations that takes one figure onto the other. There are different sequences that will accomplish this. Students will justify the sequence they have arrived at and communicate this to classmates. They will also have the opportunity to consider alternative sequences of others and decide whether these sequences do in fact verify the similarity of the two triangles. They will also have the opportunity to identify correspondences between the different sequences.* |
| | **Model with mathematics.** | Animators need your help coming up with the coordinate rule that would reduce the size of the dragon's head to half its size when it enters a cave.<br>1. Write your proposed coordinate rule in the table below.<br>2. Write the new coordinates for your rule.<br>3. Graph the new coordinates.<br><br>  <br><br><br>*In this problem, students determine how to scale the figure shown above (i.e. make it a different size while maintaining its shape). Scaling is something we see and use constantly in the world around us. Many professionals such as architects and computer animators rely on scaling techniques in order to create scaled down versions of real-life objects.* |



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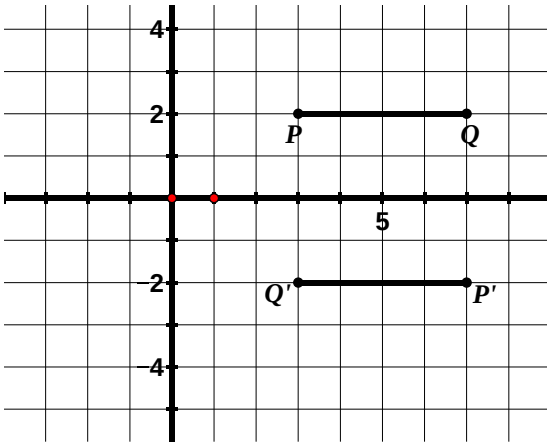
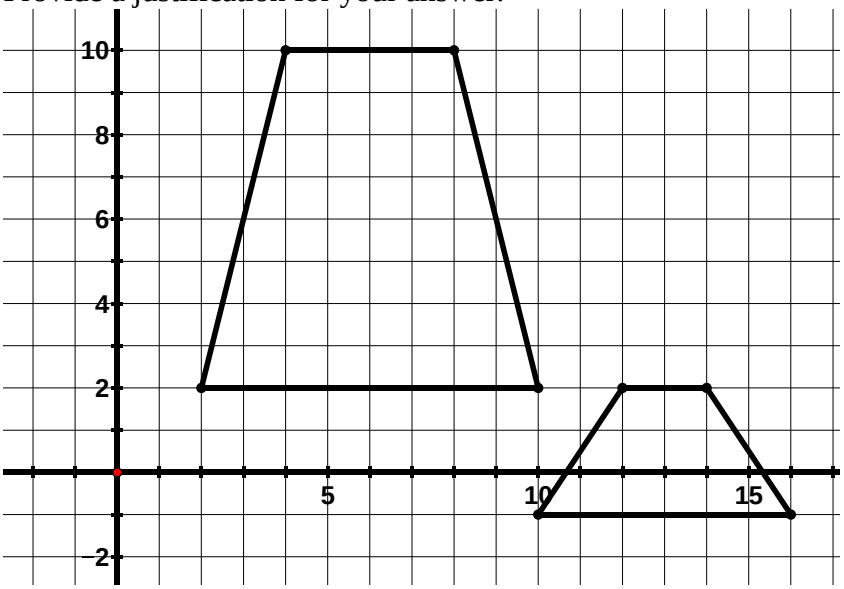
| | | |
|---|---|---|
| | **Use appropriate tools strategically.** | Find the angle of rotation (including direction of rotation) and center of rotation for the rotation shown below.<br>  <br><br>*Students use a variety of tools in this chapter. These tools include straight edge, patty (or tracing) paper, compass, protractor, additional graph paper, colored pencils, and dynamic geometry software. In the problem above, one way to find the center of rotation is to trace the figures on patty paper and fold their paper so that P lines up with P' and fold a second time so that Q lines up with Q'. The intersection of these folds is the center of dilation. They can also use the patty paper to find the angle of rotation.* |
| | **Attend to precision.** | Determine whether the figures below are congruent, similar, or neither. Provide a justification for your answer.<br>  <br><br>*In order to answer this question, students must be clear about their understanding of what establishes congruence and similarity between two figures and they must be able to clearly communicate this to others.* |



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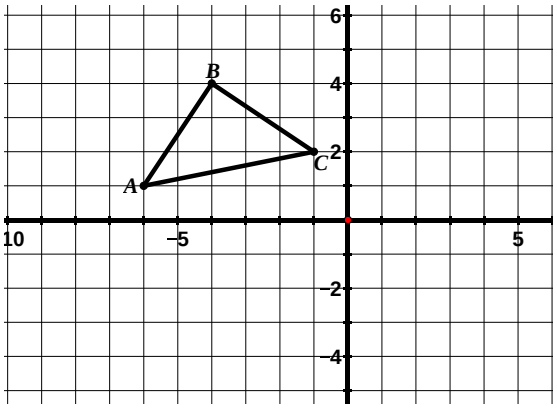
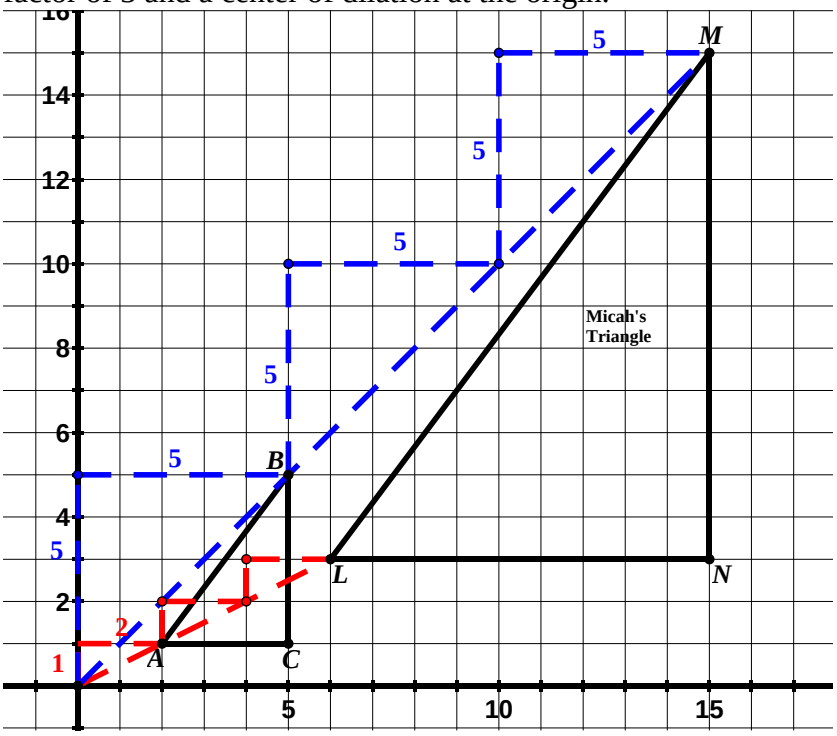
| | | |
|---|---|---|
| | **Look for and make use of structure.** | Reflect $\Delta ABC$ across the $x$-axis and label the image.<br>  <br><br>Write a coordinate rule to describe this reflection. Compare the slopes of the corresponding segments of the image and pre-image. What do you notice about the slopes? How does this connect to the coordinate rule $(x, y) \rightarrow (x, -y)$?<br>*In this problem, students determine a coordinate rule to describe a reflection across the x-axis. In doing so, students examine the structure of the ordered pairs, realizing that under a reflection across the x-axis, the x-coordinates remain unchanged while the y-coordinates change sign. Following this, students use the coordinate rule to explain the effect this transformation has on the slope of a segment.* |
| | **Look for and express regularity in repeated reasoning.** | The figure below shows a triangle that has been dilated with a scale factor of 3 and a center of dilation at the origin.<br>  <br><br>*As students solve problems throughout the chapter, they use ideas about slope over and over again to discover the properties of rigid motions and dilations. They also use ideas about slope to translate, reflect, rotate, and dilate figures. In the picture above, Micah is using slope triangles to dilate $\Delta ABC$ with a scale factor of 3 and center of dilation at the origin.* |



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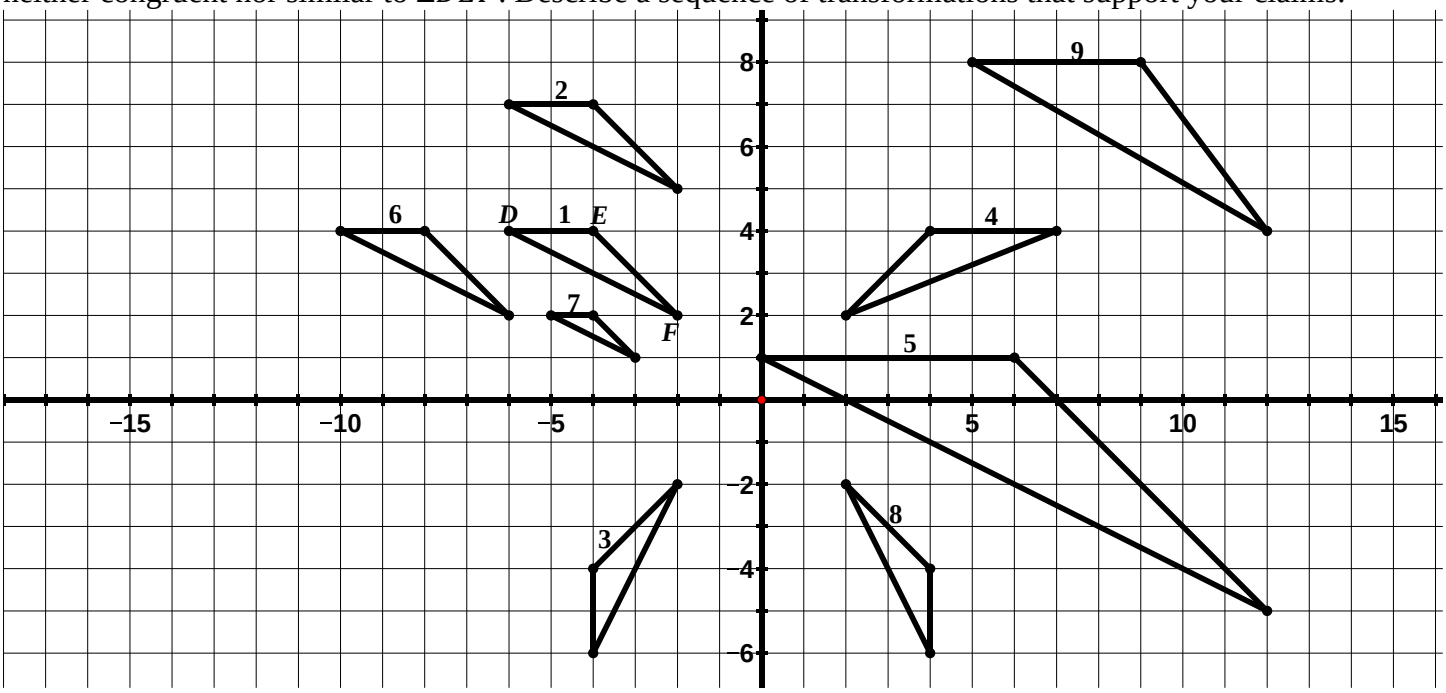
# 9.0 Anchor Problem: Congruence and Similarity

This could be used as a quiz instead of an anchor problem. Provide tracing paper.

**Directions:** Determine whether the triangles pictured below are **congruent** to $\Delta DEF$, **similar** to $\Delta DEF$, or neither congruent nor similar to $\Delta DEF$. Describe a sequence of transformations that support your claims.




In order to solve this problem, students must understand the definitions of congruence and similarity:

- In earlier grades, students learn that congruent figures are the same size and shape. In 8th grade, students understand that two figures are **congruent** if there is a sequence of rigid motions that takes one figure onto the other.
- In earlier grades, students learn that similar figures are the same shape. In 8th grade, students understand that two figures are **similar** if there is a sequence of rigid motions and dilations that takes one figure onto the other.

*Note in the key below that students may have a different justification (they may describe a different sequence of transformations). You can expand on this problem and have students describe properties, write coordinate rules for the translations and reflections, etc. You may also choose to divide the work up and have different groups of students investigate the relationships of different triangles to $\Delta DEF$ and present their findings to the class. You can then have the other groups try the transformations suggested by the group to determine if it results in the desired figure.*

| Congruent/Similar/Neither | Justification |
|---|---|
| $\Delta 1 \cong \Delta 2$ | Figure 2 is a translation 3 units up of Figure 1. |
| $\Delta 1 \cong \Delta 3$ | Figure 3 is a rotation 90˚ counterclockwise with center of rotation at the origin of Figure 1. |
| Figure 4 is neither | Students may give a variety of reasons. The ratios of corresponding sides are not congruent. It appears to be a reflection over the *y*-axis but one point (the point that would correspond to *D* has been pulled to the right one unit). |
| $\Delta 1 \sim \Delta 5$ | One possible justification: Figure 1 was translated down 3 units and then dilated by a factor of 3 with the center of dilation at (-9, 1) in order to obtain Figure 5. |
| $\Delta 1 \cong \Delta 6$ | Figure 6 is a translation 4 units to the left of Figure 1. |
| $\Delta 1 \sim \Delta 7$ | One possible justification: Figure 1 was dilated by a factor of $\frac{1}{2}$ with the center of dilation at (-4, 0) in order to obtain Figure 7. |
| $\Delta 1 \cong \Delta 8$ | Figure 8 is a reflection of Figure 1 across the line $y = x$. |
| Figure 9 is neither | Students may give a variety of reasons. One is that corresponding angles are not congruent. |



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# 9.1 Rigid Motion and Congruence

**Section Overview:**
In this section, students study the different types of rigid motion: translations, reflections, and rotations. Students begin their study of rigid motion with translations. Students describe translations that have taken place, both in words and with a coordinate rule. Students execute various translations, given a coordinate rule. Then, students summarize the properties of a translation based on the work they have done. These first few lessons also introduce students to some of the vocabulary used in transformational geometry. Next, students turn to reflections, again discovering the properties of reflections (including those over horizontal/vertical lines and the lines $y = x$ and $y = -x$). They write coordinate rules for reflections, connecting these rules to the slopes of the corresponding segments in the image and pre-image. Lastly, students draw lines of reflection and write the equations for these lines. Students then study rotations, with the emphasis on rotations of $90^\circ$ increments. Students describe the properties of rotations and use these properties to solve problems. They start with rotations where the center of rotation is at the origin, again describing and executing rotations. Then, students study rotations where the center of rotation is not at the origin. Throughout the study of translations, reflections, and rotations, students articulate which properties hold for all of the rigid motions and which are specific to a given rigid motion. Students also perform a sequence of rigid motions and identify sequences of rigid motions that carry one figure to another. Students will then apply this knowledge to determine if two figures are congruent, understanding that two figures are congruent if there is a sequence of rigid motions that takes one figure onto the other.

**Concepts and Skills to Master:**
*By the end of this section, students should be able to:*

1. Given a pre-image and its image under a translation, describe the translation in words and using a coordinate rule.
2. Perform a translation of a figure given a coordinate rule.
3. Describe the properties of a translation and the effects a translation has on a figure and use this knowledge to solve problems.
4. Given a pre-image and its image under a reflection, describe the reflection in words and using a coordinate rule.
5. Perform a reflection of a figure given a line of reflection.
6. Describe the properties of a reflection and the effects a reflection has on a figure and use this knowledge to solve problems.
7. Find a reflection line for a given reflection and write the equation of the reflection line.
8. Given a pre-image and its image under a rotation, describe the rotation in words and using a coordinate rule (coordinate rule for rotations centered at the origin only).
9. Perform a rotation of a figure given a center of rotation, an angle of rotation, and a direction.
10. Describe the properties of a rotation and the effects a rotation has on a figure and use this knowledge to solve problems.
11. Connect ideas about slopes of perpendicular lines and rotations.
12. Understand what it means for two figures to be congruent.
13. Determine if two figures are congruent based on the definition of congruence.
14. Given two figures that are congruent, describe the sequence of transformations that exhibits the congruence between them.


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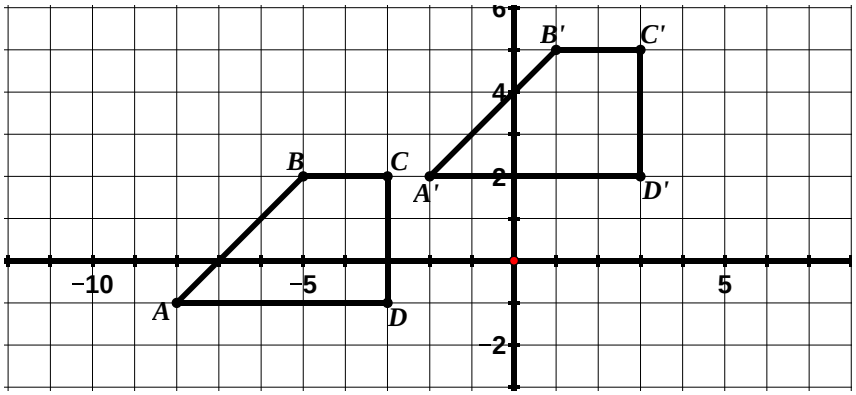
## 9.1a Class Activity: Properties of Translations

1. In the grid below, $ABCD$ has been transformed to obtain $A'B'C'D'$.




a. This type of transformation is called a **translation**. Describe in your own words the movement of a figure that has been translated.
Answers will vary. Students may use the words slide/shift, a movement up, down, left, right.

b. Show on the picture how you would move on the coordinate plane to get from $A$ to $A'$, $B$ to $B'$, $C$ to $C'$, and $D$ to $D'$.
To move between corresponding vertices, go over 6 and up 3. Show on grid.

$ABCD$ is called the **pre-image** and $A'B'C'D'$ is called the **image**. The **pre-image** is the figure prior to the transformation and the **image** is the figure after the transformation. $A$ and $A'$, $B$ and $B'$, $C$ and $C'$, and $D$ and $D'$ are **corresponding vertices**.

c. In the table below, write the coordinates for the vertices of the pre-image and image.

| Pre-Image | Image |
|---|---|
| $A$: $(-8,-1)$ | $A'$: $(-2,2)$ |
| $B$: $(-5,2)$ | $B'$: $(1,5)$ |
| $C$: $(-3,2)$ | $C'$: $(3,5)$ |
| $D$: $(-3,-1)$ | $D'$: $(3,2)$ |

d. The coordinate rule for this translation is $(x,y) \rightarrow (x+6, y+3)$. Connect this notation to your answer for part b. and to the coordinates of corresponding vertices in the table.

Students should see that in order to get from point to point, you would move to the right 6 units and up three units. If they map the coordinates of the pre-image and image, they will also see the rule. A common mistake is for students to interchange the horizontal movement and the vertical movement in the coordinate rule. Another common mistake is that students will give the coordinate rule that maps the **image to the pre-image** instead of the **pre-image to the image**. Watch for these common errors.

Honor's Extension: Examine translations as vectors. See the mathematical foundation for a detailed discussion of this.



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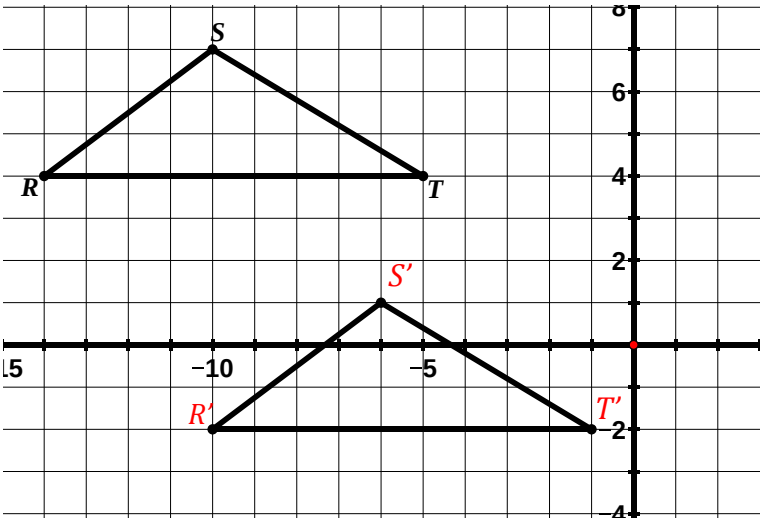
2. In the grid below, $\Delta RST$ has been translated to obtain $\Delta R'S'T'$.




a. Label the corresponding vertices of the image on the grid.

b. Describe or show on the picture how you would move on the coordinate plane to get from the vertices in the pre-image to the corresponding vertices in the image.

Right 4, down 6. Show on grid.

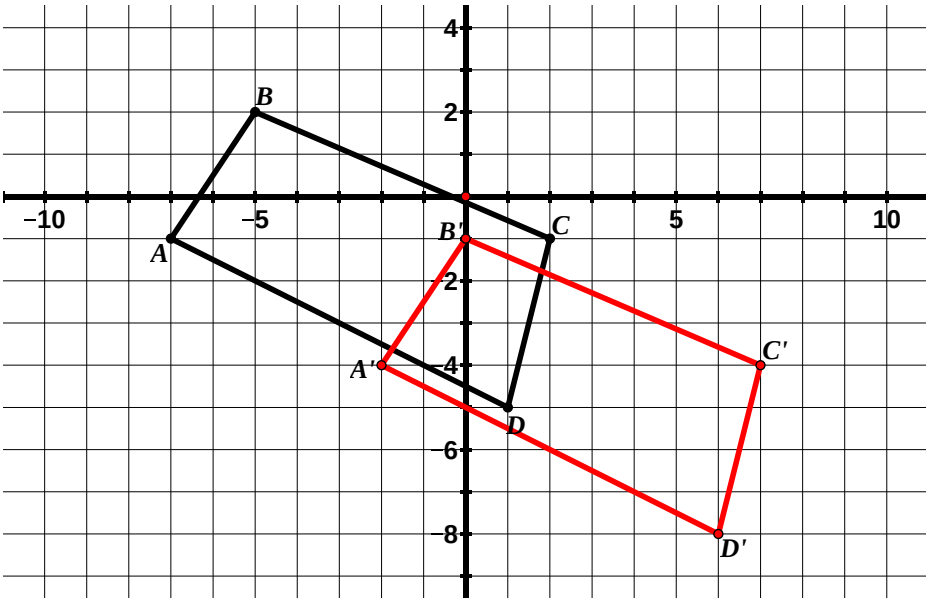
c. In the table below, write the coordinates for the vertices of the pre-image and image.

| Pre-Image | Image |
|---|---|
| $R$: $(-14, 4)$ | $R'$: $(-10, -2)$ |
| $S$: $(-10, 7)$ | $S'$: $(-6, 1)$ |
| $T$: $(-5, 4)$ | $T'$: $(-1, -2)$ |

d. Write the coordinate rule that describes this translation.

$(x, y) \rightarrow (x + 4, y - 6)$

3. Draw and label the image of the figure below for the translation $(x, y) \rightarrow (x + 5, y - 3)$



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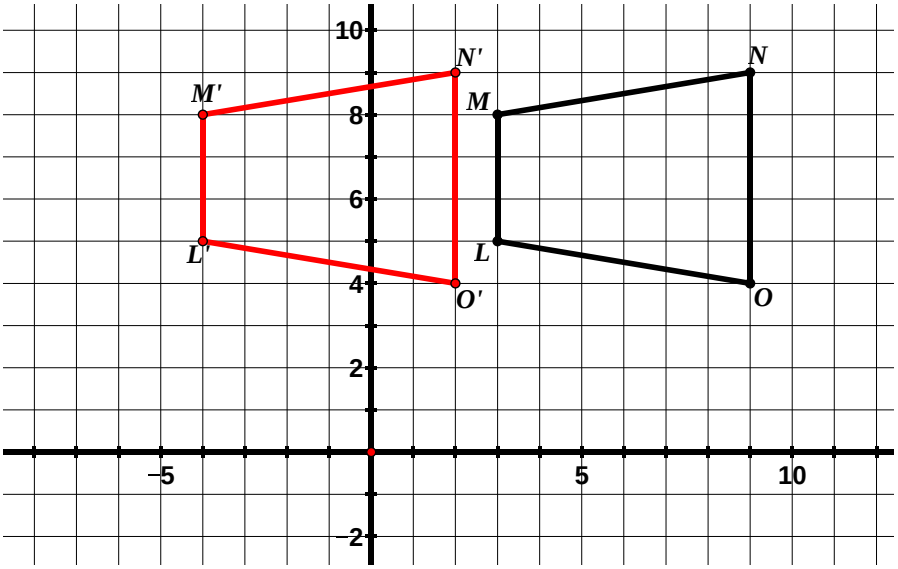
4. Draw and label the image of the figure below for the translation $(x, y) \rightarrow (x - 7, y)$




Determine the slopes for:

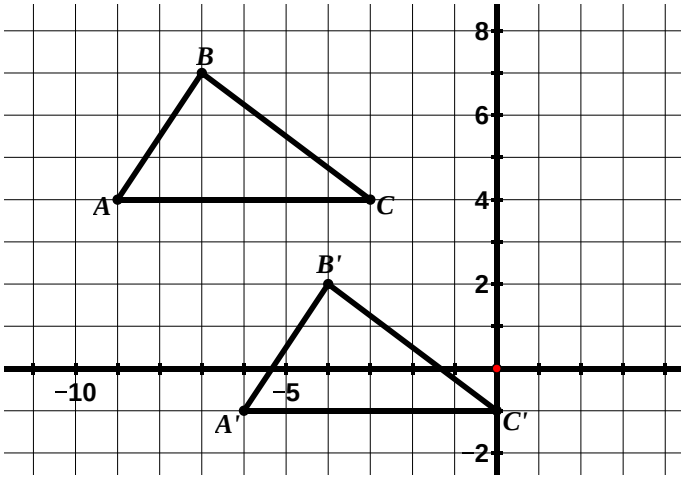
$\overline{MN}$: $\frac{1}{6}$ $\overline{M'N'}$: $\frac{1}{6}$

$\overline{NO}$: und. $\overline{N'O'}$: und.

$\overline{LO}$: $-\frac{1}{6}$ $\overline{L'O'}$: $-\frac{1}{6}$

$\overline{ML}$: und. $\overline{M'L'}$: und.

5. Write a coordinate rule to describe the translation below. ___$(x, y) \rightarrow (x + 3, y - 5)$_____



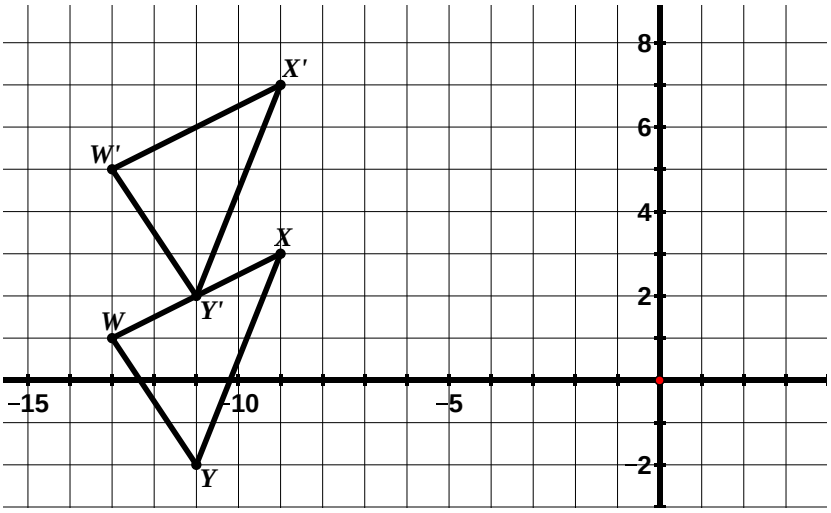

Determine the slopes for:

$\overline{AB}$: $\frac{3}{2}$ $\overline{A'B'}$: $\frac{3}{2}$

$\overline{BC}$: $-\frac{3}{4}$ $\overline{B'C'}$: $-\frac{3}{4}$

$\overline{CA}$: 0 $\overline{C'A'}$: 0

6. Write a coordinate rule to describe the translation below _____$(x, y) \rightarrow (x, y + 4)$_____




Determine the slopes for:

$\overline{WX}$: $\frac{1}{2}$ $\overline{W'X'}$: $\frac{1}{2}$

$\overline{XY}$: $\frac{5}{2}$ $\overline{X'Y'}$: $\frac{5}{2}$

$\overline{YW}$: $-\frac{3}{2}$ $\overline{Y'W'}$: $-\frac{3}{2}$



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7. Use questions #1 – 6 to explore some **properties of translations** and write your observations below.

Probe students to think about the properties of translations. It may be helpful for students to think about what we are doing when we preform a translation – we trace a figure and then slide the piece of paper that the figure is traced on around (without picking it up or turning it) in order to draw our new figure.

Let them talk about what they are seeing that is common in all of the translations. Then, help them to describe what they are seeing using precise vocabulary. The following are the properties of translations students should articulate and know:

- **Corresponding segments** in the image and pre-image are the same length.
- **Corresponding segments** in the image and pre-image have the same slope.
- **Corresponding angles** in the image and pre-image have the same measure.
- Parallel lines in the pre-image remain parallel lines in the image.
- **Segments connecting corresponding vertices** of the image and pre-image are parallel (they have the same slope). *We can see this by observing the slope triangles of the lines connecting corresponding vertices. This can also be connected to the coordinate rule that describes the translation.*
- The figure's location in the plane changes during a translation.
- The figure's orientation does not change. One way to think about orientation is to start at one of the vertices in the pre-image and move around the figure clockwise, noting the order of the vertices. If the order of the vertices is the stay on the image (starting at the same vertex and moving in a clockwise direction) then the orientation has not changed.



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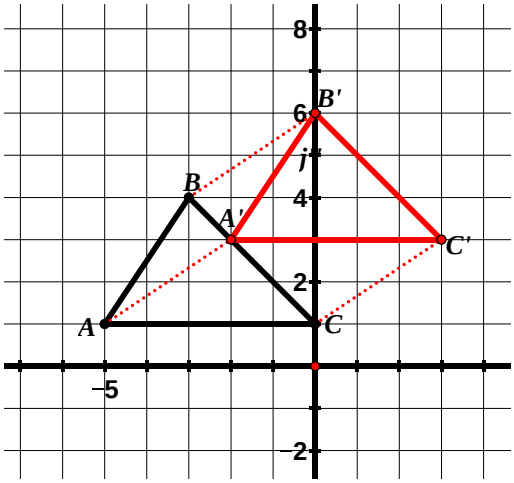
**9.1a Homework: Properties of Translations**

**Directions:** For #1 – 3, draw and label the image for the coordinate rule given. Then answer the questions.

1. Translate the figure below according to the rule $(x, y) \rightarrow (x + 3, y + 2)$ and label the image.




a. If the slope of $\overline{BC}$ is $-1$, determine the slope of $\overline{B'C'}$ without doing any calculations. $-1$

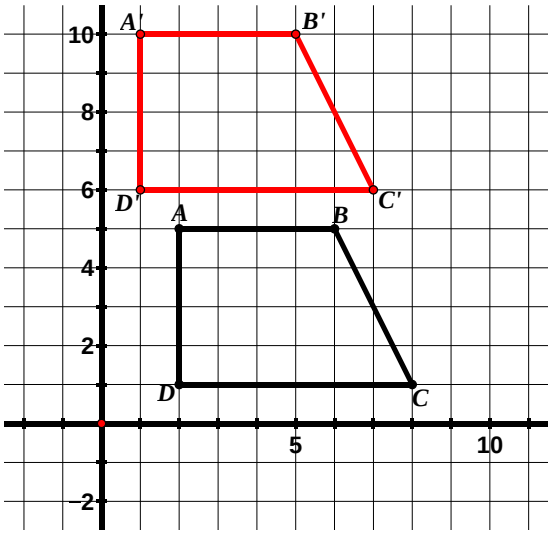
b. If the length of $\overline{BC}$ is $3\sqrt{2}$, determine the length of $\overline{B'C'}$ without doing any calculations. $3\sqrt{2}$

c. Determine the slopes of $\overline{AB}$ and of $\overline{A'B'}$. What do you notice about the slopes of **corresponding segments of a translated figure**? Both of the segments have a slope of $\frac{3}{2}$. Slopes of corresponding segments are parallel.

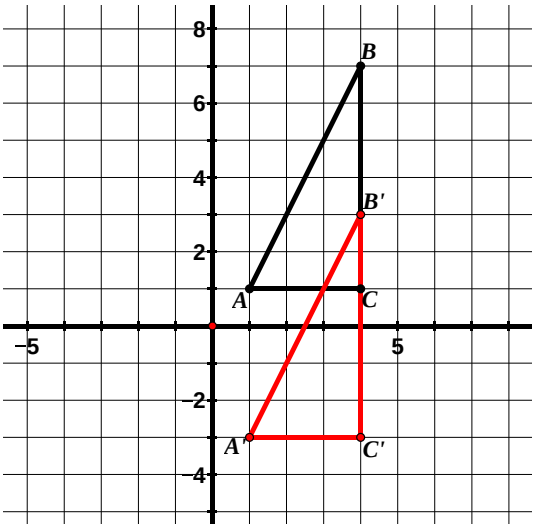
d. Using a ruler, draw a line connecting corresponding vertices in the image and pre-image ($A$ to $A'$, $B$ to $B'$, and $C$ to $C'$). Find the slopes of $\overline{AA'}$, $\overline{BB'}$, and $\overline{CC'}$. What do you notice about the slopes of the **segments connecting corresponding vertices** of the image and pre-image of a translated figure?

All of the segments have a slope of $\frac{2}{3}$ (segments connecting corresponding vertices are parallel). Point out that the slope of these segments comes from the translation – all points are moved up 2 and to the right 3 - see coordinate rule. Interesting that the picture above shows how to draw a triangular prism.

2. Translate the figure below according to the rule $(x, y) \rightarrow (x - 1, y + 5)$ and label the image.




3. Translate the figure below according to the rule $(x, y) \rightarrow (x, y - 4)$ and label the image.




Reinforce the properties using examples 2 and 3: 1) Lines that are parallel in the pre-image remain parallel in the image. 2) Corresponding segments are parallel and have the same length. 3) Segments connecting corresponding vertices are parallel and have the same length.



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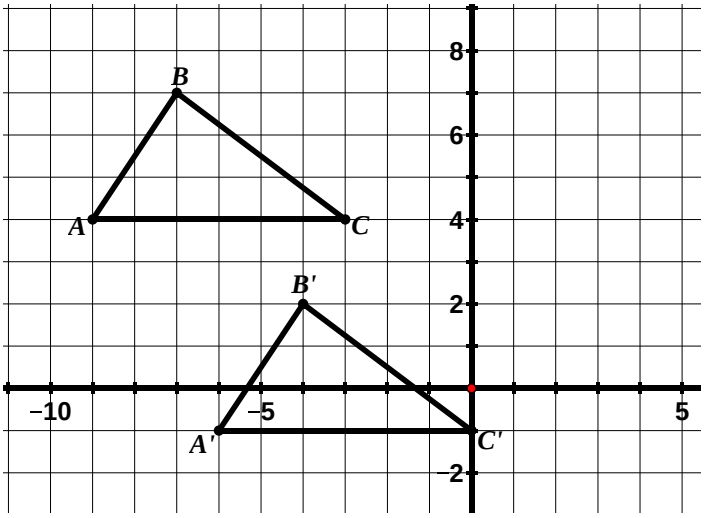
**Directions:** For #4 – 7, write a coordinate rule to describe the translation. Then answer the questions.

4. Coordinate Rule: $(x, y) \rightarrow (x + 3, y - 5)$



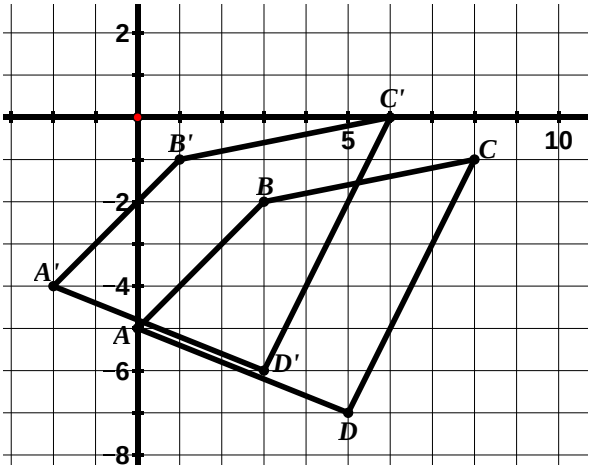

a. The slope of $\overline{BB'}$ is $-\frac{5}{3}$. Name two other segments that also have a slope of $-\frac{5}{3}$.
$\overline{AA'}$ and $\overline{CC'}$

b. If the length of $\overline{BB'}$ is $\sqrt{34}$, determine the length of $\overline{CC'}$ without doing any calculations.
$\sqrt{34}$

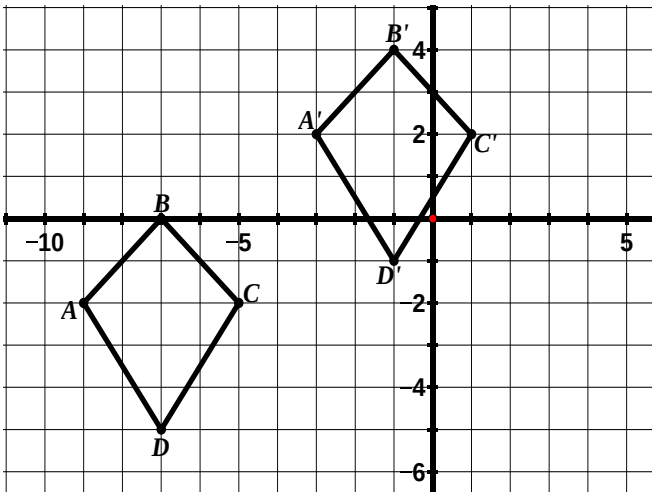
c. Determine the length of $\overline{AC}$ and of $\overline{A'C'}$.
Both have a length of 6 units.

d. Determine the slope of $\overline{AC}$ and of $\overline{A'C'}$.
Both have a slope of 0.

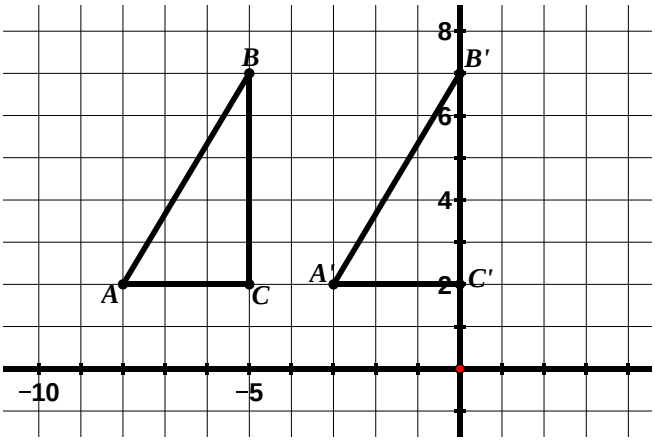
5. Coordinate Rule: $(x, y) \rightarrow (x - 2, y + 1)$




6. Coordinate Rule: $(x, y) \rightarrow (x + 6, y + 4)$




7. Coordinate Rule: $(x, y) \rightarrow (x + 5, y)$



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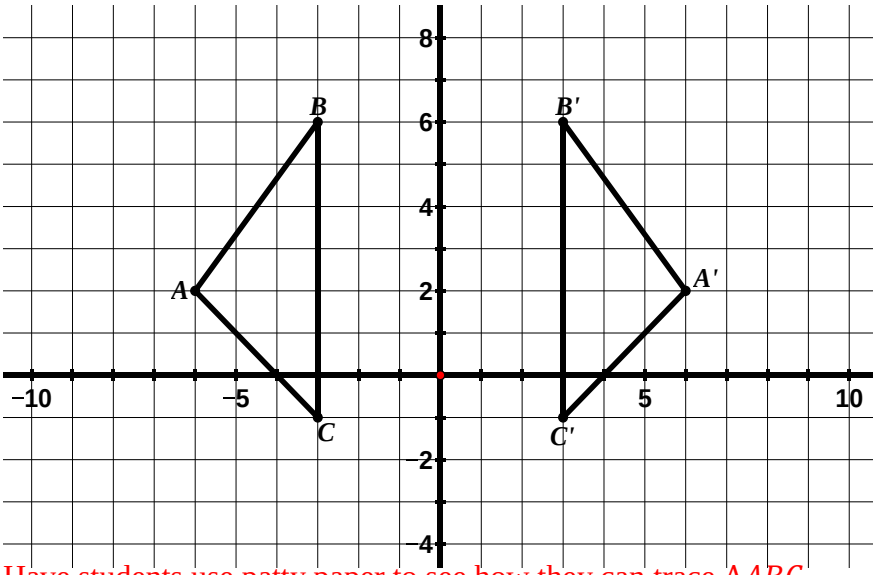
## 9.1b Class Activity: Properties of Reflections

1. In the grid below, $\Delta ABC$ has been **reflected** over the $y$-axis to obtain $\Delta A'B'C'$.




a. Describe the movement of a figure that has been **reflected**.

Answers will vary. The figures are mirror images. The object was flipped over the $y$-axis.

b. In the table below, write the coordinates for the vertices of the pre-image and image.

| Pre-Image | Image |
|---|---|
| $A$: $(-6, 2)$ | $A'$: $(6, 2)$ |
| $B$: $(-3, 6)$ | $B'$: $(3, 6)$ |
| $C$: $(-3, -1)$ | $C'$: $(3, -1)$ |

Have students use patty paper to see how they can trace $\Delta ABC$ and its image $\Delta A'B'C'$ as well as the line of reflection. How can they map the two figures onto each other?

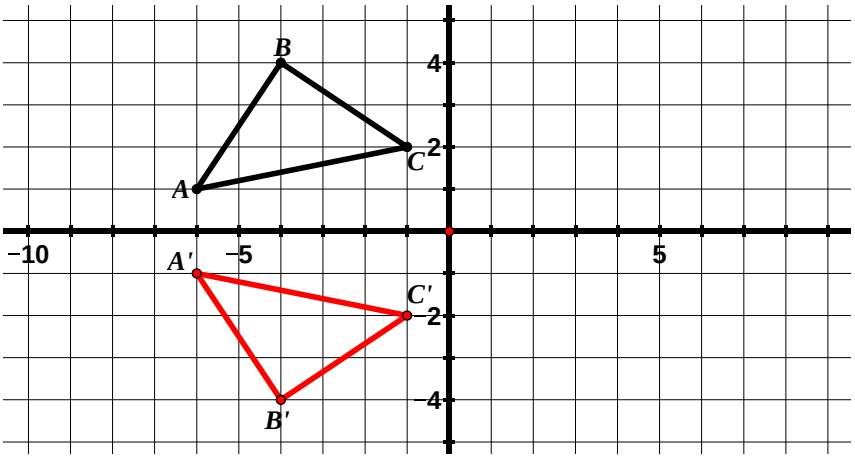
c. Write a coordinate rule to describe this reflection. $(x, y) \rightarrow (-x, y)$

d. Will this coordinate rule hold true for any figure reflected over the $y$-axis? Why or why not?

Yes, when you reflect over the $y$-axis, you are placing the points on the opposite side of the $y$-axis (changing their $x$-coordinate) but the $y$-coordinate does not change.

**Directions:** Draw and label the image of each figure for the reflection given. Then, answer the questions.

2. Reflect $\Delta ABC$ across the $x$-axis and label the image.




a. In the table below, write the coordinates for the vertices of the pre-image and image.

| Pre-Image | Image |
|---|---|
| $A$: $(-6,1)$ | $A'$: $(-6, -1)$ |
| $B$: $(-4,4)$ | $B'$: $(-4, -4)$ |
| $C$: $(-1,2)$ | $C'$: $(-1, -2)$ |

b. Write a coordinate rule to describe this reflection.
$(x, y) \rightarrow (x, -y)$

c. Will this coordinate rule hold true for any figure reflected over the $x$-axis? Why or why not?

Yes, when you reflect over the $x$-axis, you are placing the points on the opposite side of the $x$-axis (changing their $y$-coordinate) but the $x$-coordinate does not change. This can be a confusing concept for students.



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3. Use questions #1 – 2 to explore some **properties of reflections**.
   a. Go back to problem #1. Draw a segment connecting B and B', A and A', and C and C'. Make at least two conjectures about the relationship between **the line of reflection and the segments connecting corresponding vertices** in the image and pre-image of a reflection.

Segments connecting corresponding vertices are parallel (have the same slope) and are perpendicular to the line of reflection. The line of reflection cuts the segments connecting corresponding vertices into two equal parts. Another way of saying this is that the line of reflection is the perpendicular bisector of all segments connecting corresponding points of the image and pre-image. Show this on the image.

Compare this to the segments that connect corresponding vertices of a figure under a translation – these segments are also parallel but they are also the same length whereas in a reflection they are not – discuss this with students.

   b. Do your conjectures hold true in problem #2?
      Yes, show this on the image.

   c. Go back to problem #1. For a *translation* we learned that corresponding segments are parallel (have the same slope). Is this property also true for reflections?
      No – students can observe this fact in problem #1. Compare the slope of $\overline{AB}$ $(\frac{3}{2})$ to that of $\overline{A'B'}(\frac{3}{2})$. Allow students to observe this on the picture and talk about how folding the figure across the line of reflection causes the slopes of corresponding segments to change.
   d. Now, go to problem #2. Find the slopes of the following segments:

| | | |
|---|---|---|
| $\overline{AB} = \frac{3}{2}$ | $\overline{AC} = \frac{1}{5}$ | $\overline{BC} = -\frac{2}{3}$ |
| $\overline{A'B'} = -\frac{3}{2}$ | $\overline{A'C'} = -\frac{1}{5}$ | $\overline{B'C'} = \frac{2}{3}$ |

   e. Compare the slopes of the corresponding segments of the image and pre-image. What do you notice about the slopes? How does this connect to the coordinate rule $(x, y) \rightarrow (x, -y)$? n#

You can see in the coordinate rule that the sign of the *y*-coordinate changes under a reflection across the *x*-axis; therefore the sign of our rise will change, changing the sign of our slope. Help students to see this by also looking at the picture. Discuss this same concept for problem #1. This time, the sign of the *x*-coordinate changes; therefore the sign of our run will change, changing the sign of our slope. In both cases, the absolute values of the slope do not change; however the sign of the slopes change. This question is an excellent exercise in reasoning abstractly and quantitatively. The coordinates and slope calculations give students a numeric way to see how the slope has changed; whereas the coordinate rule provides an abstract way to look at the change in slope under these types of reflections.

   f. Examine problems #1 and #2. What do you notice about the **lengths of corresponding segments** in the image and pre-image?

The lengths of corresponding segments are the same (or congruent). Ask students about the corresponding angles as well – corresponding angles have the same measure (are congruent).


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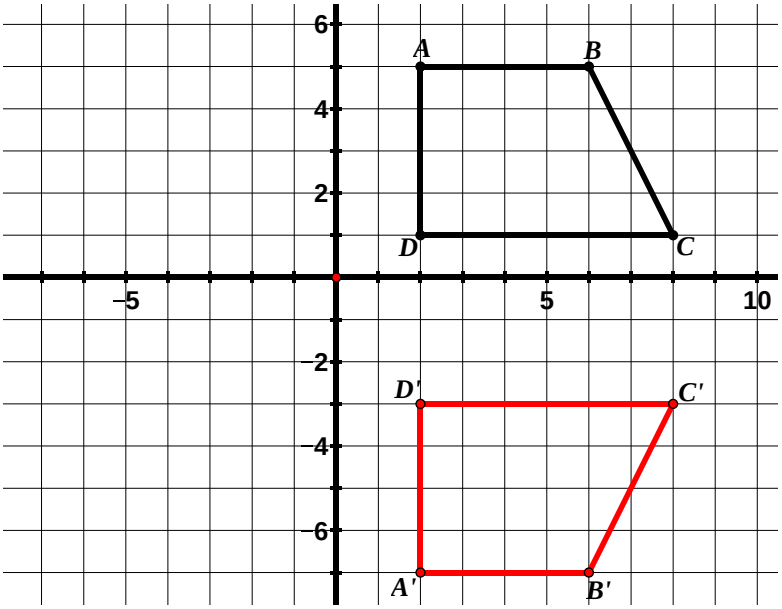
4. Reflect $ABCD$ across the line $y = -1$ and label the image.



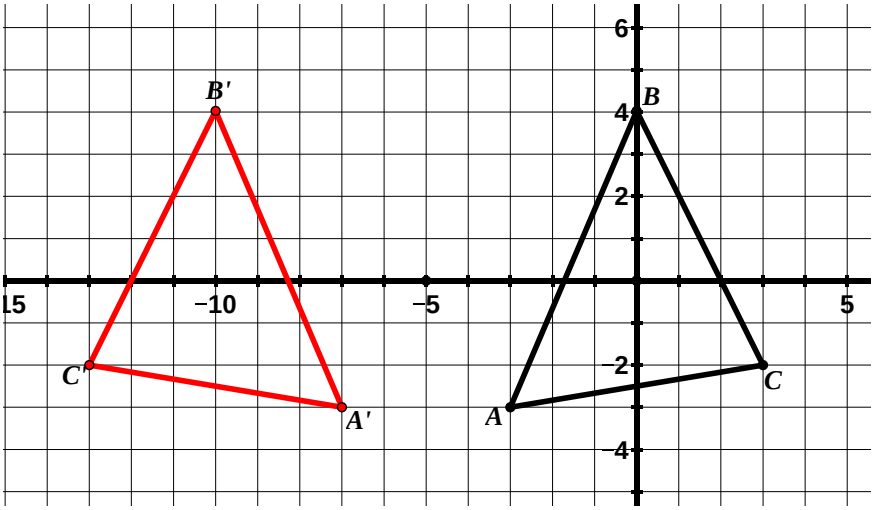

a. In the table below, write the coordinates for the vertices of the pre-image and image.

| Pre-Image | Image |
|---|---|
| $A$: $(2, 5)$ | $A'$: $(2, -7)$ |
| $B$: $(6, 5)$ | $B'$: $(6, -7)$ |
| $C$: $(8, 1)$ | $C'$: $(8, -3)$ |
| $D$: $(2, 1)$ | $D'$: $(2, -3)$ |

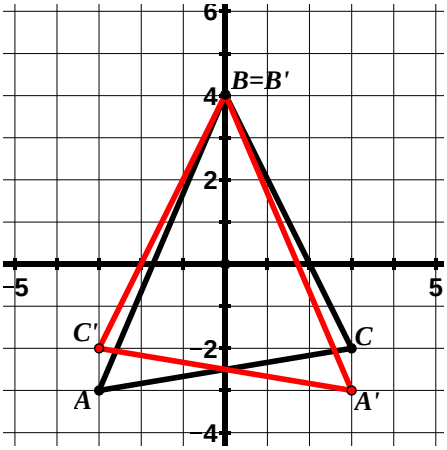
b. Write a coordinate rule to describe this reflection. $(x, y) \rightarrow (x, (-y - 2))$

This rule may be a little more difficult for students to see. One thing they should see is that a reflection over a horizontal line will cause a change in the *y*-coordinate but not the *x*-coordinate. Another way to think about the reflection above is as a reflection over the *x*-axis first and then a translation down 2 units.

5. Reflect $\Delta ABC$ across the line $x = -5$ and label the image.




6. Reflect $\Delta ABC$ over the *y*-axis and label the image.



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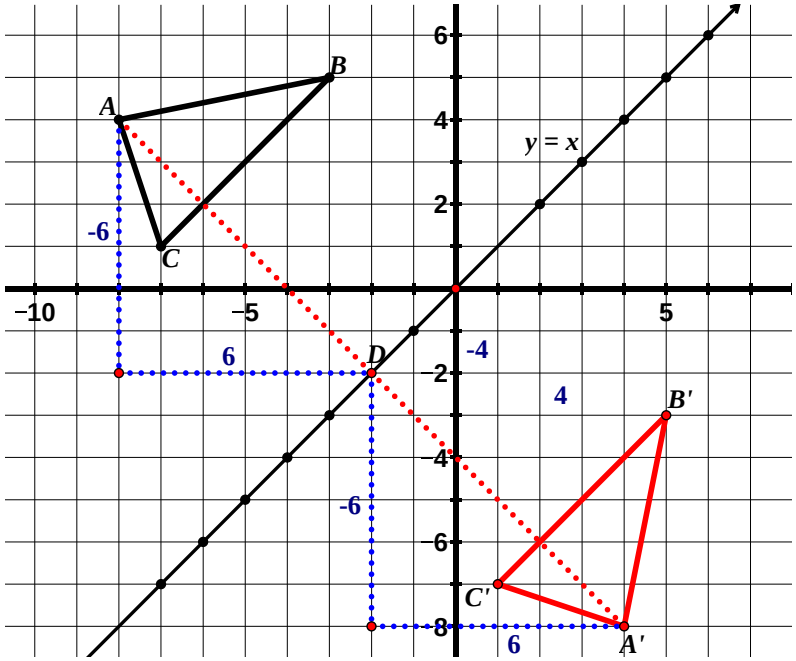
7. Reflect $\Delta ABC$ across the line $y = x$ and label the image.




a. Describe the method you used to solve this problem.

Students may use an informal technique and fold their paper along the line of reflection and trace where the resulting image would be. If students use an informal technique such as paper folding, encourage them to check that the properties hold true – it is easy to be off when folding the paper and tracing. Students can also use the properties they have discovered about reflections and concepts about slope. In the figure above, the slope of the reflection line is 1. That tells us that the slope of all of our segments connecting corresponding vertices should be -1. We also know that corresponding vertices are equidistant from the line of reflection. Putting these ideas together, we can start at point *A* and move on a line with a slope of -1 until we hit the line of reflection (labeled *D*). If we look at the slope triangle for this movement (shown in blue), we can see that we go down 6 and right 6. If we move this same distance from point *D*, we can mark the location of *A'*. We can use a similar process to locate the other vertices of the image.

b. In the table below, write the coordinates for the vertices of the pre-image and image.

| Pre-Image | Image |
|---|---|
| $A$: $(-8, 4)$ | $A'$: $(4, -8)$ |
| $B$: $(-3, 5)$ | $B'$: $(5, -3)$ |
| $C$: $(-7, 1)$ | $C'$: $(1, -7)$ |

c. Write a coordinate rule to describe this reflection.
$(x, y) \rightarrow (y, x)$

A common mistake here is that students will think that these corresponding segments are perpendicular – the rule is close but not quite – we are missing a sign change.

d. Will this coordinate rule hold true for any figure reflected over the line $y = x$? Why or why not?
Yes, allow students to explain their reasoning

e. Find the slopes of the following segments:

$\overline{AB} = \frac{1}{5}$ $\quad$ $\overline{AC} = -3$ $\quad$ $\overline{BC} = 1$

$\overline{A'B'} = 5$ $\quad$ $\overline{A'C'} = -\frac{1}{3}$ $\quad$ $\overline{B'C'} = 1$

f. Compare the slopes of the corresponding segments of the image and pre-image. What do you notice? How does this connect to the coordinate rule?
The *x* and *y* values switch under a reflection across the line $y = x$; therefore the rise and run will swap in the slope.

g. **Bonus:** What is the coordinate rule for a figure reflected across the line $y = -x$? $(x, y) \rightarrow (-y, -x)$

Encourage students to make a conjecture and then test their conjecture by doing this reflection.



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8. The following table lists the properties of translations discovered in the previous lesson. Put a checkmark in the box if the property is also true for reflections. Add additional statements to the table that are only true for reflections.

| Properties of Translations | Also True for Reflections? |
|---|---|
| Segments connecting the corresponding vertices of the image and pre-image are the same length. | |
| Segments connecting the corresponding vertices of the image and pre-image are parallel to each other. | ✓ |
| *Corresponding segments in the image and pre-image are the same length. | ✓ |
| *Corresponding angles in the image and pre-image have the same measure. | ✓ |
| *Parallel lines in the pre-image remain parallel lines in the image. | ✓ |
| Corresponding segments in the image and pre-image have the same slope. | |
| | Line of reflection is the perpendicular bisector of all segments connecting corresponding vertices of the image and pre-image. |
| | The slopes of corresponding segments may change under a reflection. |
| | Orientation changes under a reflection. Orientation is a commonly misunderstood concept. If you move clockwise around the pre-image and image and the order of the vertices changes, the orientation has changed. Orientation changes under a reflection; however it does **not** change under a rotation as you will see in lessons that follow. |
| | |

We are building the properties of congruent figures as we discover the properties of the different types of rigid motion. Students should clearly understand the properties with an * above as these are the properties common to all types of rigid motion as we will see after our study of rotations; therefore they are also the properties of congruent figures.



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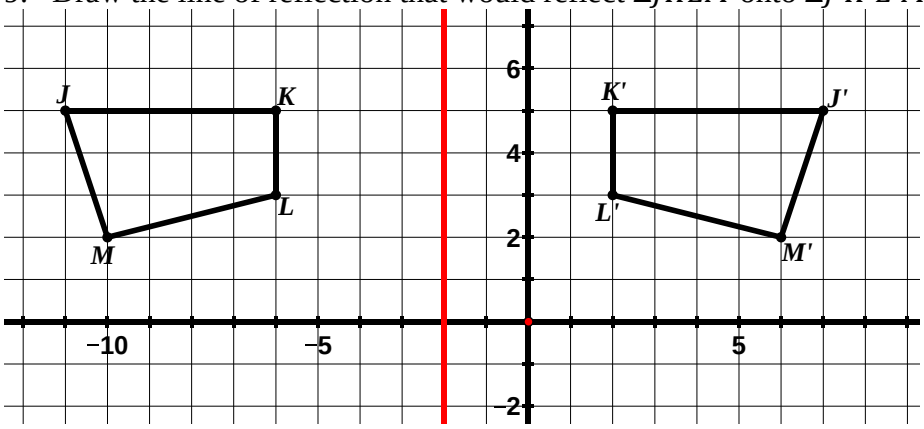
**Directions:** For #9 – 11, draw the line of reflection that would reflect one figure onto the other. Then, write the equation for the line of reflection and the coordinate rule that describes the reflection.

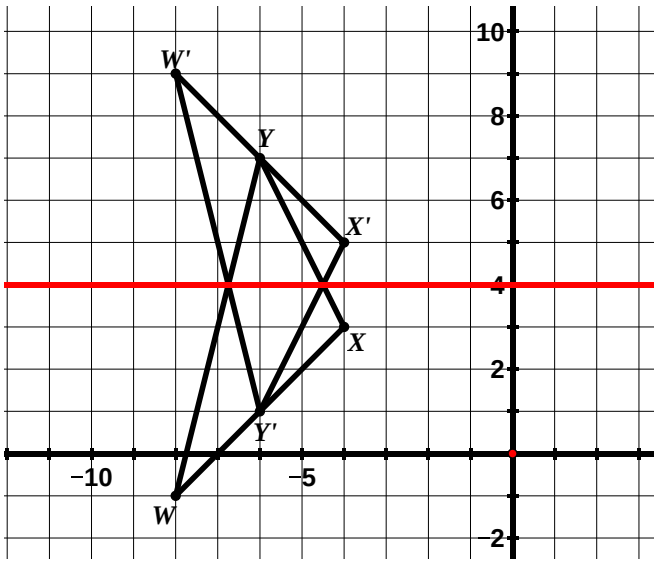
9. Draw the line of reflection that would reflect $\Delta JKLM$ onto $\Delta J'K'L'M'$.




a. Write the equation for the line of reflection.
$x = -2$

b. Write a coordinate rule for the reflection. $(x, y) \rightarrow ((-x - 4), y)$ *Encourage students to write the coordinates on the grid or put them into a table.

10. Draw the line of reflection that would reflect $\Delta WXY$ onto $\Delta W'X'Y'$.




a. Write the equation for the line of reflection.
$y = 4$

b. Write a coordinate rule for the reflection.
$(x, y) \rightarrow (x, (-y + 8))$

**A connection to think about:** We can make a great connection to reflections and the work we did in Chapter 7 with the number line. In 6th grade students developed an understanding of negative numbers as the opposite (reflection) of positive numbers. The reflection is one-dimensional with the point of reflection at 0. Orientation changes--for positive numbers 0 is on the left and for negative numbers it's on the right.



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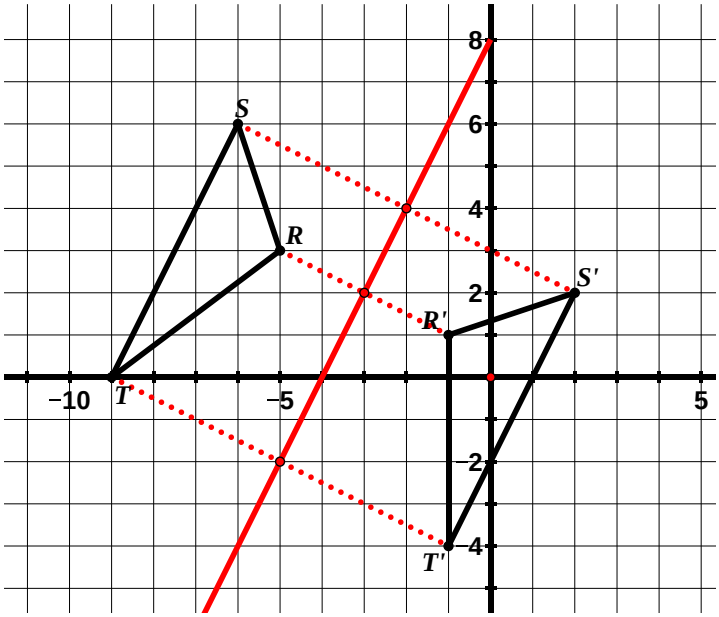
11. Draw the line of reflection that would reflect $\Delta RST$ onto $\Delta R'S'T'$.




a. Write the equation for the line of reflection.

$y = 2x + 8$

Students may use a variety of strategies to solve this problem, including folding their paper to match the images, guess and check, or they may utilize the properties as shown above. Have students share strategies. If they use folding methods or guess and check, make sure they verify that the properties hold true for the line of reflection determined using these informal methods.



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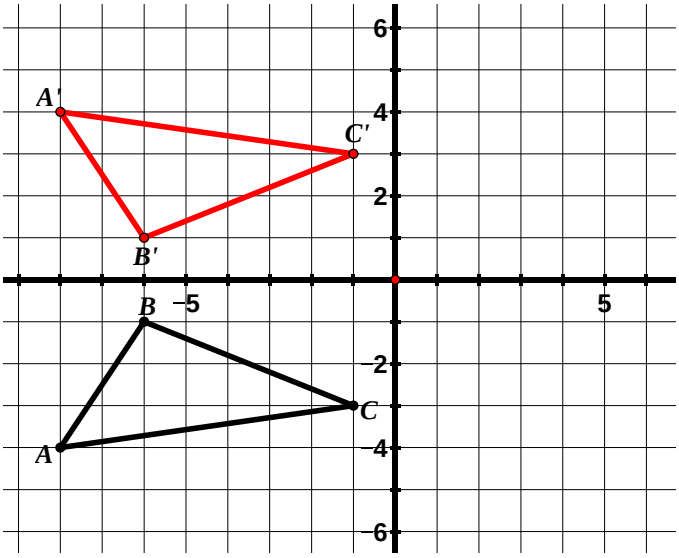
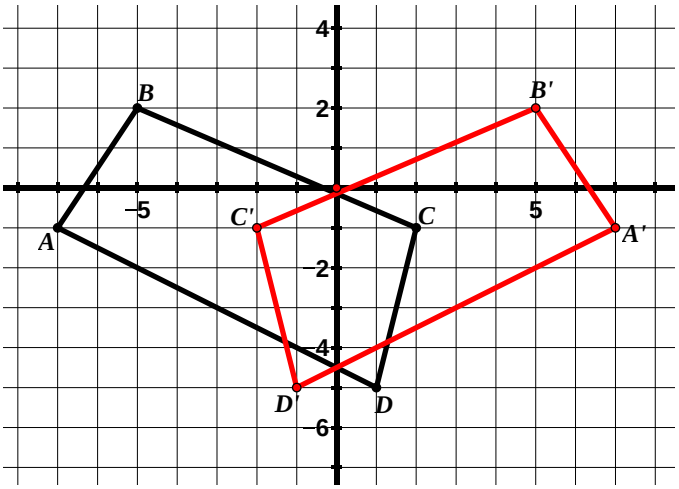
## 9.1b Homework: Properties of Reflections

1. Reflect $\Delta ABC$ across the $x$-axis and label the image.



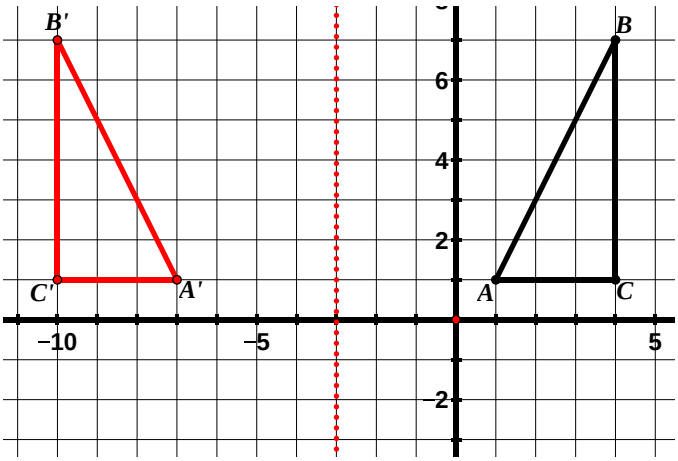

a. Write a coordinate rule to represent this transformation.
$(x, y) \rightarrow (x, -y)$

2. Reflect $ABCD$ across the $y$-axis and label the image.



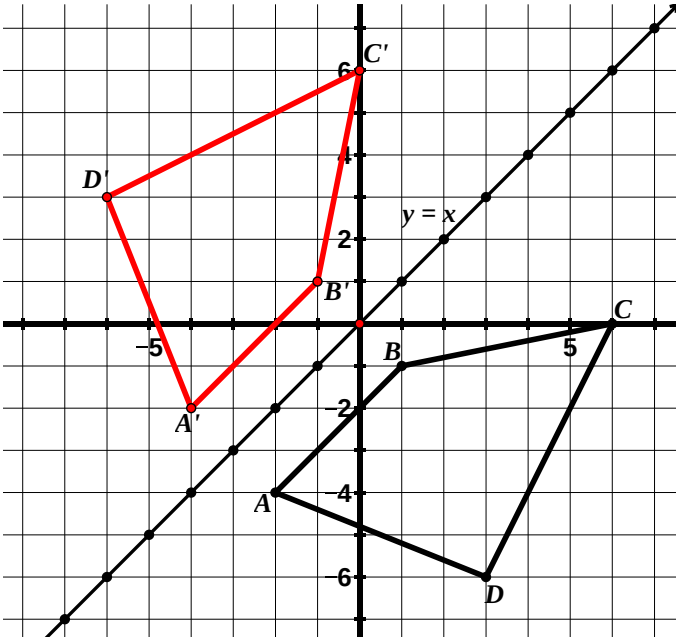

a. Write a coordinate rule to represent this transformation.
$(x, y) \rightarrow (-x, y)$

3. Reflect $\Delta ABC$ across the line $x = -3$ and label the image.




a. Write a coordinate rule to represent this transformation.
$(x, y) \rightarrow ((-x - 6), y)$

4. Reflect $ABCD$ across the line $y = x$.




a. Write a coordinate rule to represent this transformation.
$(x, y) \rightarrow (y, x)$



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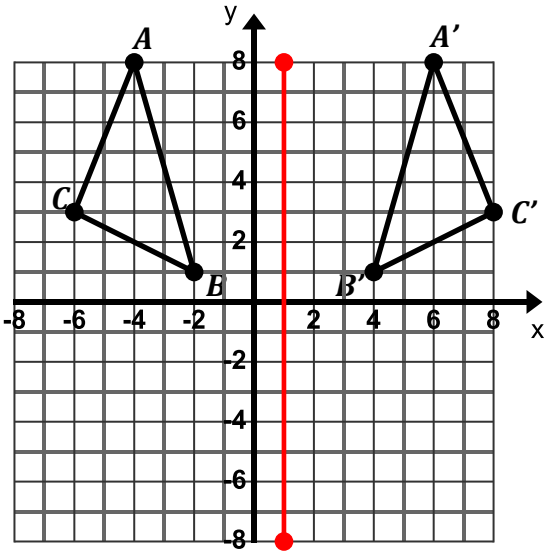
5. For each of the following:

- Draw the line of reflection that would reflect the pre-image onto the image.
- Find the equation for the line of reflection.
- Write a coordinate rule to describe the reflection.

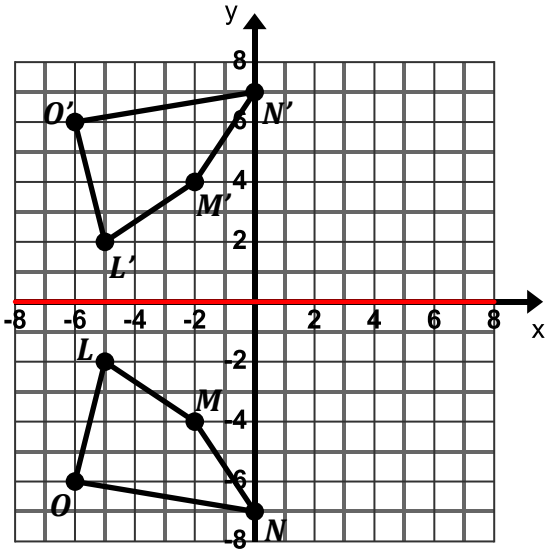
a. Equation of line of reflection: $x = 1$

Coordinate Rule: $(x, y) \rightarrow ((-x + 2), y)$



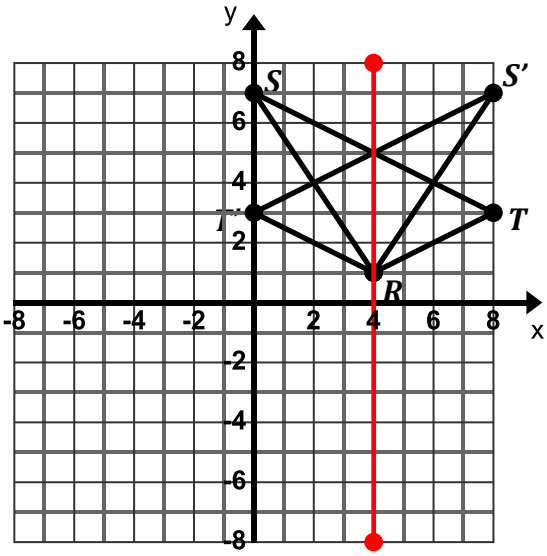

b. Equation of line of reflection: $y = 0$

Coordinate Rule: $(x, y) \rightarrow (x, -y)$



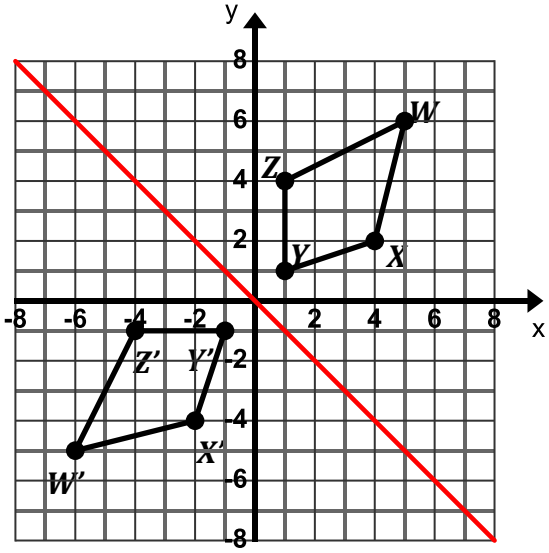

c. Equation of line of reflection: $x = 4$

Coordinate Rule: $(x, y) \rightarrow ((-x + 8), y)$




d. Equation of line of reflection: $y = -x$

Coordinate Rule: $(x, y) \rightarrow (-y, -x)$



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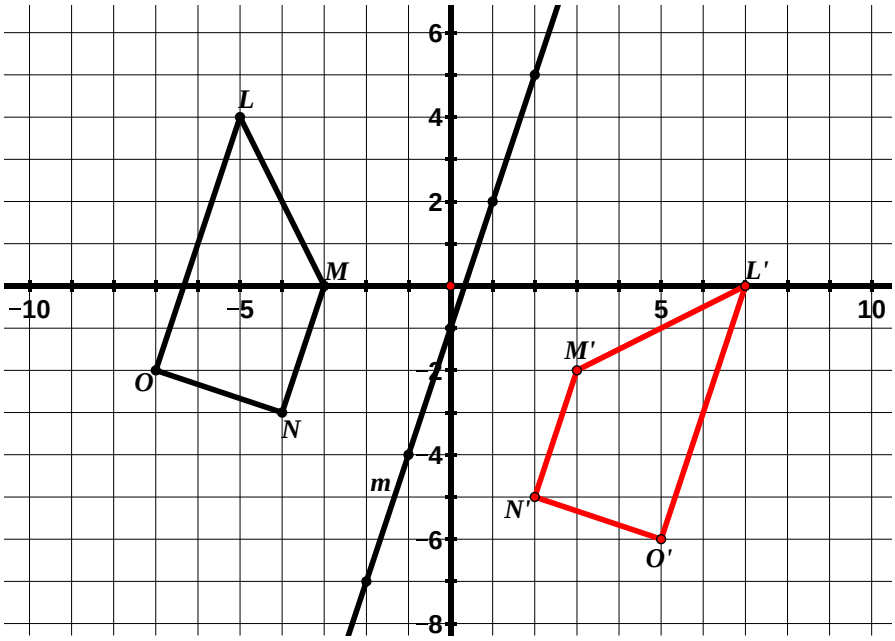
6. Given *LMNO*.




a. What is the equation of the line of reflection? $y = 3x - 1$

b. Based on the slope of the line of reflection, determine what the slope of the segments connecting corresponding points of the image and pre-image should be. $-\frac{1}{3}$

c. Reflect *LMNO* over the line *m* and label the image.

d. What is the relationship between $\overline{LO}$ and $\overline{MN}$ before the transformation? What is the relationship between these two segments after the transformation? Use numerical evidence to support your answer.
These lines are parallel before the transformation and after the transformation. The slope of these lines before and after the transformation is 3.



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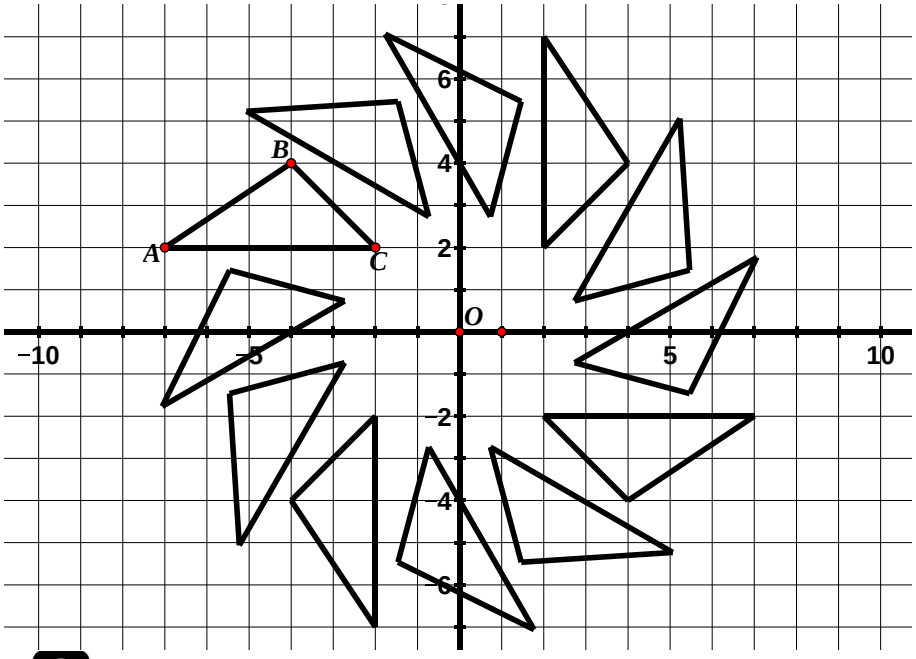
**9.1c Class Activity: Properties of Rotations**

1. In the grid below, $\Delta ABC$ has been **rotated** counterclockwise with the **center of rotation** at the origin $O$. This process was repeated several times to create the images shown.




a. Using tracing paper, trace $\Delta ABC$ and the $x$-axis. Holding your pencil as an anchor on the origin, rotate the triangle counterclockwise to see how the images were created.

b. Label the corresponding vertices of the images of $\Delta ABC$.

c. Describe the relationship between ***C* and its images to the center of rotation *O***. Do the same for A and its images. Does this relationship to the center of rotation hold true for B and its images?

C and its images are equidistant from the center of rotation. This is also true for A and its images and B and its images. Another way to say this is that corresponding vertices lie on the same circle whose center is the center of rotation (in this case $O$). We also see that the circles are concentric – meaning that they have the same center. Illustrate these ideas on the diagram and allow students to use a compass to draw the concentric circles. Having them draw the circles reinforces the point that a rotation is a rigid motion that leaves one point in the plane fixed.

d. If there are $360^0$ in one full rotation, determine the angle of rotation from one image to the next in the picture above. $30^0$



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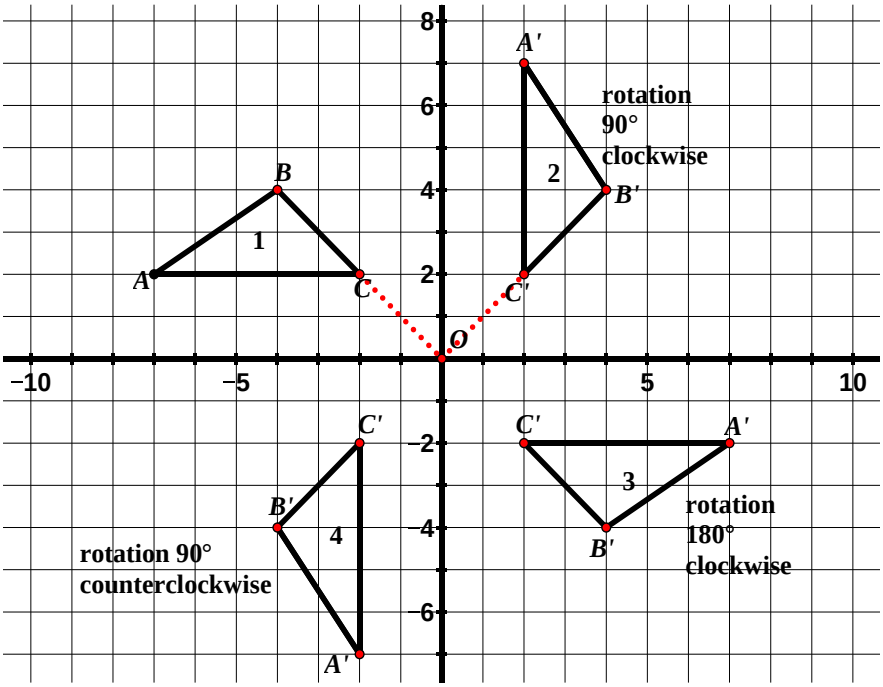
2. The picture from the previous page was modified so that only the images that are increments of $90^0$ rotations of the pre-image $\Delta ABC$ are shown. The center of rotation is the origin $O$.




a. Verify using tracing paper that the descriptions of the rotations are accurate.

It may help students to position their patty paper so the sides are vertical and horizontal and not tilted. Trace figure 1. Students should be able to articulate how to use patty paper to perform rotations that are increments of $90^0$. Students can also observe the angle from the origin (in this case $\angle COC'$ is $90^0$). Students can verify this informally using the corner of an index card as a $90^0$ angle.

b. The rotation from figure 1 to figure 4 has been described as a rotation $90^0$ counterclockwise. How would you describe this rotation in the clockwise direction? $270^0$ clockwise

c. Consider the rotation from Figure 1 to Figure 2, a rotation $90^0$clockwise. Find the slopes of the following segments:

$\overline{AB} = \frac{2}{3}$ $\quad$ $\overline{BC} = -1$ $\quad$ $\overline{AC} = 0$

$\overline{A'B'} = -\frac{3}{2}$ $\quad$ $\overline{B'C'} = 1$ $\quad$ $\overline{A'C'} =$ und.

d. Use the slopes from the previous question to determine the relationship between corresponding segments in a $90^0$ rotation.

The lines are perpendicular. We can see from part c. that the slopes are opposite reciprocals. Have students compare what happens to the slopes of the segments under a reflection and a rotation. A common mistake is for students to think these transformations have the same effect on the slopes of segments. Discuss the differences.

e. Which segments would you expect to be perpendicular in the rotation from Figure 1 to Figure 4, the rotation $90^0$ counterclockwise? Use slope to support your answer.

$\overline{AB}$ and $\overline{A'B'}$ ; $\overline{BC}$ and $\overline{B'C'}$ ; $\overline{AC}$ and $\overline{A'C'}$

f. Determine the coordinate rule for a $90^0$ rotation clockwise about the origin. Connect this rule to the slopes of perpendicular lines. n#

$(x, y) \rightarrow (y, -x)$

The rise and run have been interchanged. Also, the sign of the run has changed which would change the sign of the slope. This is easiest to see comparing the slopes of $\overline{AB}$ and $\overline{A'B'}$.

g. Determine the coordinate rule for a $90^0$ rotation counterclockwise about the origin. Connect this rule to the slopes of perpendicular lines.

$(x, y) \rightarrow (-y, x)$ The rise and run have been interchanged. Also, the sign of the rise has changed which would change the sign of the slope.

h. Describe what happens in a $180^0$ rotation of a figure. What is the relationship of the corresponding segments?

The corresponding segments are parallel.

i. Determine the coordinate rule for a rotation of $180^0$. Connect this rule to your answer for part h.

$(x, y) \rightarrow (-x, -y)$

The rise and run both change their sign (not their value) but this will not change the resulting slope because taking the opposite of both of them is the same as multiplying by positive one.



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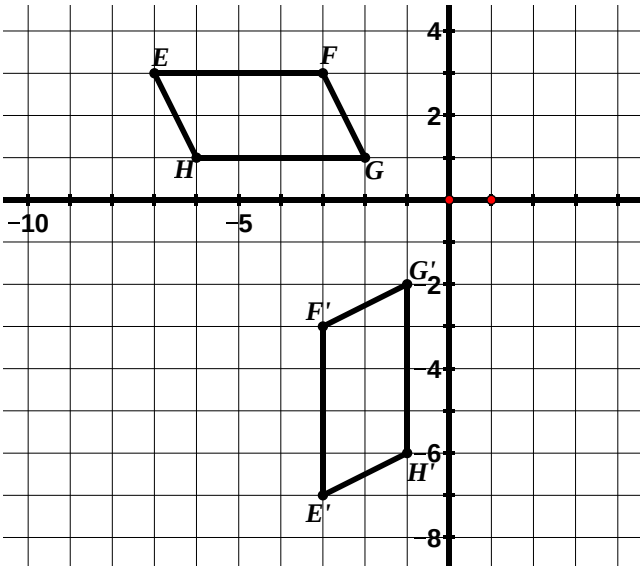
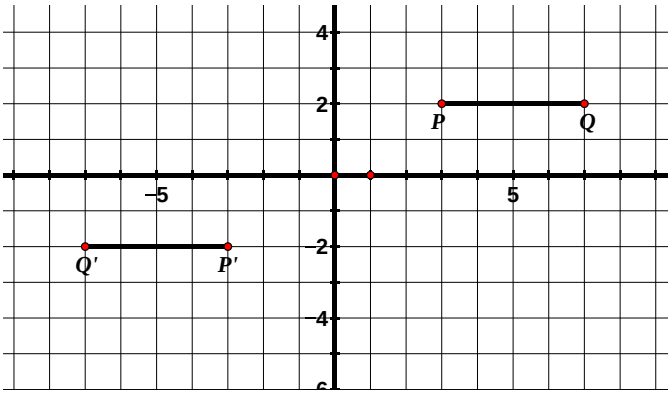
3. For the following rotation, the center of rotation is the origin.




a. Determine the angle of rotation. Be sure to also indicate a direction of rotation.
$90^0$ rotation counterclockwise

b. If the slope of $\overline{EH}$ is $-2$, determine the slope of $\overline{E'H'}$ without doing any calculations. $\frac{1}{2}$

4. For the following rotation, the center of rotation is the origin.




a. Determine the angle of rotation. Be sure to also indicate a direction of rotation.
$180^0$ rotation counterclockwise or clockwise (the direction does not matter in the case of a $180^0$ rotation)



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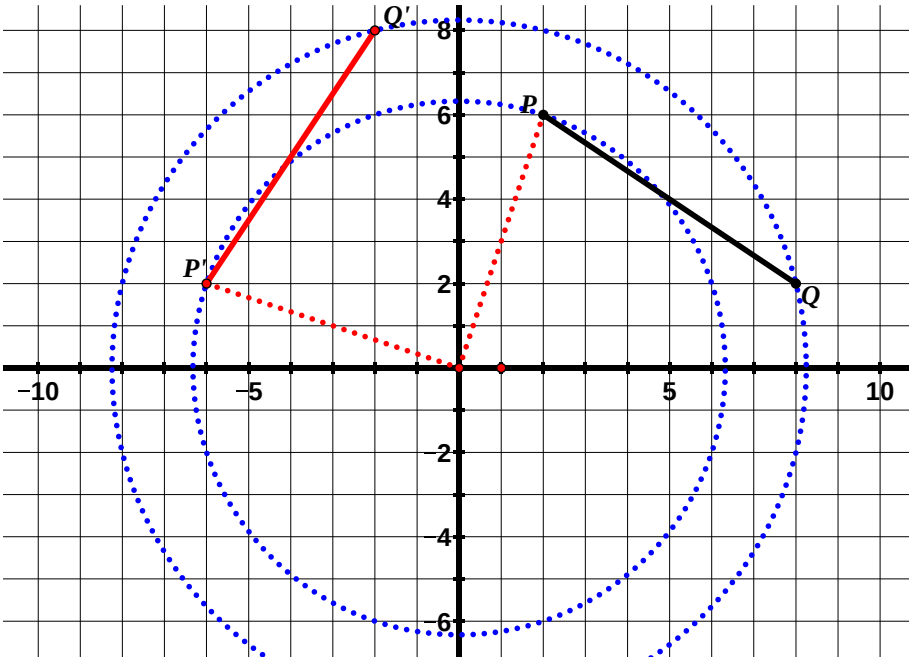
5. Rotate $\overline{PQ}$ $90^0$ counterclockwise with the center of rotation at the origin and label the image.




a. How can you verify using slope that your image is in fact a $90^0$ rotation?

The slopes should be opposite (negative) reciprocals of each other. The slope of $\overline{PQ}$ is $-\frac{2}{3}$ and the slope of $\overline{P'Q'}$ is $\frac{3}{2}$.

b. How can you verify using distance that the center of rotation is the origin?

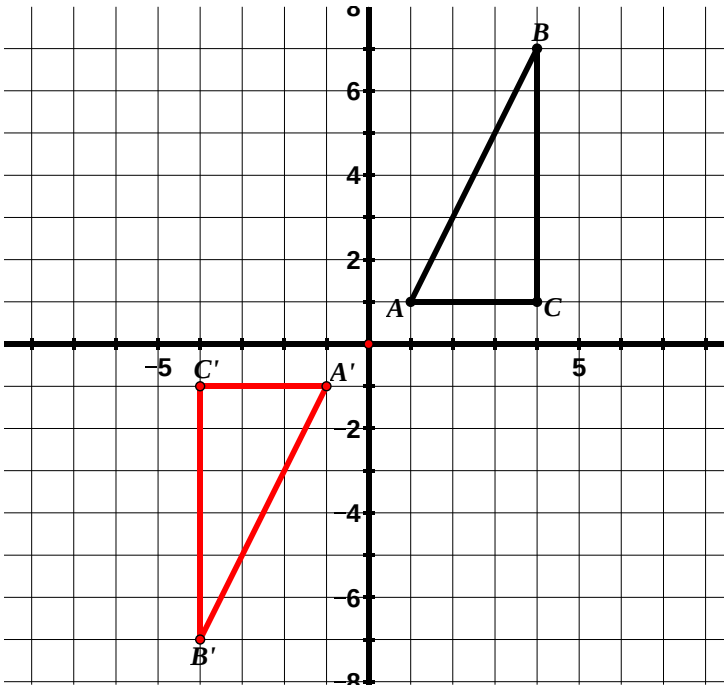
Corresponding vertices should be equidistant from the center of rotation (origin in this case). Verify on diagram. Show students the concentric circles that can be drawn.

c. Use a compass to draw the concentric circles of this rotation.

d. What do the concentric circles prove?

Points that lie on the same circle are equidistant from the center of the circle; therefore corresponding points are equidistant from the center of rotation.

6. Rotate $\Delta ABC$ $180^0$ counterclockwise with the center of rotation at the origin and label the image.




You may want to take time to discuss the properties of rotations with students. Which properties do rotations have in common with translations, reflections, or both? What are properties unique to rotations? Note: A common error is that people think that orientation changes under a rotation because the slopes of the segments change. Again, think of orientation as starting at one vertex and moving around in a clockwise fashion. Will the vertices appear in the same order in the pre-image and image? The answer is yes **so the orientation of a figure does <u>not</u> change under a rotation.**



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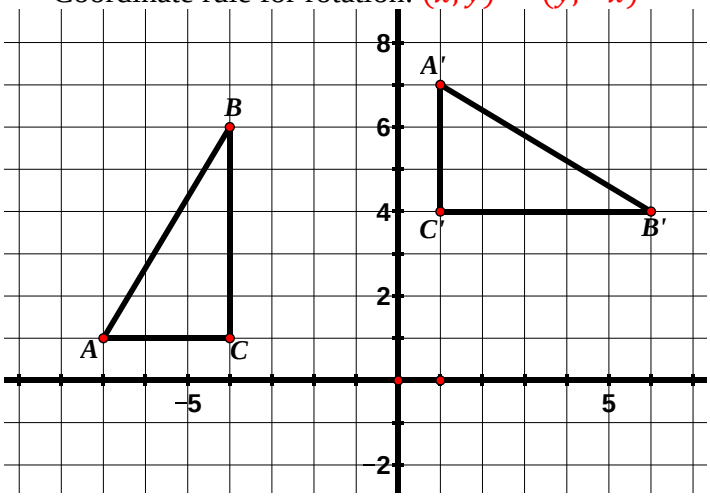
**9.1c Homework: Properties of Rotations**

**Directions:** For each of the following rotations, the center of rotation is the origin. Determine the angle of rotation (be sure to also indicate a direction of rotation). Write a coordinate rule for the transformation.

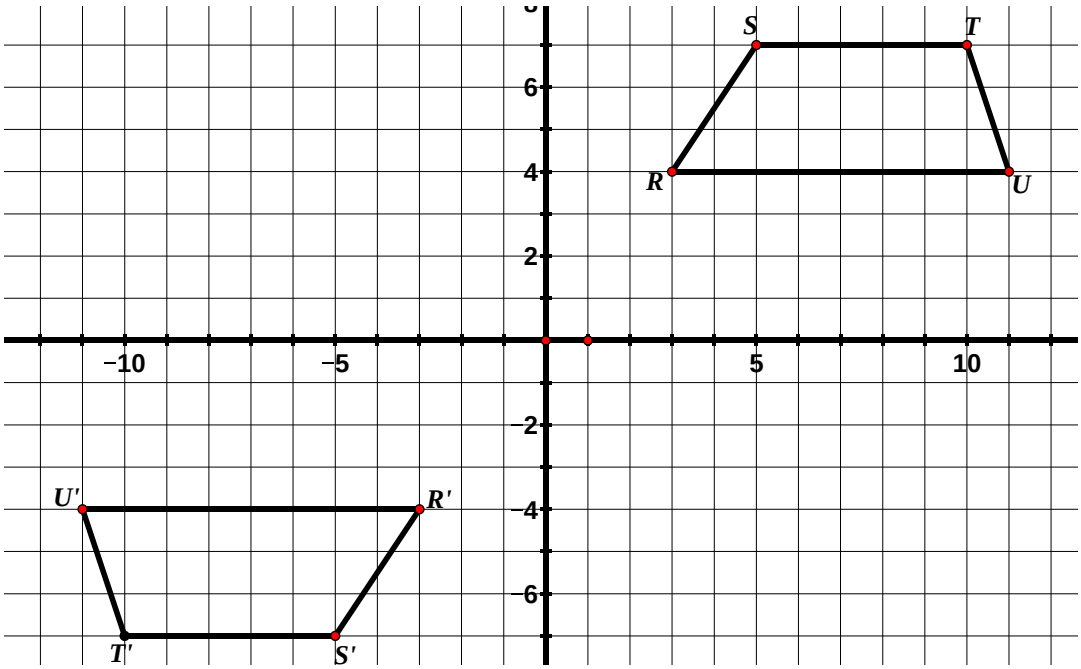
1. Angle of Rotation (including direction of rotation): $90^0$ clockwise (or $270^0$ counterclockwise)

   Coordinate rule for rotation: $(x, y) \rightarrow (y, -x)$




2. Angle of Rotation (including direction of rotation): $180^0$ clockwise

   Coordinate rule for rotation: $(x, y) \rightarrow (-x, -y)$



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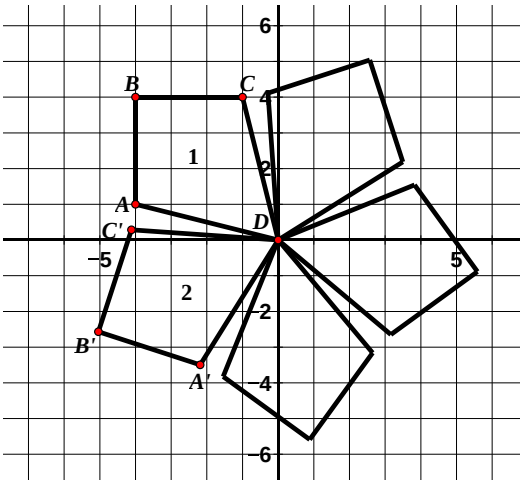
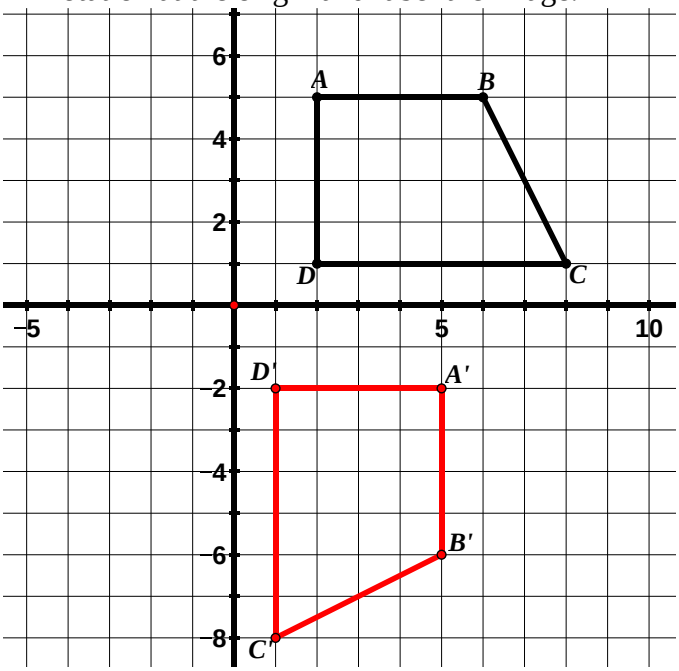
3. Find the angle of rotation from Figure 1 to Figure 2. Be sure to include a direction.

   $72^0$ counterclockwise




4. Rotate *ABCD* $90^0$ clockwise with the center of rotation at the origin and label the image.




   a. How can you verify using slope that your image is in fact a $90^0$ rotation? The slopes of corresponding segments should be opposite reciprocals.

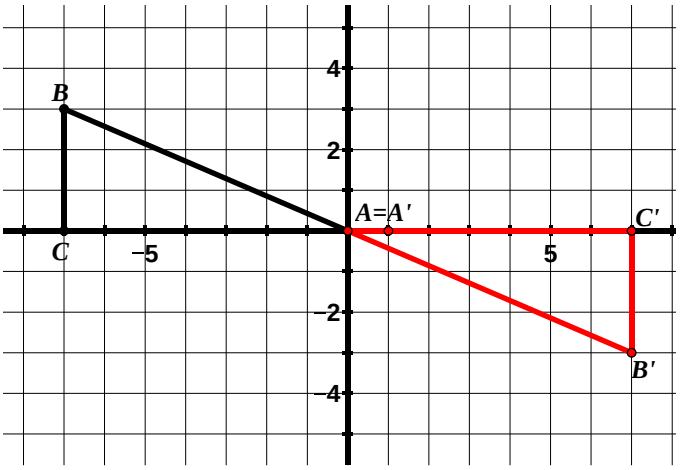
   b. How can you verify using distance that the center of rotation is the origin?

   Corresponding vertices lie on the same circle. They are equidistant from the center of rotation.

   c. Write a coordinate rule for the rotation.

   $(x, y) \rightarrow (y, -x)$

5. Rotate $\Delta ABC$ $180^0$ clockwise with the center of rotation at the origin.



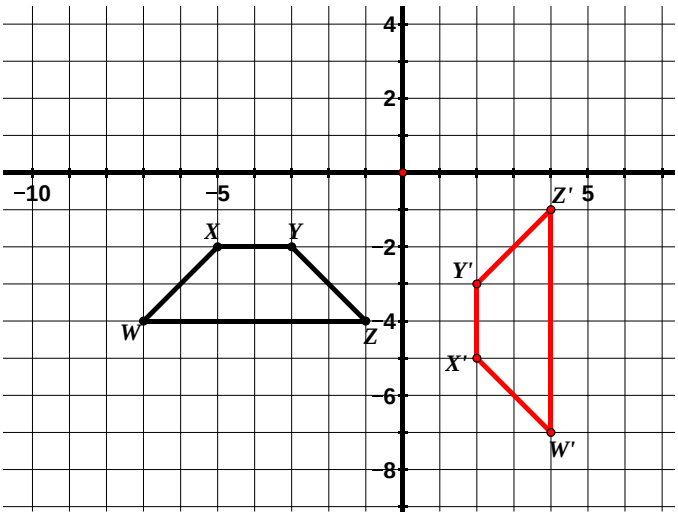

   a. Write a coordinate rule for the rotation.

   $(x, y) \rightarrow (-x, -y)$

   b. Compare the slopes of the segments of the pre-image to the image.

   The slopes are the same.

6. Rotate $WXYZ$ $90^0$counterclockwise with the center of rotation at the origin.




You can use the examples from the homework to reinforce concepts as needed about the properties of rotations studied so far.

   a. Compare the slopes of the segments of the pre-image to the image.

   The slopes are opposite reciprocals.



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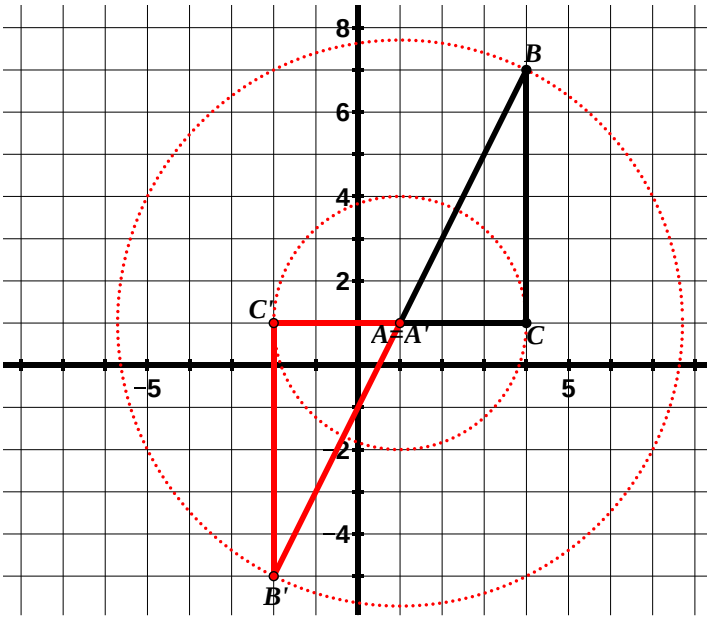
**9.1d Class Activity: Properties of Rotations cont.**

In our prior work with rotations, the center of rotation was always at the origin. Today, we will look at rotations where the center may not be at the origin.

1. **Quiet Write**: In the space below, write everything you have learned about rotations so far.

Gauge where students are at in their understanding. The following ideas may be expressed: In a rotation, corresponding points are equidistant from the center of rotation. Corresponding points lie on the same circle. A rotation of 90˚results in corresponding segments that are perpendicular. Corresponding segments are congruent. Corresponding angles are congruent. Parallel lines remain parallel. Have students share out their ideas and refer to previous problems as necessary to help reinforce concepts.

2. Rotate $\Delta ABC$ $180^0$ counterclockwise with the center of rotation at (1, 1) and label the image.




a. How can you verify that your center of rotation is at (1, 1)?

Using a compass, students can create circles with the center at (1, 1) and show that the corresponding points lie on the same circle (are equidistant from the center of rotation (1, 1)). Similarly, students can use the compass to measure the lengths of corresponding segments to show that they are equidistant from (1, 1). They may also use ideas about distance that have surfaced. They should easily see that $\overline{AC}$ and $\overline{A'C'}$ are the same length as are $\overline{BC}$ and $\overline{B'C'}$. Using slope triangles, they may also conclude that $\overline{AB}$ and $\overline{A'B'}$ are the same length.

If students use a compass here, allow them access to a compass for the remaining rotation problems.
**This lesson is important to show that rotations do not only occur around the origin.

This is a lesson that connects nicely to the Pythagorean Theorem and the equation of a circle that students will see in Secondary II.



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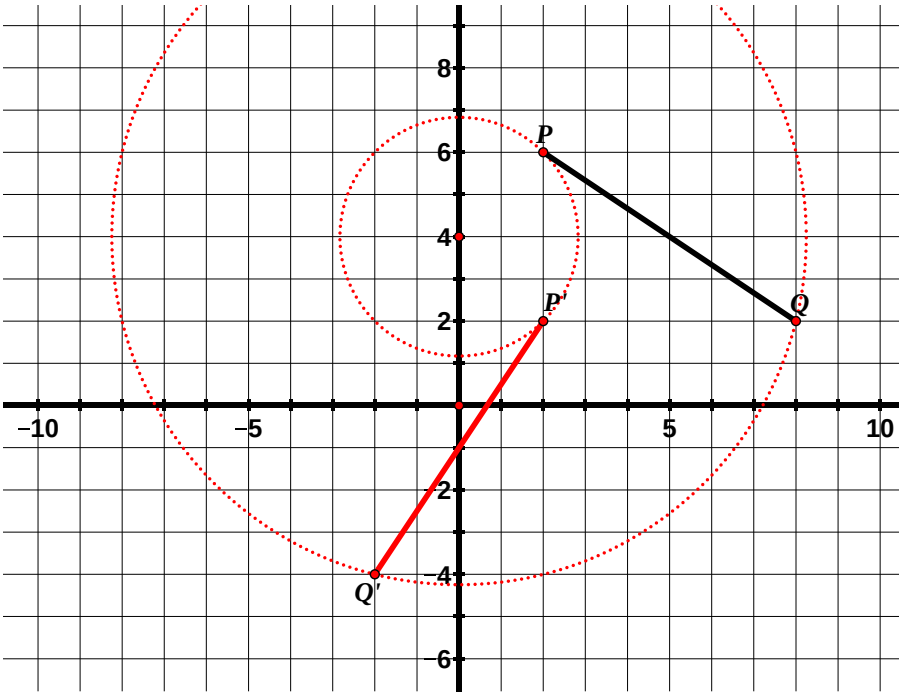
3. Rotate $\overline{PQ}$ $90^0$clockwise with the center of rotation at (0, 4).



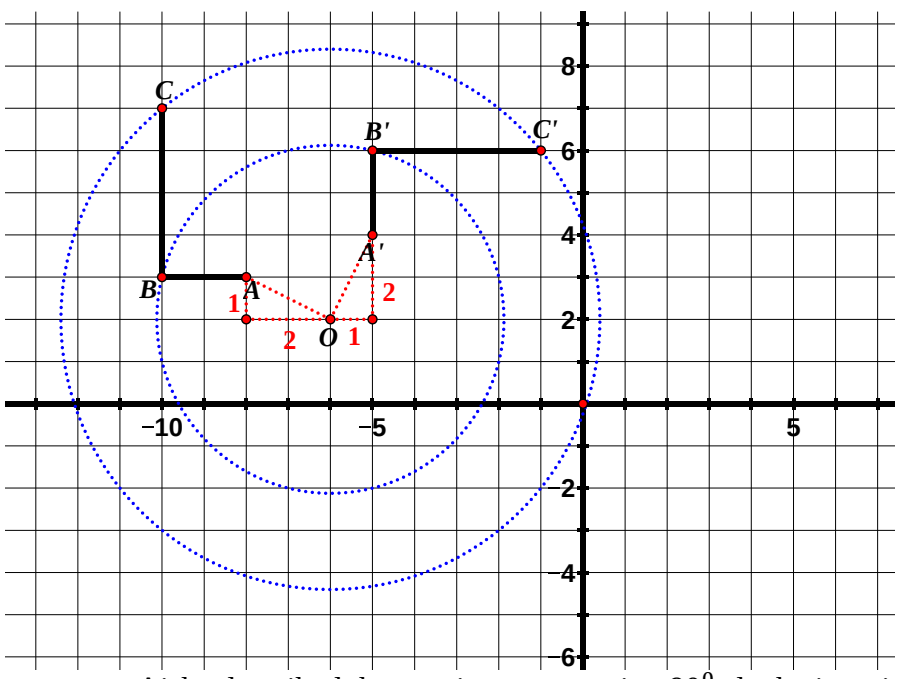

a. How can you verify using slope that your image is in fact a $90^0$ rotation?

Slopes should be opposite reciprocals (or the product of the slopes should be 1). The slope of $\overline{PQ}$ is $-\frac{2}{3}$ and the slope of $\overline{P'Q'}$ is $\frac{3}{2}$.

b. How can you verify using distance that the center of rotation is at (0, 4)?

See 2a.

4. A teacher asked her students to determine the center of rotation and angle of rotation for the rotation shown below.




a. Aisha described the rotation as a rotation $90^0$ clockwise with the center at $O\ (-6, 2)$. Do you agree with Aisha? Use the properties of rotations and numerical evidence to support your answer.

Aisha is correct – see the diagram above for a few different strategies students might use to prove this using numerical evidence.

For an extra challenge, you may ask students to write a coordinate rule for this rotation. The coordinate rule for a 90˚ about the origin is $(x, y) \rightarrow (y, -x)$. If we first apply this rule to the coordinates of the pre-image, we see that the coordinate rule for the rotation picture above is $(x, y) \rightarrow ((y - 8), (-x - 4))$



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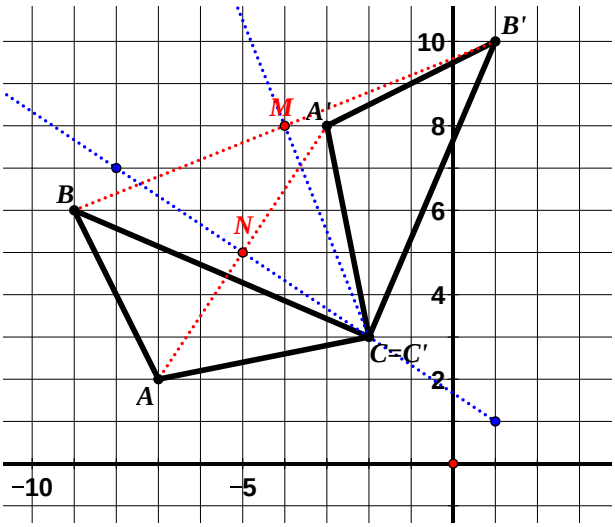
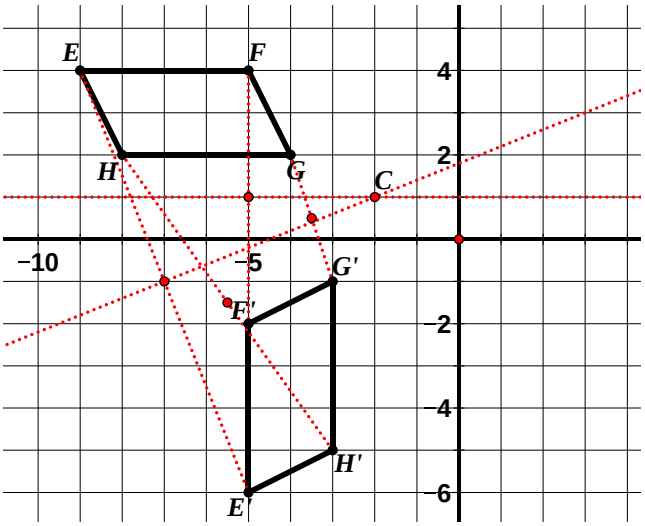
**Directions:** For #5 – 7, find the angle of rotation (including the direction) and the center of rotation.

5. Angle of Rotation (including direction of rotation): 90˚ clockwise (270˚ counterclockwise)
   Center of Rotation: $C: (-2, 3)$



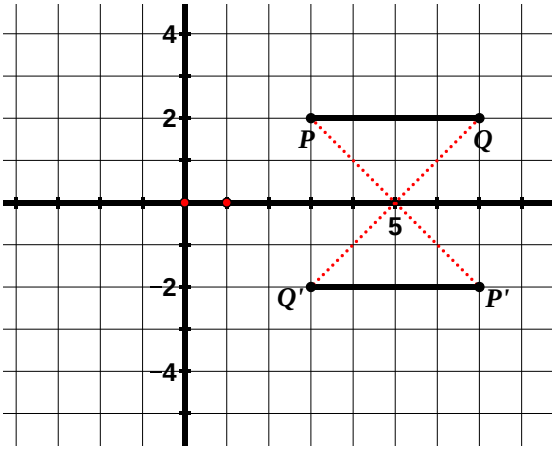

6. Angle of Rotation (including direction of rotation): 90˚ counterclockwise (270˚ clockwise)

   Center of Rotation: $C: (-2, 1)$




7. Angle of Rotation (including direction of rotation): 180˚ counterclockwise (or clockwise)

   Center of Rotation: $(5, 0)$




Students may find the center of rotation using guess and check (using logic to approximate the region where the center of rotation is located). Students can use ideas about slope to solve this problem. Alternatively, you may discuss with students that the center of the circle is the intersection point of the perpendicular bisectors of the segments joining corresponding vertices. This concept is beyond the scope of 8th grade; however students should be developing an intuitive sense of this fact. They will build on these ideas in Secondary I and II. You may want to explain what a perpendicular bisector is and then encourage students to find the midpoint of each segment connecting corresponding vertices. Then students can use coordinate geometry to graph the line perpendicular to the segment connecting corresponding vertices that passes through the midpoint of the segment. Students will study constructions in Secondary I but you may choose to show honors' classes a few simple constructions (in this case, how to find the perpendicular bisector of a segment).

Alternatively, students can fold their papers so that corresponding vertices match up (for example in #5 fold so that *A* matches up with *A'*, *B* with *B'*, and *C* with *C'*). The lines of the folds will intersect at the center of rotation. This is because every rotation can be written as the succession of two reflections.



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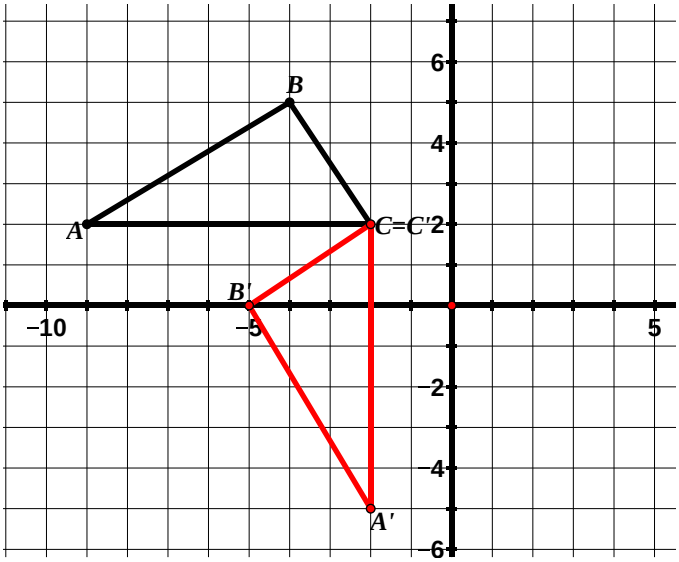
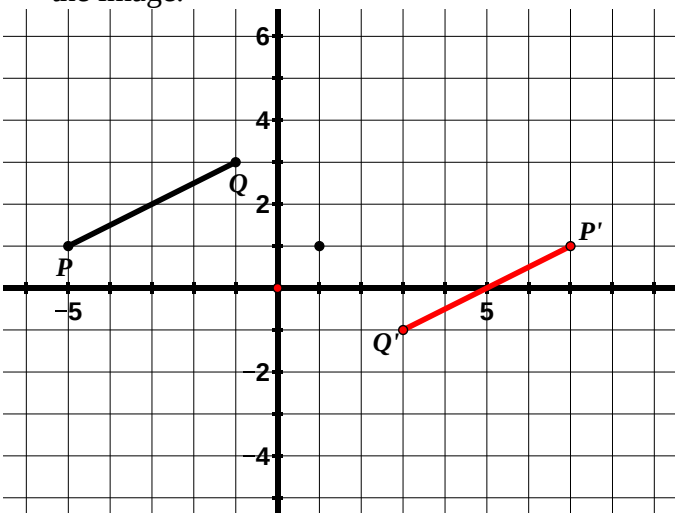
## 9.1d Homework: Properties of Rotations cont.

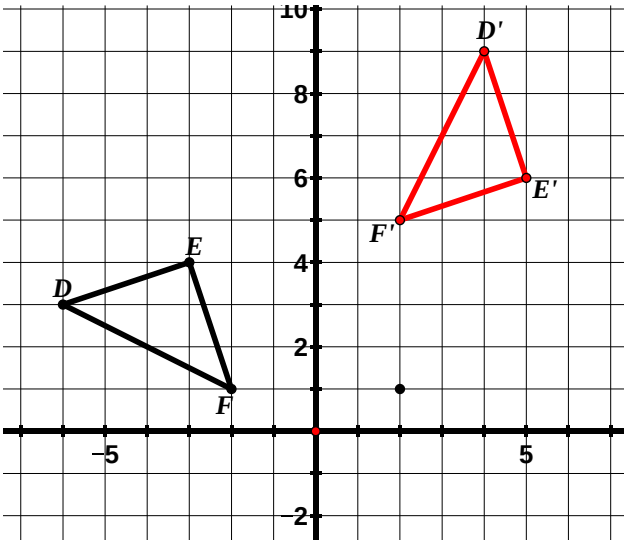
1. Rotate $\Delta ABC$ $90^0$ counterclockwise about $C$ and label the image.




2. Rotate $PQ$ $180^0$ clockwise about (1, 1) and label the image.

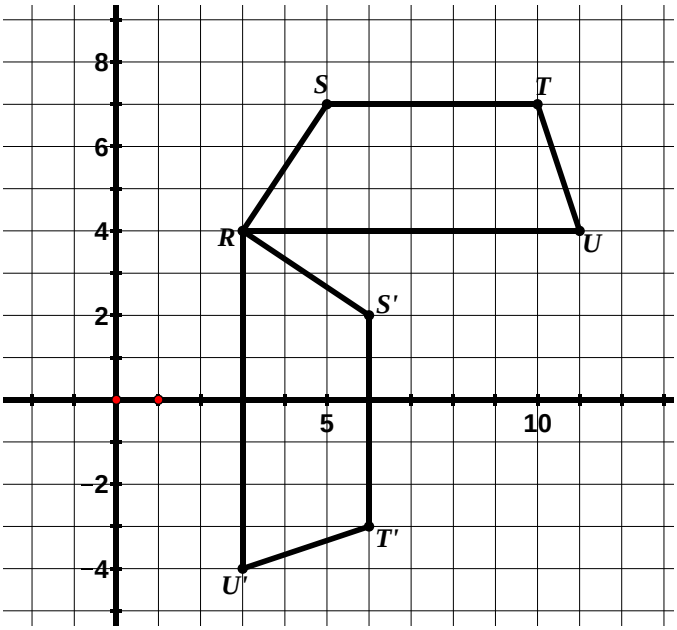



3. Rotate $\Delta DEF$ $90^0$ clockwise about (2, 1) and label the image.




**Directions:** For #4 – 6, find the angle of rotation (including the direction) and the center of rotation.

4. Angle of Rotation (including direction of rotation): 90˚ clockwise (270˚ counterclockwise)
   Center of Rotation: $R$: (3, 4)



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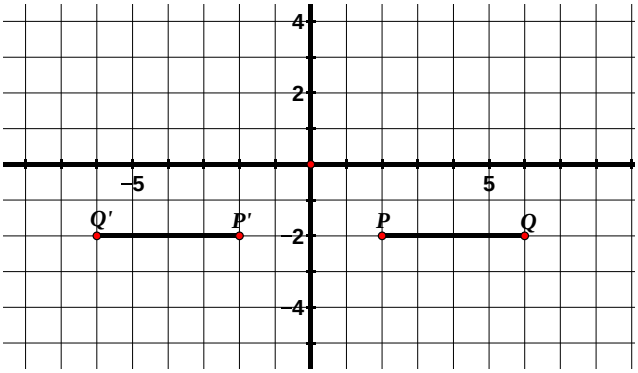
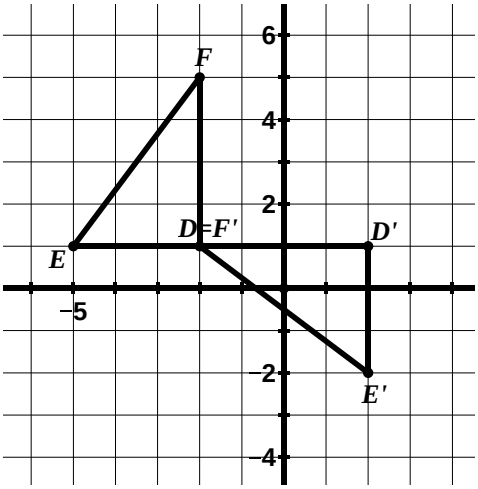
5. Angle of Rotation (including direction of rotation): 180˚ clockwise (or counterclockwise)

   Center of Rotation: $(0, -2)$



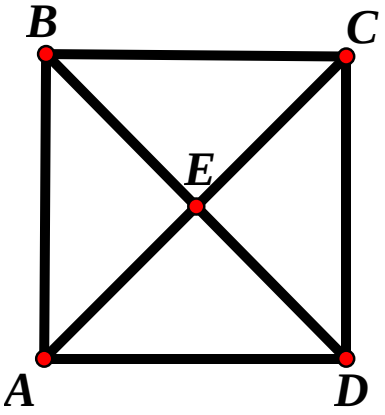

6. Angle of Rotation (including direction of rotation): 90˚ counterclockwise (270˚ clockwise)
   Center of Rotation: _____$(0, 3)$_____




7. *ABCD* is a square.
   a. What is the image of *B* under a $90^0$ rotation counterclockwise about *C*? *D*

   b. What is the image of *B* under a $180^0$ rotation about E? *D*

   c. Name three different rotations for which the image of *A* is *C*.
   $180^0$ rotation about E
   $270^0$ rotation clockwise about *B*
   $90^0$ rotation clockwise about *D*
   *There are others*




Possible honors' extension: Talk about the set of all rigid motions that carry the figure onto itself.



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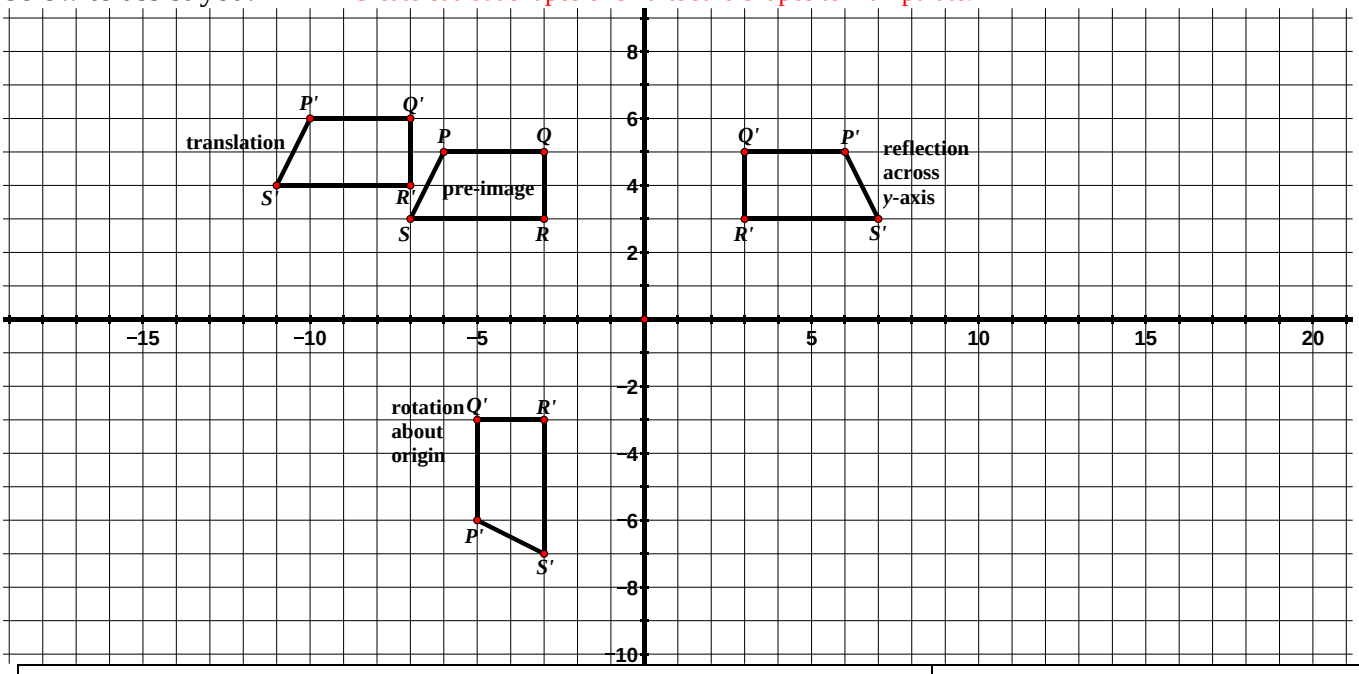
## 9.1e Class Activity: Congruence

The following phrases and words are properties or descriptions of one or more of the transformations we have studied so far: translation, reflection, and rotation. Determine which type of transformation(s) the statements describe and write your answer(s) on the line. An example of each type of transformation has been provided below to assist you. Create cut-out shapes or smartboard shapes to manipulate.




| Property/Description | Type of Transformation(s) |
|---|---|
| Flip | reflection |
| Slide | translation |
| Turn | rotation |
| Image has the same orientation as pre-image. | Translation, rotation |
| Specified by a figure, a center of rotation, and an angle of rotation. | rotation |
| Specified by a figure and a line of reflection. | reflection |
| Specified by a figure, a distance, and a direction. | translation |
| Segments connecting corresponding vertices of image and pre-image are the same length. | translation |
| Corresponding image and pre-image vertices lie on the same circle. | rotation |
| Segments connecting corresponding vertices of image and pre-image are parallel to each other. | translation and reflection |
| Line of reflection is the perpendicular bisector of all segments connecting corresponding vertices of the image and pre-image. | reflection |
| Concentric circles | rotation |
| Orientation of the figure does not change. | translation and rotation |
| The slopes of corresponding segments may change. | reflection, rotation |
| Corresponding segments in the image and pre-image are the same length. | translation, reflection, rotation |
| Corresponding angles in the image and pre-image have the same measure. | translation, reflection, rotation |
| Parallel lines in the pre-image remain parallel lines in the image. | translation, reflection, rotation |



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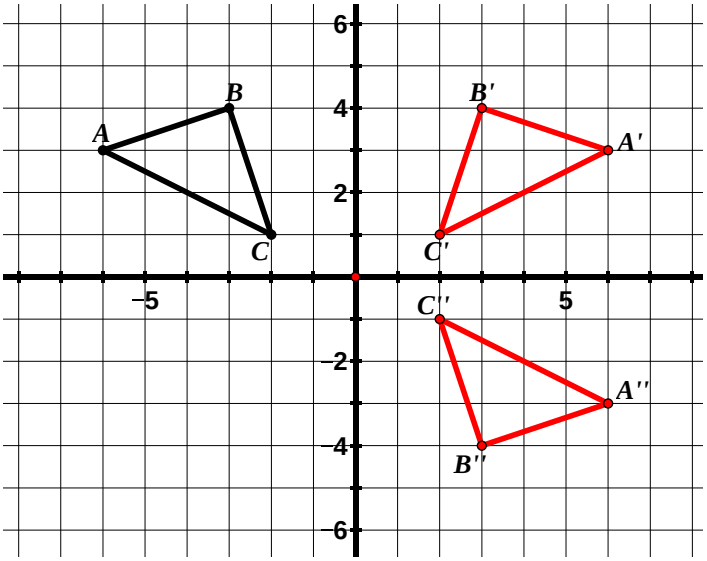
In the examples we have studied so far, we have only performed one transformation on a figure. We can also perform more than one transformation on a figure. In the following problems, you will perform a **sequence of transformations** on a figure.

1. $\Delta ABC$ has been plotted below.



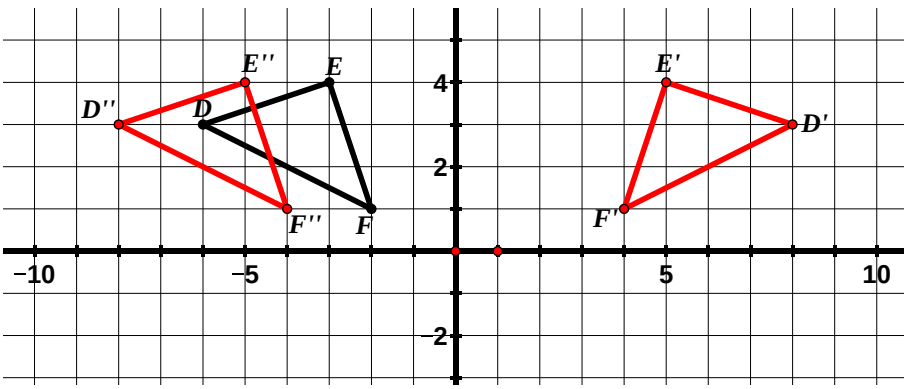

a. Reflect $\Delta ABC$ over the $y$-axis and label the image $\Delta A'B'C'$.

b. Reflect $\Delta A'B'C'$ over the $x$-axis and label the image $\Delta A''B''C''$.

c. What **one** transformation is the same as this double reflection?

a 180˚ rotation clockwise (or counterclockwise) about the origin

2. $\Delta DEF$ has been plotted below.



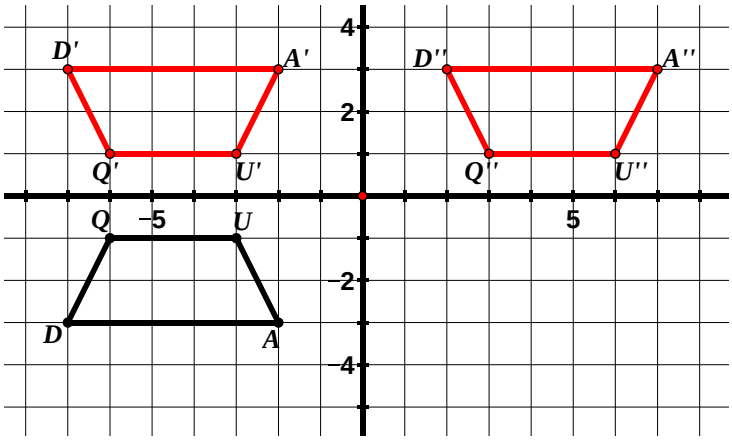

a. Reflect $\Delta DEF$ over the line $x = 1$ and label the image $\Delta D'E'F'$.

b. Reflect $\Delta D'E'F'$ over the $y$-axis and label the image $\Delta D''E''F''$.

c. What **one** transformation is the same as this double reflection?
A translation 2 units to the left

d. Write a coordinate rule for the transformation of $\Delta DEF$ to $\Delta D''E''F''$.
$(x, y) \rightarrow (x - 2, y)$

3. $QUAD$ has been plotted below.




a. Reflect $QUAD$ over the $x$-axis and label the image $Q'U'A'D'$.

b. Translate $Q'U'A'D'$ according to the rule $(x, y) \rightarrow (x + 9, y)$ label the image $Q''U''A''D''$.



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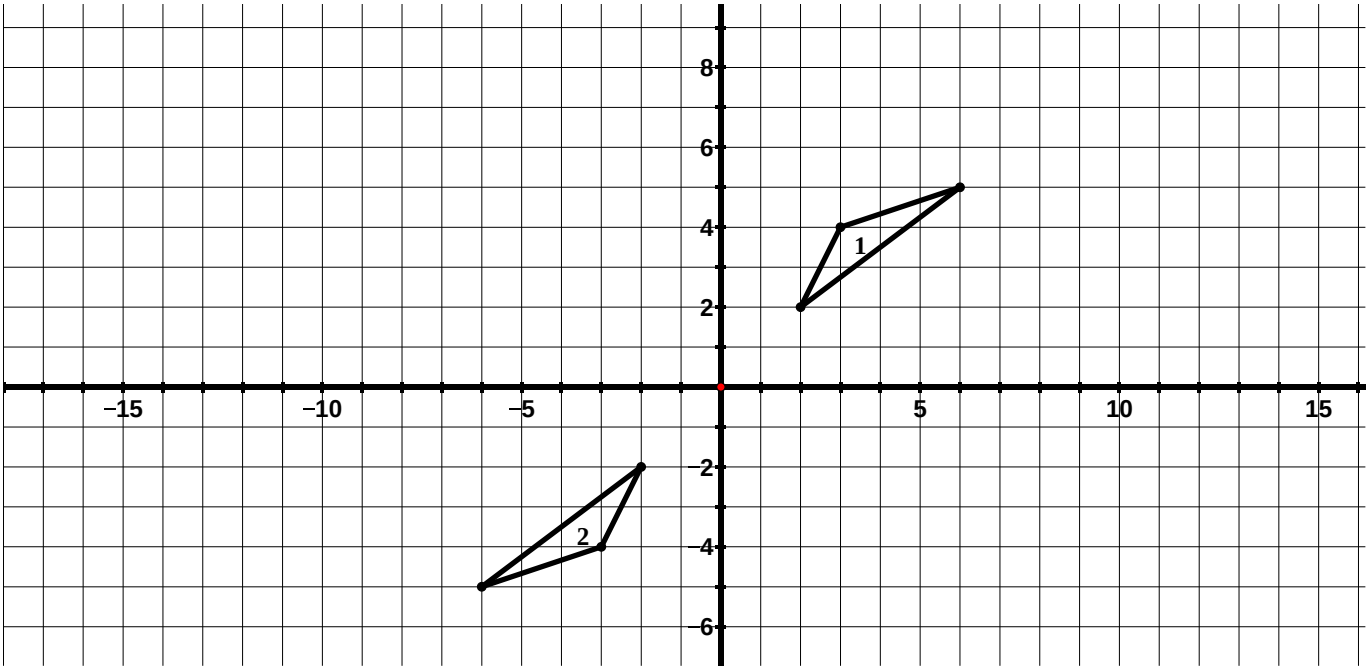
Throughout this lesson and the one that follows, students will see that there are multiple ways to describe the rigid motion or sequence of rigid motions that carry one object to another. Students will have the opportunity to consider the different approaches taken by others, compare the approaches, and identify correspondences between the different approaches.

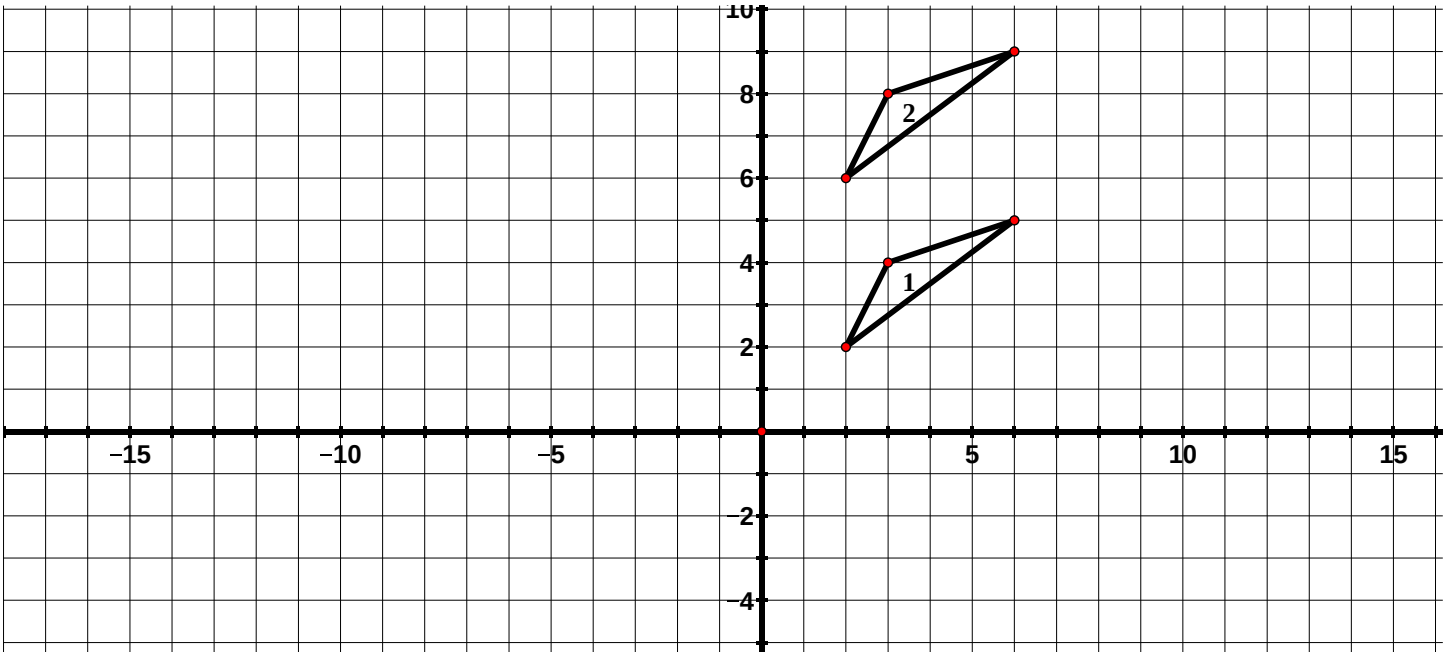
4. In the picture below, triangle 1 has been transformed to obtain triangle 2.




a. Which **two** transformations in succession would carry triangle 1 onto triangle 2? Multiple answers. One possible answer: a reflection over the *y*-axis and then a reflection over the *x*-axis.

b. Which **one** transformation would carry triangle 1 onto triangle 2? 180˚ rotation about the origin

5. In the picture below, triangle 1 has been transformed to obtain triangle 2.




a. Which **two** transformations in succession would carry triangle 1 onto triangle 2? One possible answer: a translation down 1 unit followed by a translation up 5 units. Another possible answer is a reflection in the line $y = 4$ followed by a reflection in the line $y = 6$

b. Which **one** transformation would carry triangle 1 onto triangle 2? A translation up 4 units.



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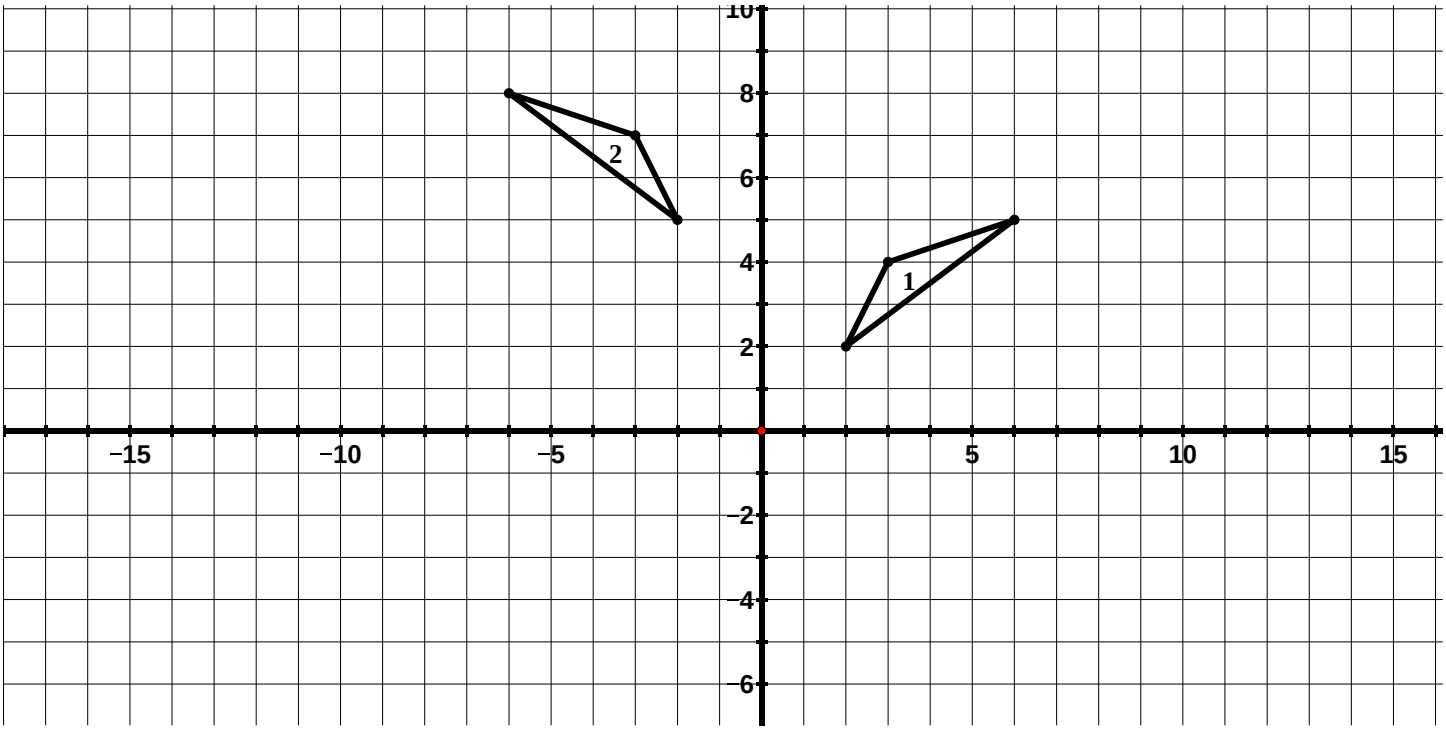
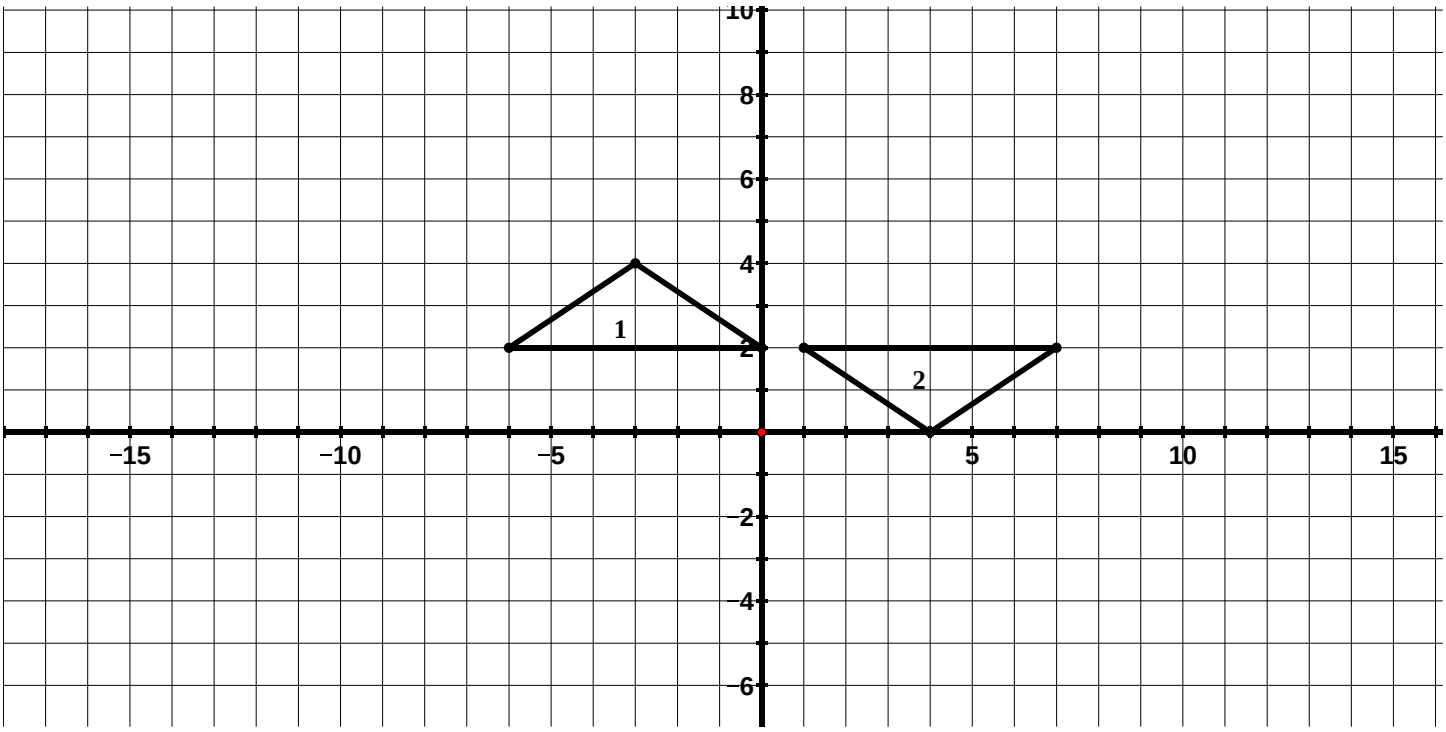
6. Describe a transformation or sequence of transformations that would carry triangle 1 onto triangle 2.
One possible answer: A translation up 3 units followed by a reflection across the $x$-axis. Again, if students are struggling to write coordinate rules, you can have them practice here. The coordinate rule that maps triangle 1 to triangle 2 is $(x, y) \rightarrow (-x, y + 3)$.




7. Describe a transformation or sequence of transformations that would carry triangle 1 onto triangle 2.
One possible answer: a reflection across the line $y = 2$, followed by a translation to the right 7 units



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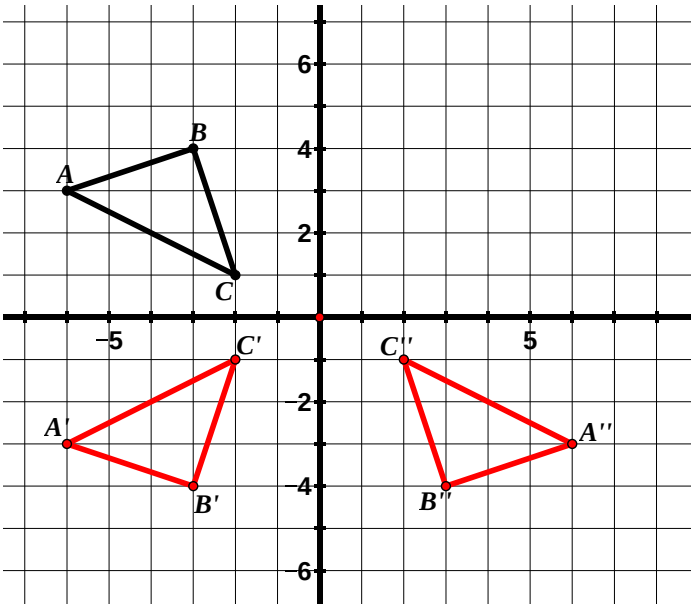
## 9.1e Homework: Congruence

1. $\Delta ABC$ has been plotted below.



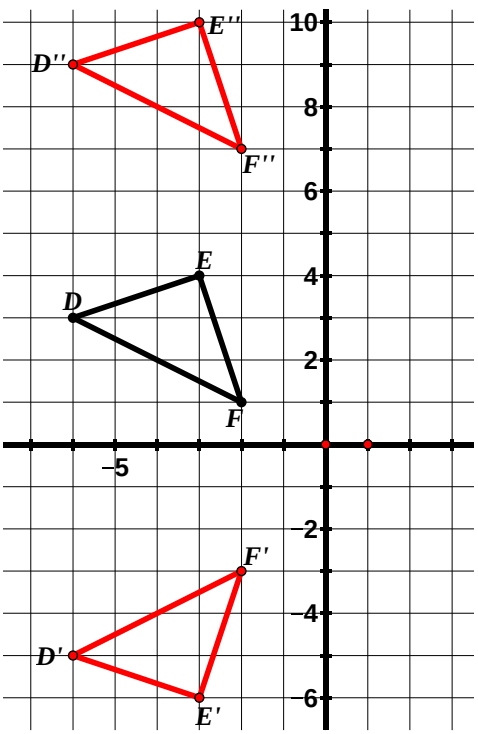

a. Reflect $\Delta ABC$ over the $x$-axis and label the image $A'B'C'$.
b. Reflect $\Delta A'B'C'$ over the $y$-axis and label the image $A''B''C''$.
c. What **one** transformation is the same as this double reflection?

a 180˚ rotation clockwise (or counterclockwise) about the origin

d. In #1 of your class work, we performed a similar series of transformation; however we reflected over the $y$-axis first and then the $x$-axis. Compare these transformations.

Changing the order of these reflections results in the same location of the image – both are a 180˚ rotation clockwise (counterclockwise) about the origin.

2. $\Delta DEF$ has been plotted below.




a. Reflect $\Delta DEF$ over the line $y = -1$ and label the image $\Delta D'E'F'$.
b. Reflect $\Delta D'E'F'$ over the line $y = 2$ and label the image $\Delta D''E''F''$.
c. What **one** transformation is the same as this double reflection?

A translation 6 units up

d. Write a coordinate rule for the transformation of $\Delta DEF$ to $\Delta D''E''F''$.

$(x, y) \rightarrow (x - 2, y)$

e. In #2 of your class work, we performed a similar series of transformation; however we reflected over vertical lines. Compare these transformations and write your observations below.

A double reflection over horizontal lines (as shown above) results in a vertical translation. A double reflection over vertical lines (as shown in #2 of the classwork) results in a horizontal translation.



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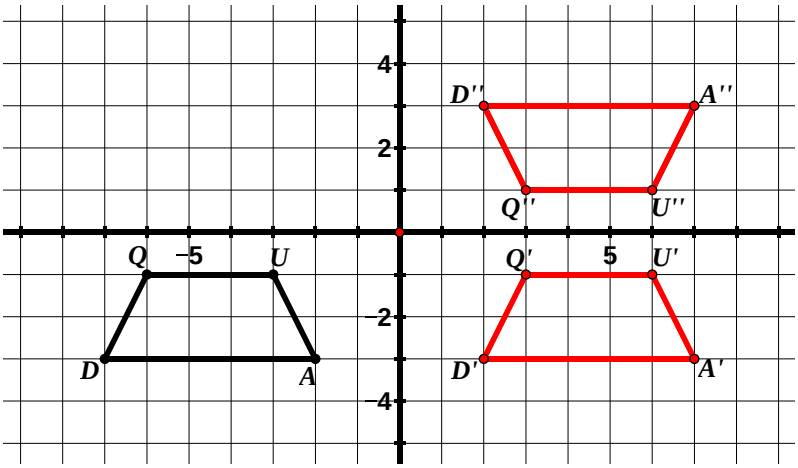
3. $QUAD$ has been plotted below.



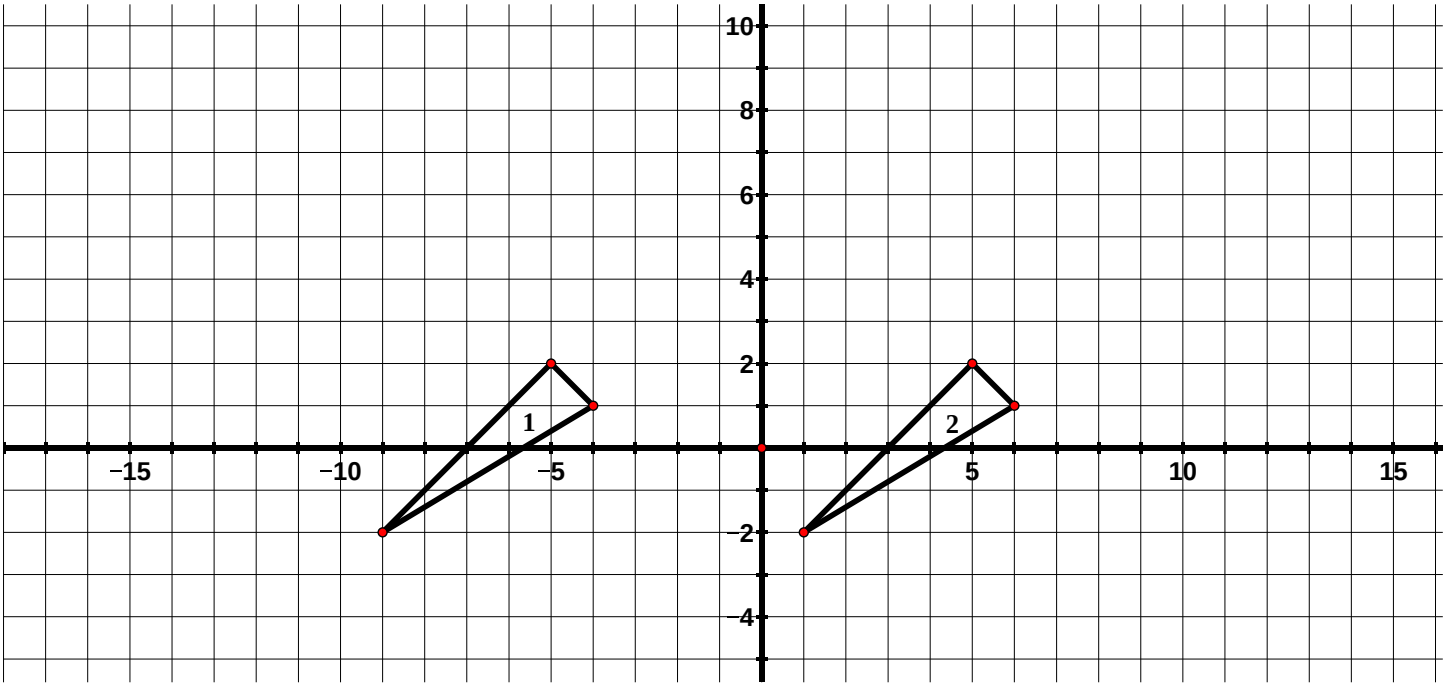

a. Translate $QUAD$ according to the rule $(x, y) \rightarrow (x + 9, y)$.

b. Reflect $Q'U'A'D'$ over the $x$-axis and label the image $Q''U''A''D''$.

c. In #3 of your class work, we performed a similar series of transformation; however we did the reflection first and then the translation. Compare these transformations and write your observations below.

Changing the sequence of these transformations results in the same image.

4. In the picture below, triangle 1 has been transformed to obtain triangle 2.




a. Which **two** transformations in succession would carry triangle 1 onto triangle 2?

One possible answer: a reflection over the line $x = -2$ followed by a reflection over the line $x = 3$

b. Which **one** transformation would carry triangle 1 onto triangle 2?

a translation 10 units to the right



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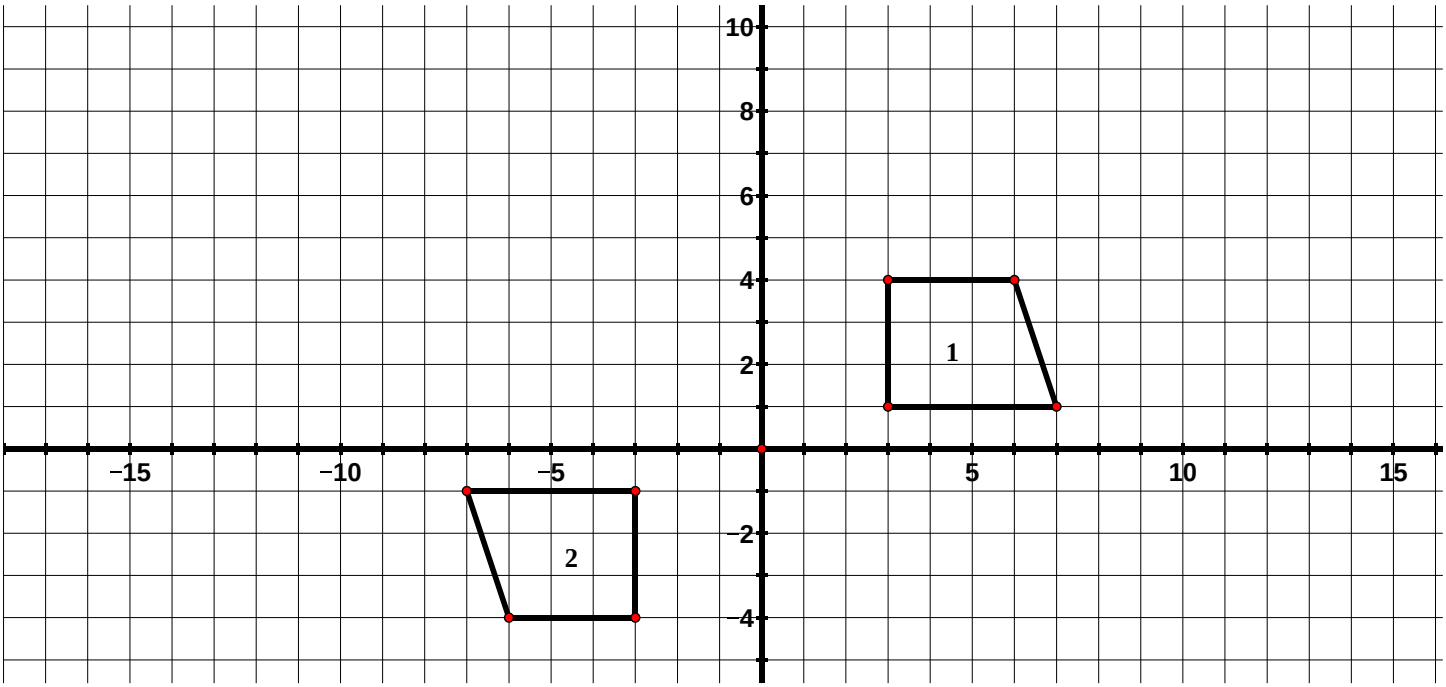
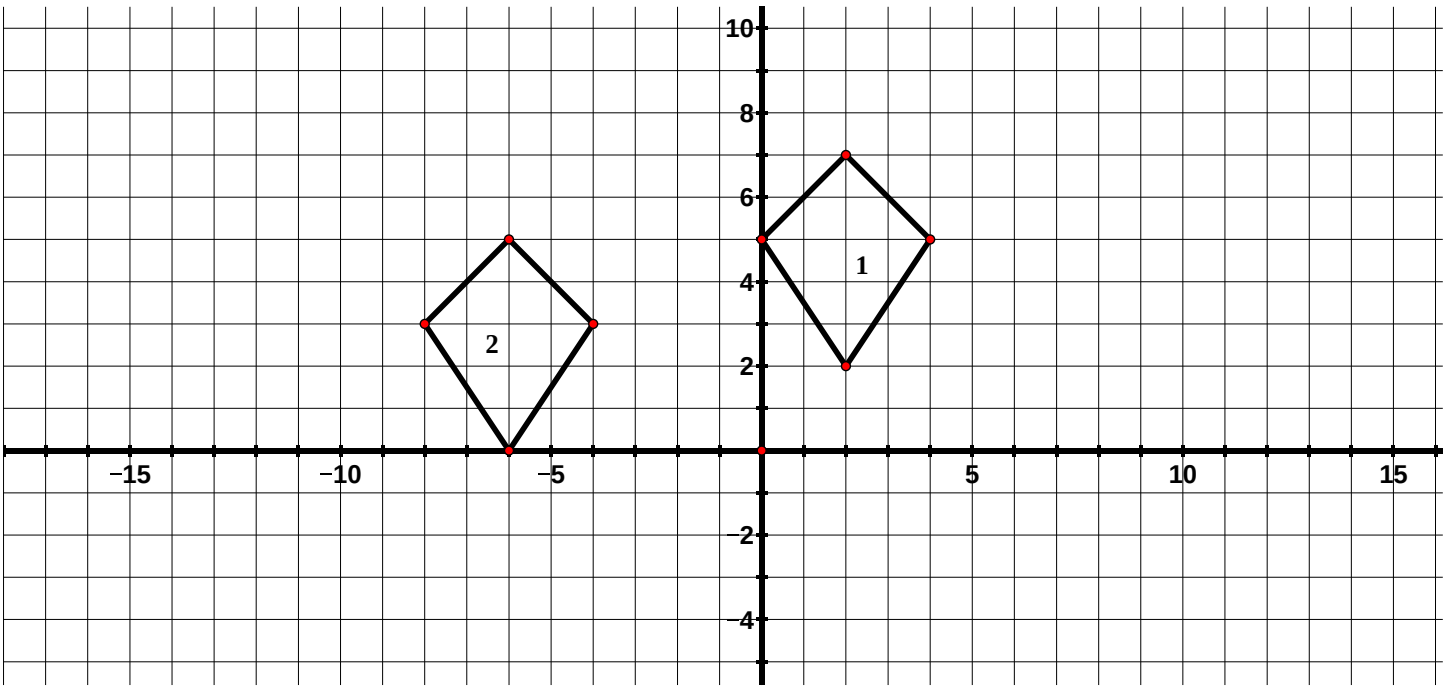
5. In the picture below, trapezoid 1 has been transformed to obtain trapezoid 2.




a. Which **two** transformations in succession would carry trapezoid 1 onto trapezoid 2?
Possible answer – a reflection over the *x*-axis followed by a reflection over the *y*-axis

b. Which **one** transformation would carry trapezoid 1 onto trapezoid 2?
a 180˚ rotation about the origin

6. Describe a transformation or sequence of transformations that would carry figure 1 onto figure 2.
Answers may vary. Students may describe the transformation in words (a translation 2 units down and 8 units to the left) or using a coordinate rule:
$(x, y) \rightarrow (x - 8, y - 2)$



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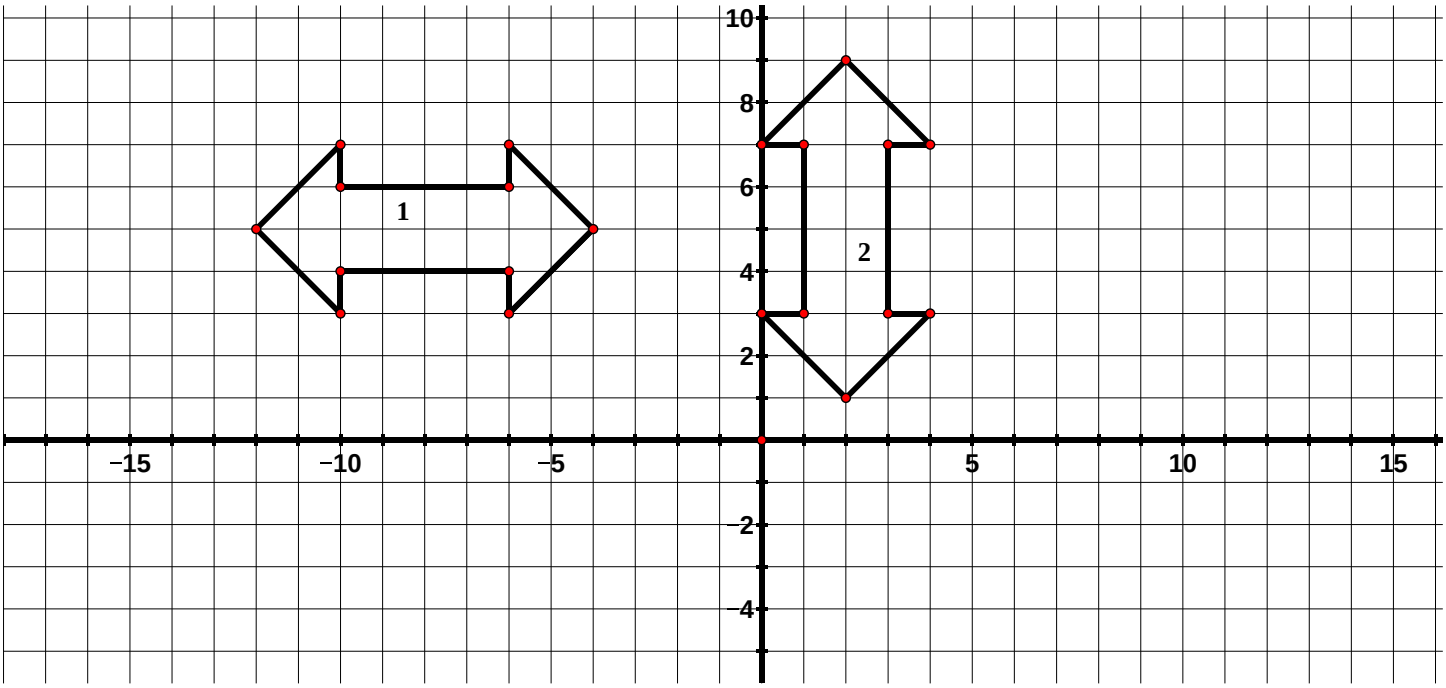
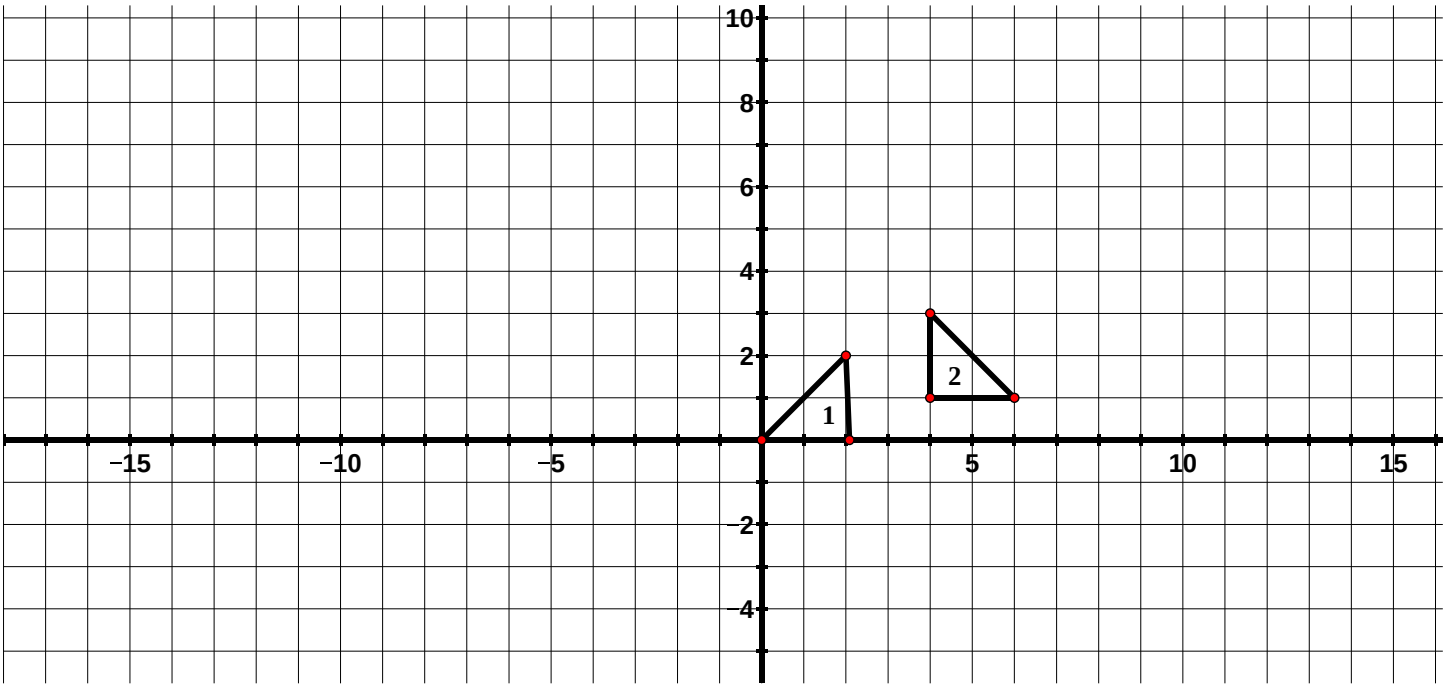
7. Describe a transformation or sequence of transformations that would carry figure 1 onto figure 2.
There are many answers. One possible answer is to first rotate figure 1 $270^\circ$ counterclockwise about the point $(-4, 5)$ followed by a translation according to the rule $(x, y) \rightarrow (x + 6, y - 4)$




8. Describe a transformation or sequence of transformations that would carry triangle 1 onto triangle 2.

One possible answer: a reflection over the line $x = 3$ followed by a translation 1 unit up



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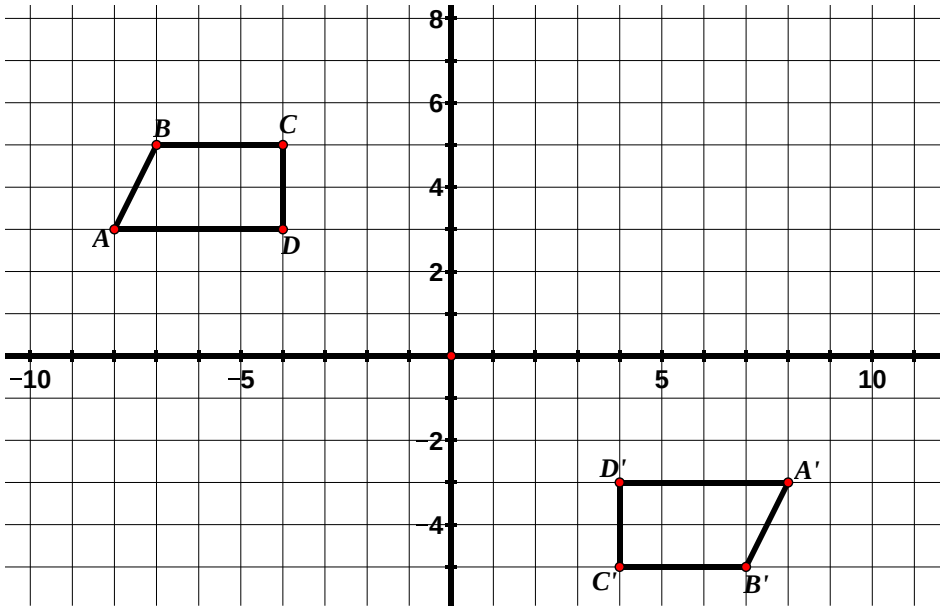
**9.1f Class Activity: Congruence cont.**

1. Observe the two figures below.




a. Describe the ways in which the figures are the same and the ways in which they are different.

Refer back to the table on pg. 48.
Possible answers:
Corresponding segments in the image and pre-image are the same length.
Corresponding angles in the image and pre-image have the same measure.
Parallel lines in the pre-image remain parallel lines in the image.
They are in a different location in the plane.

The two figures above are said to be **congruent**. In 7th grade, you learned that two figures are congruent if they have the **same shape** and are the **same size**. In 8th grade, we define congruence in terms of transformations. A two-dimensional figure is congruent to another if the second can be obtained from the first by a rigid motion (rotation, reflection, or translation) or a sequence of rigid motions. If you can move one of the figures using one of these transformations or a series of these transformations so that it fits exactly on the other one, the two figures are congruent.

b. In this case, there are several different transformations that will carry one figure onto the other. Describe one transformation (or sequence of transformations) that will carry *ABCD* onto *A'B'C'D'*.

A rotation 180˚ about the origin

c. Can you think of a different transformation (or sequence of transformations) that will carry *ABCD* onto *A'B'C'D'*?

A reflection across the *x*-axis followed by a reflection across the *y*-axis

d. A translation, reflection, and rotation are described as rigid motions. Describe in your own words what this means.

Answers will vary – have students share out. Some possible responses – motions that preserve the lengths of segments and the measure of angles. Rigid motions preserve dimension and shape. You can trace the original shape and then move the paper on the plane (sliding it, flipping it, or turning it) so that it fits exactly on the image without stretching the paper.



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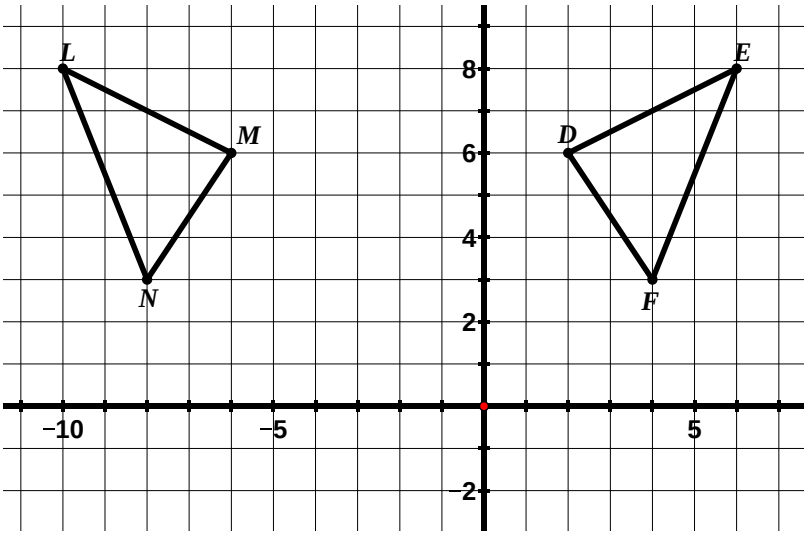
2. The two figures below are congruent.




a. Describe the transformation or sequence of transformations that will carry $\Delta LMN$ onto $\Delta EDF$.
Answers will vary. One possible answer is a reflection across the line $x = -2$

b. Congruent figures have corresponding parts – their matching sides and angles. For example, in the figure above, $\overline{LM}$ corresponds to $\overline{ED}$ and $\angle D$ corresponds to $\angle M$. List the other corresponding parts below.

$\overline{LN}$ corresponds to __$\overline{EF}$_____ $\angle E$ corresponds to ____$\angle L$ _____

$\overline{MN}$ corresponds to ___$\overline{DF}$_____ $\angle F$ corresponds to ___$\angle N$ _____

We can write a congruence statement for the two triangles. You can denote that two figures are congruent by using the symbol ≅ and listing their vertices in corresponding order.

In the example above, we would write this symbolically as $\Delta LMN \cong \Delta EDF$. The order the vertices is written tells us which segments and angles are corresponding in the figures. Have students articulate how to write a congruence statement in their own words.

Corresponding parts of congruent figures are congruent (corresponding segments have the same length and corresponding angles have the same measure). We can show this symbolically in the following way:

$\overline{LM} \cong \overline{ED}$ $\angle D \cong \angle M$
$\overline{LN} \cong \overline{EF}$ $\angle E \cong \angle L$
$\overline{MN} \cong \overline{DF}$ $\angle F \cong \angle N$

We can also annotate the diagram to show which parts are congruent. Do this on the diagram above.
Show students how we use marks to show on a diagram which segments and angles are congruent.



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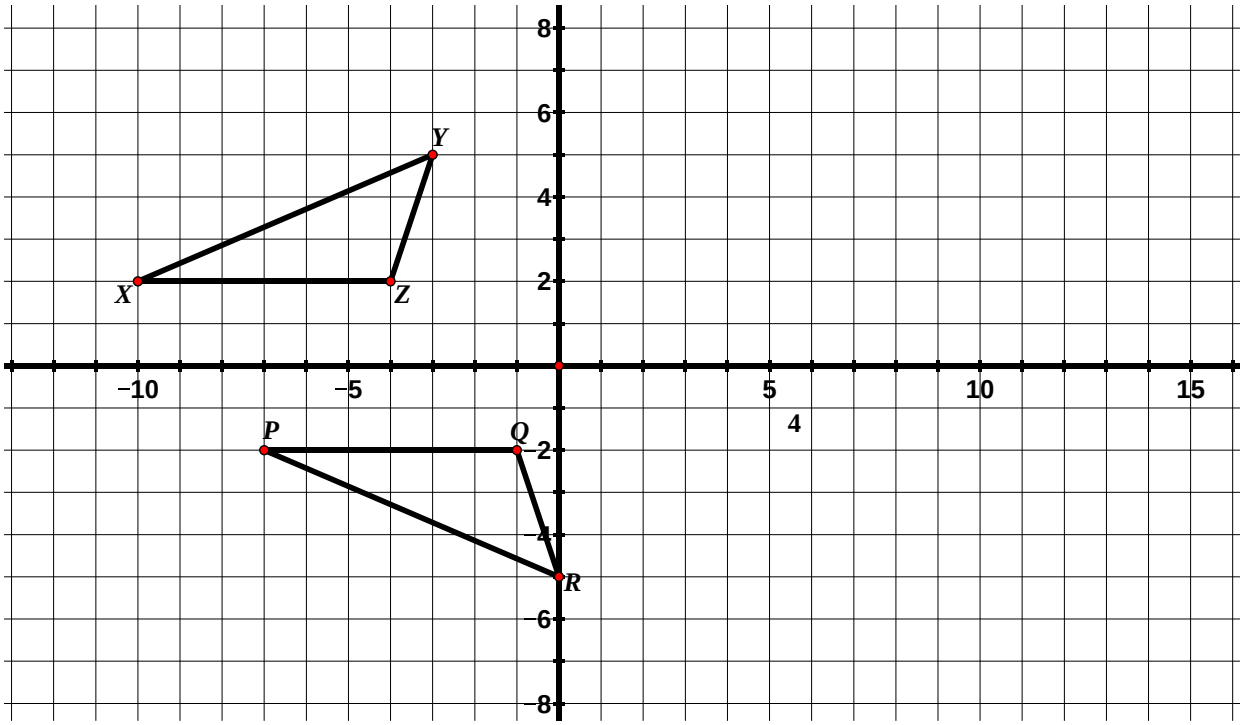
3. The two objects below are congruent.




a. Describe the transformation or sequence of transformations that will carry $\Delta XYZ$ onto $\Delta PRQ$.

One possibility: Reflect $\Delta PRQ$ over the $x$-axis and then translate it 3 units to the left

b. List the congruent corresponding parts.

| | |
|---|---|
| $XY \cong PR$ | $\angle X \cong \angle P$ |
| $XZ \cong PQ$ | $\angle Y \cong \angle R$ |
| $YZ \cong RQ$ | $\angle Z \cong \angle Q$ |

c. Write a congruence statement for the triangles.

$\Delta XYZ \cong \Delta PRQ$

Remind students that the order the vertices are listed when referring to the line segments matters – they should be listed in corresponding order.

d. Annotate the diagram to show which parts are congruent.



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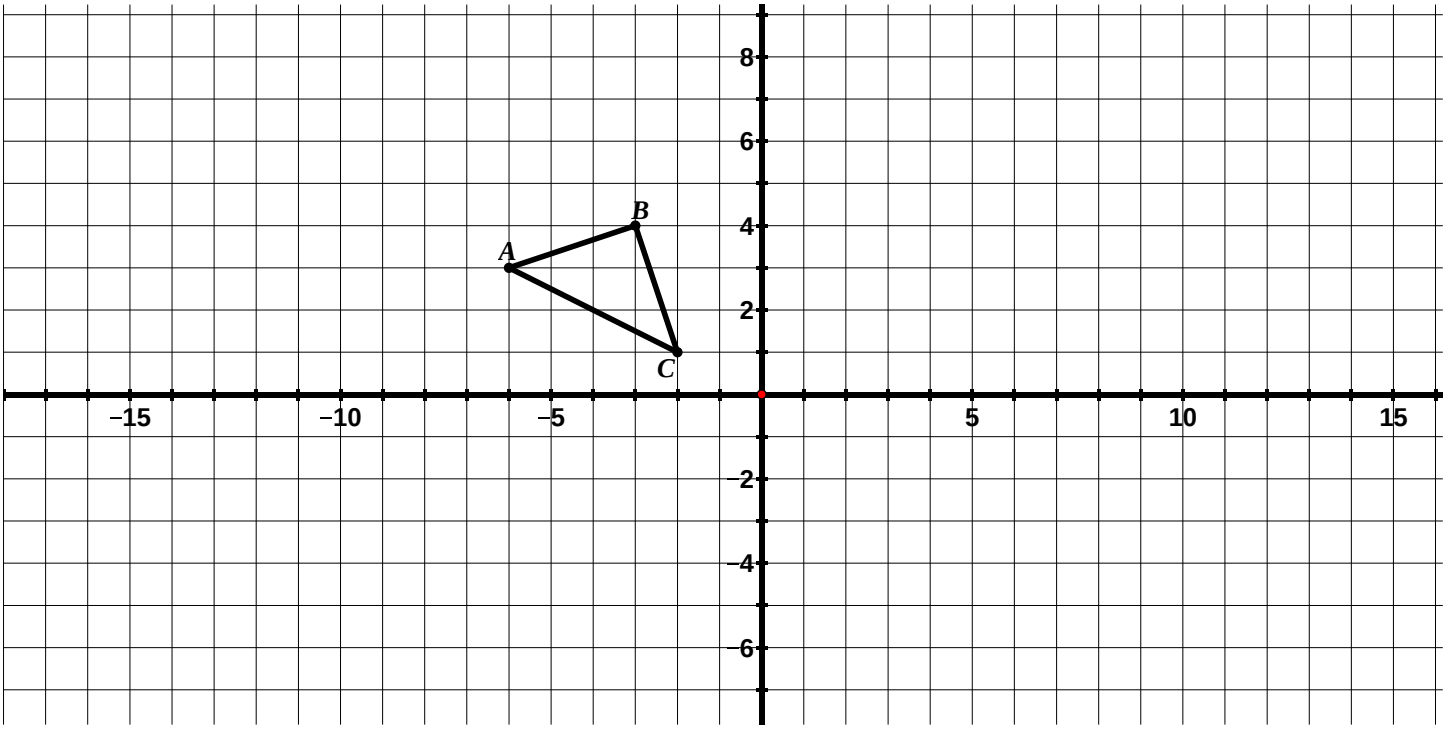
4. Using $\Delta ABC$ in the diagram below as the pre-image, apply the following rules to $\Delta ABC$ and determine whether the resulting image is congruent to $\Delta ABC$. Always start with $\Delta ABC$ as your pre-image.




a. $(x, y) \rightarrow (x, y + 7)$
Is the resulting image congruent to $\Delta ABC$? Why or why not?
Yes, this rule results in a translation. Since a translation is a rigid motion, the objects are congruent.

b. $(x, y) \rightarrow (-x, y)$
Is the resulting image congruent to $\Delta ABC$? Why or why not?
Yes, this rule results in a reflection across the $y$-axis. Since a reflection is a rigid motion, the objects are congruent.

c. $(x, y) \rightarrow (x, 2y)$
Is the resulting image congruent to $\Delta ABC$? Why or why not?
No, this is not a rigid motion. The object is being stretched in the vertical direction.

d. $(x, y) \rightarrow (2x, 2y)$
Is the resulting image congruent to $\Delta ABC$? Why or why not?
No, this is not a rigid motion. The object is being stretched in the vertical and horizontal direction. It is twice as big as the pre-image.

e. Write your own coordinate rule that is different than the ones above that would result in an image that **is congruent** to $\Delta ABC$. How do you know that the resulting image is congruent to $\Delta ABC$?
Answers will vary. Have students share out. As long as their rule results in a rigid motion, the image will be congruent.

f. Write your own coordinate rule that is different than the ones above that would result in an image that **is <u>not</u> congruent** to $\Delta ABC$. How do you know that the resulting image is not congruent to $\Delta ABC$? Answers will vary. Have students share out. As long as their rule does **not** result in a rigid motion, the resulting image will not be congruent.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5. Which of the following properties of a figure can change during a rigid motion? Explain.
   a. Interior angles
      No

   b. Slope of a side
      Yes, we see this in a reflection and a rotation

   c. Parallel lines in the pre-image
      No, lines that are parallel in the pre-image remain parallel lines in the image

   d. Orientation
      Yes, we see this in a reflection

   e. Side lengths
      No

   f. Location in the plane
      Yes, it is possible to see this in all of the rigid motions

   g. Perimeter
      No, if the side lengths do not change, the perimeter will not change

   h. Area
      No, if the side lengths and angles do not change, the area will not change

Review with students the definition of congruence – a figure is congruent to another if the second can be obtained from the first by a sequence of rigid motions (rotations, reflections, and translations). It will help with the corresponding homework.



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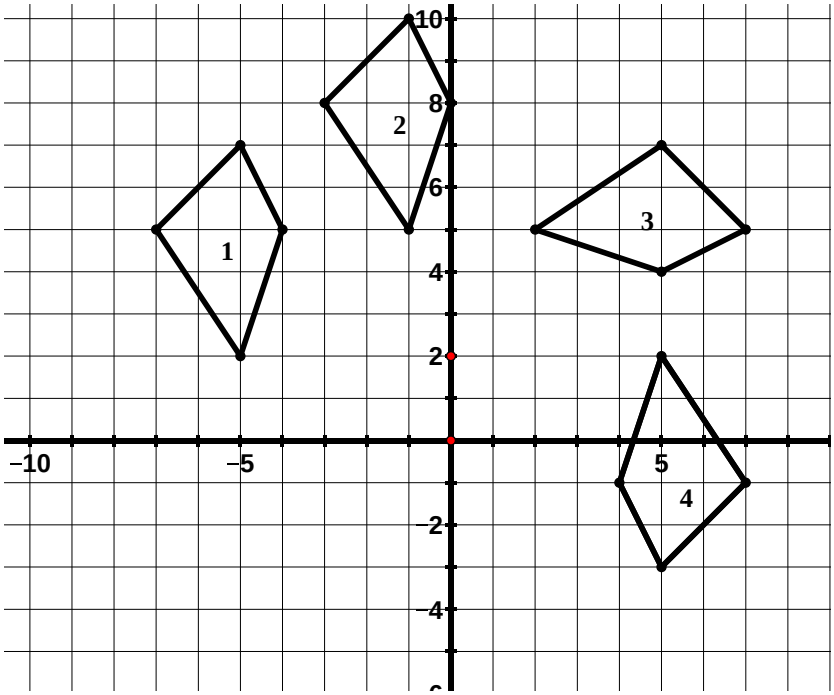
**9.1f Homework: Congruence cont.**

1. Jeff's teacher asked him to create 3 figures that were congruent to figure 1 in the picture below. Jeff created figures 2, 3, and 4.




a. Use the definition of congruence to determine if Jeff's figures are congruent to figure 1. Explain your answers.

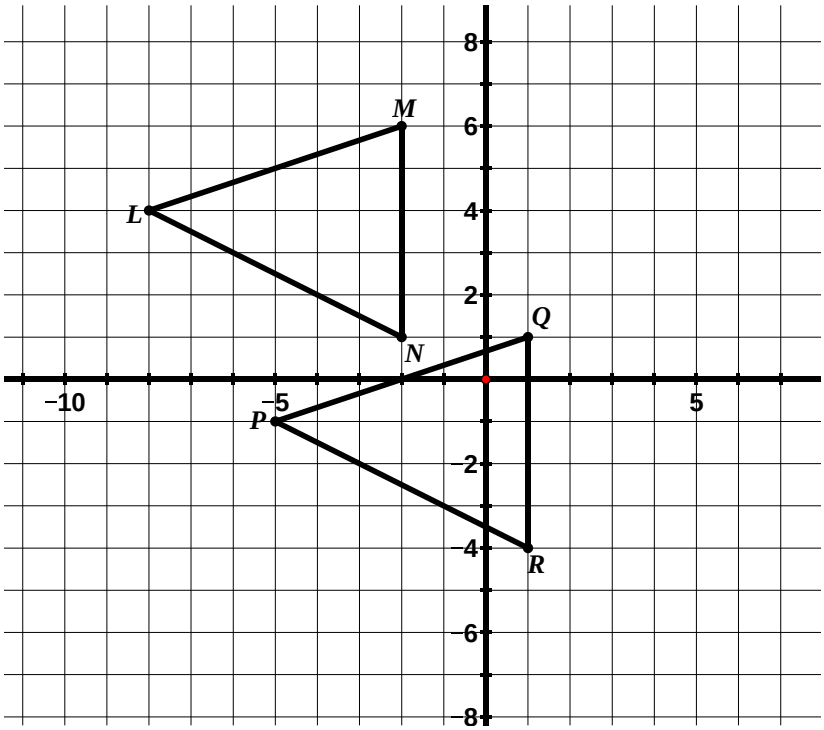
Answers may vary but students should justify the congruence of each set of figures by determining if the second figure was created from a rigid motion of Figure 1.

- Figure 2 is congruent to Figure 1 because Figure 2 is a translation of Figure 1.
- Figure 3 is congruent to Figure 1 because Figure 3 is a 90˚ rotation clockwise about the origin of Figure 1
- Figure 4 is congruent to Figure 1 because Figure 4 is a 180˚ rotation about $(0, 2)$ of Figure 1

b. Draw an additional figure that is congruent to figure 1. How do you know your figure is congruent to figure 1?

As long as the figure was created by a rigid motion, it will be congruent to figure 1.

2. The two figures below are congruent.




a. Describe the transformation or sequence of transformations that will carry $\Delta LMN$ onto $\Delta PQR$.

Answers may vary. One possible answer, a translation 5 units down and 3 units to the right.

b. List the congruent corresponding parts.

| | |
|---|---|
| $LM \cong PQ$ | $\angle L \cong \angle P$ |
| $LN \cong PR$ | $\angle M \cong \angle Q$ |
| $MN \cong QR$ | $\angle N \cong \angle R$ |

c. Write a congruence statement for the triangles.

$\Delta LMN \cong \Delta PQR$



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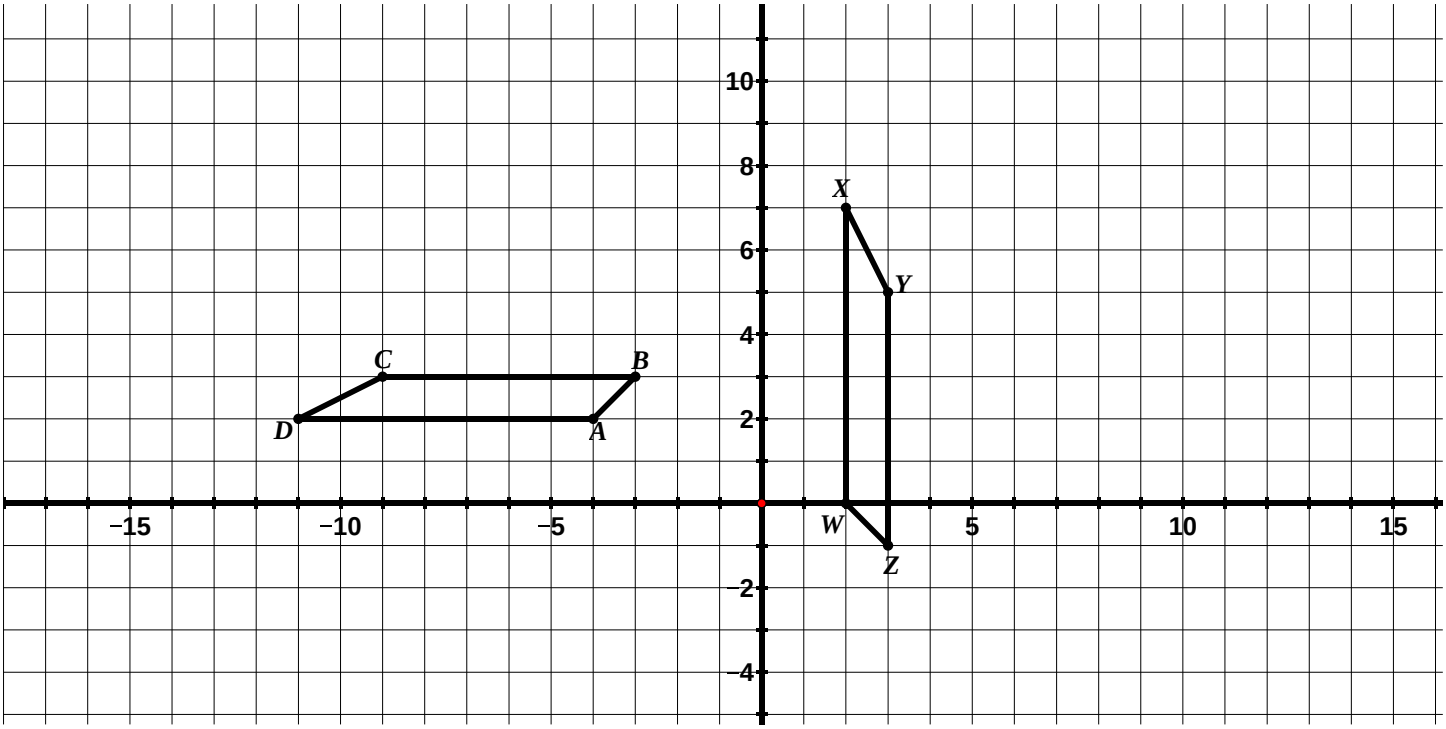
3. The two figures below are congruent.



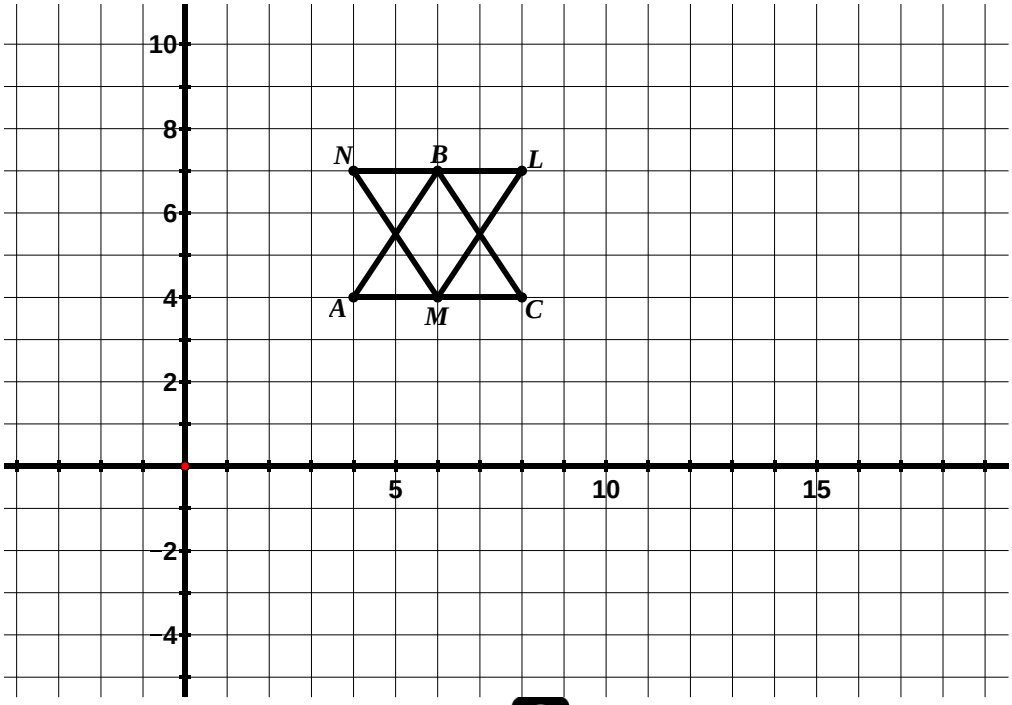

a. Describe the transformation or sequence of transformations that will carry *ABCD* onto *WXYZ*.

Answers may vary. One possible answer, rotate *ABCD* 270° counterclockwise about the origin followed by a translation 4 units down

b. Write a congruence statement for the parallelograms.

*ABCD* ≅ *WZYX*

4. Consider $\Delta ABC$ and $\Delta LMN$ below. The two triangles are congruent.




In order to prove that these two triangles are congruent, students need to describe a rigid motion or sequence of rigid motions that carry one triangle onto the other. One possible answer is: a reflection of $\Delta ABC$ across the line $y = 4$ followed by a translation 3 units up. Have students share different answers and compare the sequences.

a. Prove that $\Delta ABC \cong \Delta LMN$.



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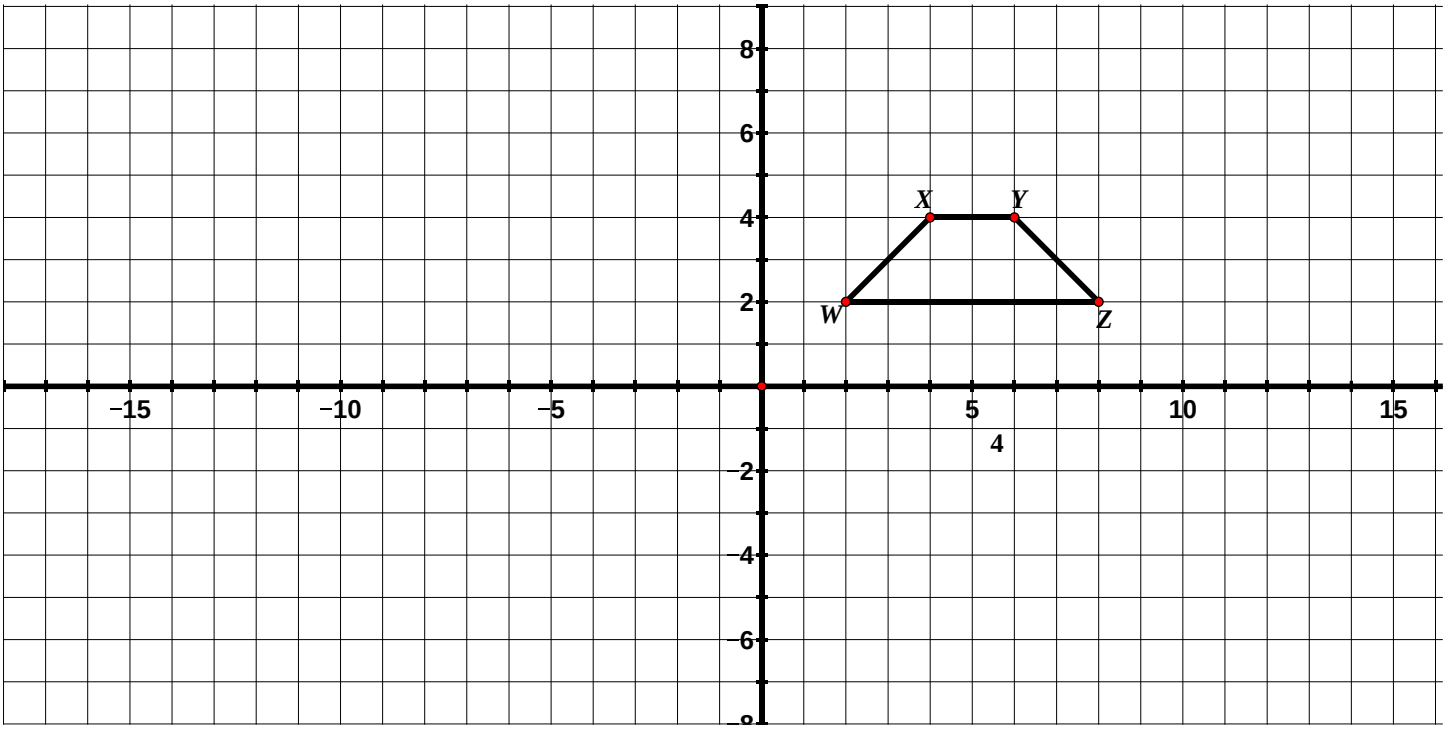
5. Using *WXYZ* in the diagram below as the pre-image, apply the following rules to *WXYZ* and determine whether the resulting image is congruent to *WXYZ*. Always start with *WXYZ* as your pre-image.




a. $(x, y) \rightarrow (x - 2, y + 1)$
Is the resulting image congruent to *WXYZ*? Why or why not?
Yes, this is a translation

b. $(x, y) \rightarrow (y, x)$
Is the resulting image congruent to *WXYZ*? Why or why not?
Yes, this is a reflection across the line $y = x$

c. $(x, y) \rightarrow (\frac{1}{2}x, \frac{1}{2}y)$
Is the resulting image congruent to *WXYZ*? Why or why not?
No, this rule reduces the size of the figure

d. Write your own coordinate rule that is different than the ones above that would result in an image that **is congruent** to *WXYZ*. How do you know that the resulting image is congruent to *WXYZ*?
Answers will vary

e. Write your own coordinate rule that is different than the ones above that would result in an image that **is <u>not</u> congruent** to *WXYZ*. How do you know that the resulting image is not congruent to *WXYZ*?
Answers will vary



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**9.1g Self-Assessment: Section 9.1**

Consider the following skills/concepts. Rate your comfort level with each skill/concept by checking the box that best describes your progress in mastering each skill/concept. Sample problems for each standard can be found on the following page(s).

| Skill/Concept | Minimal Understanding 1 | Partial Understanding 2 | Sufficient Mastery 3 | Substantial Mastery 4 |
|---|---|---|---|---|
| 1. Given a pre-image and its image under a translation, describe the translation in words and using a coordinate rule. | | | | |
| 2. Perform a translation of a figure given a coordinate rule. | | | | |
| 3. Describe the properties of a translation and the effects a translation has on a figure and use this knowledge to solve problems. | | | | |
| 4. Given a pre-image and its image under a reflection, describe the reflection in words and using a coordinate rule. | | | | |
| 5. Perform a reflection of a figure given a line of reflection. | | | | |
| 6. Describe the properties of a reflection and the effects a reflection has on a figure and use this knowledge to solve problems. | | | | |
| 7. Find a reflection line for a given reflection and write the equation of the reflection line. | | | | |
| 8. Given a pre-image and its image under a rotation, describe the rotation in words and using a | | | | |



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| | | | | |
|---|---|---|---|---|
| coordinate rule (coordinate rule for rotations centered at the origin only). | | | | |
| 9. Perform a rotation of a figure given a center of rotation, an angle of rotation, and a direction. | | | | |
| 10. Describe the properties of a rotation and the effects a rotation has on a figure and use this knowledge to solve problems. | | | | |
| 11. Connect ideas about slopes of perpendicular lines and rotations. | | | | |
| 12. Understand what it means for two figures to be congruent. | | | | |
| 13. Determine if two figures are congruent based on the definition of congruence. | | | | |
| 14. Given two figures that are congruent, describe the sequence of transformations that exhibits the congruence between them. | | | | |



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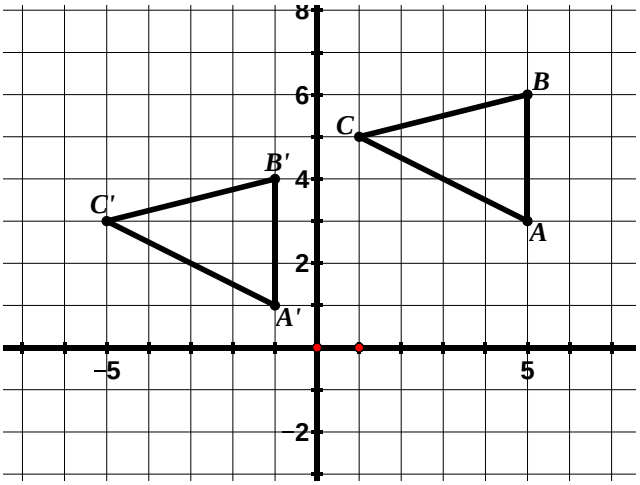
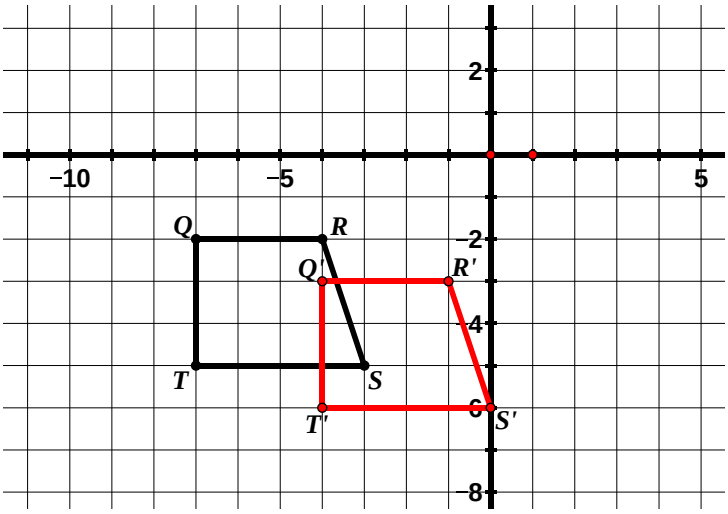
**Section 9.1 Sample Problems** (For use with self-assessment)

1. Describe the following transformation in words and using a coordinate rule.




A translation 2 units down and 6 units to the left. $(x, y) \rightarrow (x - 6, y - 2)$

2. Translate $QRST$ according to the rule $(x, y) \rightarrow (x + 3, y - 1)$.




3. Which of the following properties are true about a figure that has been translated. Check all that apply.
   - ☐ The slopes of corresponding segments are opposite reciprocals.
   - ☑ Segments connecting corresponding vertices of the image and pre-image are the same length.
   - ☐ The perimeter of the pre-image is smaller than the perimeter of the image.
   - ☑ Segments connecting corresponding vertices of the image and pre-image are parallel.
   - ☐ Corresponding vertices of the image and pre-image lie on the same circle.

Discuss additional properties of translations and correct statements above to make them true where appropriate.



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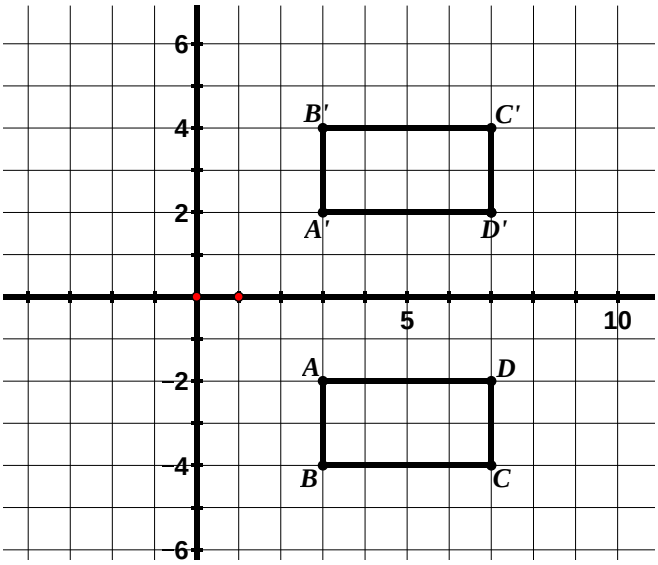
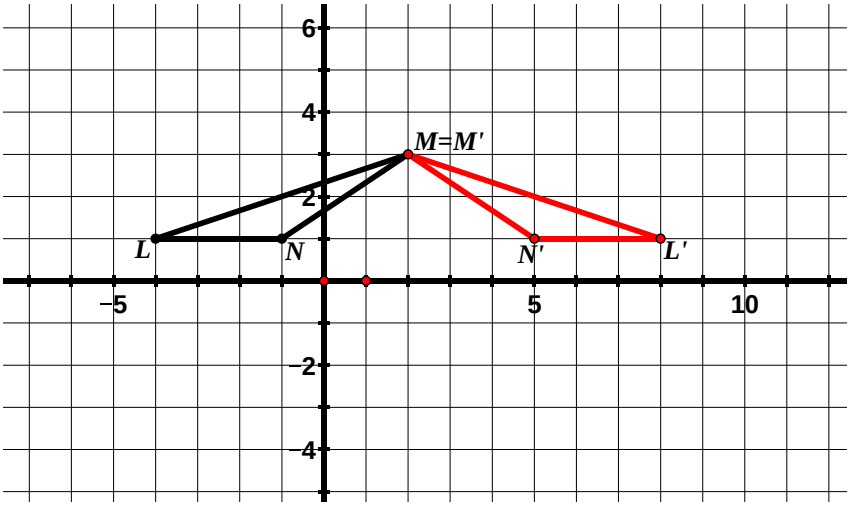
4. Describe the following transformation in words and using a coordinate rule.




A reflection across the $x$-axis. $(x, y) \rightarrow (x, -y)$

5. Reflect $\Delta LMN$ over the line $x = 2$.




6. Which of the following properties are true about a figure that has been reflected. Check all that apply.

- ☐ The slopes of corresponding segments are the same.
- ☐ Segments connecting corresponding vertices of the image and pre-image are the same length.
- ☑ Segments connecting corresponding vertices of the image and pre-image are parallel.
- ☑ The area of the pre-image is the same as the area of the image.
- ☑ Corresponding vertices are equidistant from the line of reflection.
- ☐ The measure of the interior angles of an object may change under a reflection.

Discuss additional properties of reflections and correct statements above to make them true where appropriate.



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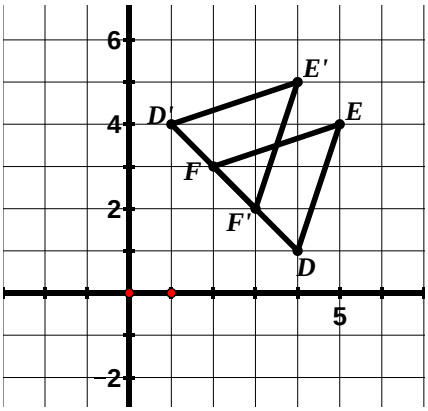
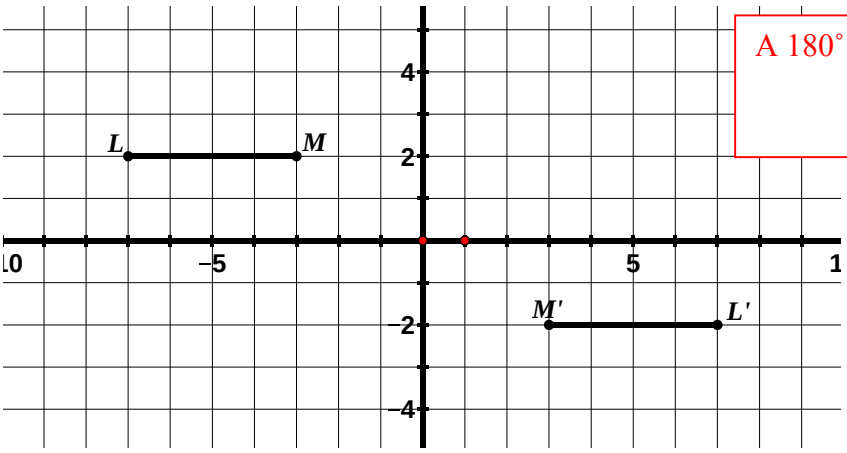
7. $\Delta EFD$ has been reflected to obtain $\Delta E'F'D'$. Write the equation for the line of reflection.



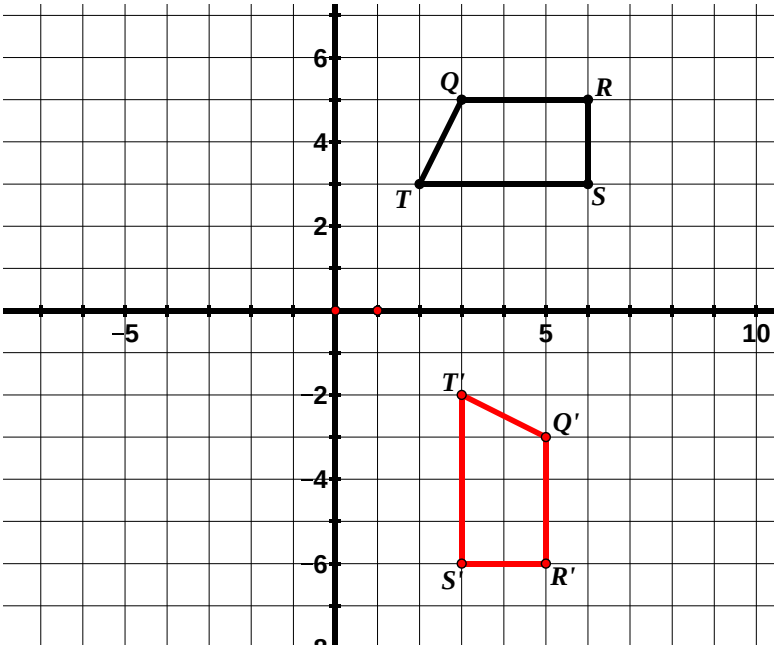

Equation for line of reflection: $y = x$

8. Describe the following rotation in words and using a coordinate rule.




A 180˚ rotation. $(x, y) \rightarrow (-x, -y)$

9. Rotate $QRST$ 90˚ clockwise about the origin.



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10. Which of the following properties are true about a figure that has been rotated 90˚. Check all that apply.

- [x] The slopes of corresponding segments are opposite reciprocals.
- [ ] Segments connecting corresponding vertices of the image and pre-image are the same length.
- [ ] Segments connecting corresponding vertices of the image and pre-image are parallel.
- [ ] Lines that are parallel in the pre-image are not necessarily parallel in the image.
- [x] Corresponding segments are perpendicular.
- [x] Corresponding vertices lie on the same circle.

Discuss additional properties of rotations and correct statements above to make them true where appropriate.

11. In a 90˚ rotation, what happens to the slopes of the corresponding segments of the image and pre-image? How does the coordinate rule show this? In a 180˚ rotation, what happens to the slopes of the corresponding segments of the image and pre-image? How does the coordinate rule show this?
In a 90˚ rotation, the slopes of corresponding segments are opposite reciprocals or the product of the slopes is 1. The coordinate rule for a 90˚ rotation is $(x, y) \rightarrow (y, -x)$. The $x$- and $y$-coordinates change places therefore the slope is inverted. The sign is changed on only the $x$-coordinates therefore changing the sign of the slope. In a 180˚ rotation, the slopes of corresponding segments are the same. The coordinate rule for a 180˚ rotation is $(x, y) \rightarrow (-x, -y)$. The signs of both the $x$- and $y$-coordinates change therefore the net result is that the slope does not change.

12. In your own words, describe what it means for two figures to be congruent.
Two figures are congruent if the second can be obtained from the first by a rigid motion (rotation, reflection, or translation) or a sequence of rigid motions. If you can move one of the figures using one of these transformations or a series of these transformations so that it fits exactly on the other one, the two figures are congruent.



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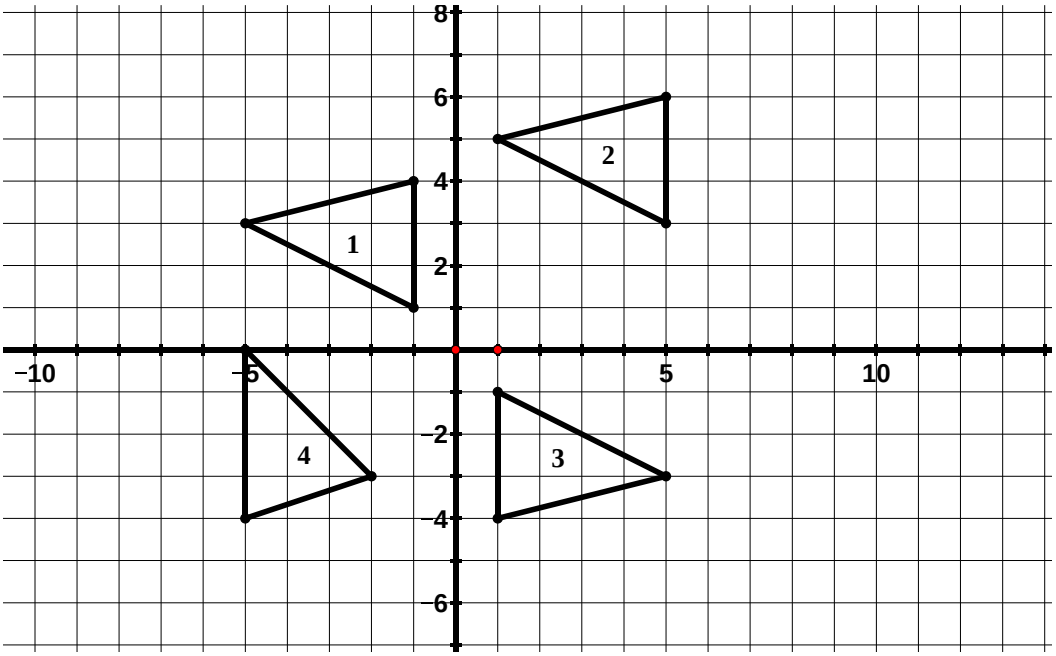
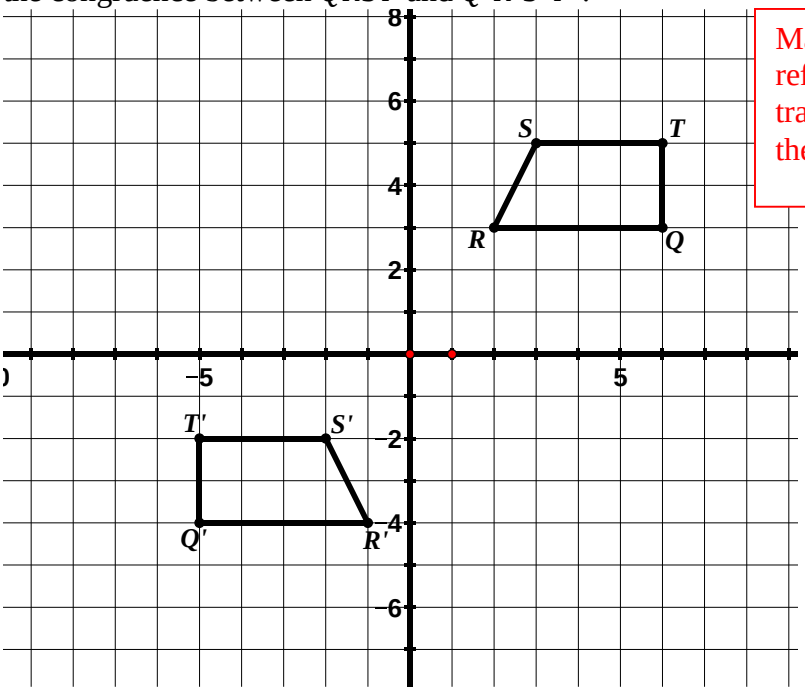
13. Which of the following triangles are congruent? Justify your answers.




Justifications may vary. Possible justifications:
1 ≅ 2 because 2 is a translation of 1 (translation of 2 units up and 6 units right)
1 ≅ 3 because 3 is a rotation of 1 (a 180˚rotation about the origin) or a double reflection (reflect 1 across *y*-axis and then across *x*-axis)
2 ≅ 3 because 3 is a rotation of 2 (a 180˚rotation about (3,1))

14. $QRST$ is congruent to $Q'R'S'T'$. Describe a transformation or sequence of transformations that exhibits the congruence between $QRST$ and $Q'R'S'T'$.




Many answers. One possible answer is a reflection across the *y*-axis followed by a translation 7 units down and one unit to the right.



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# Section 9.2 Dilations and Similarity

**Section Overview:**
Students start this section by applying different transformations (given by coordinate rules) to a figure to determine what these rules do to the shape and size of the figure. In this activity, they start to surface ideas about similarity. Students then begin to study dilations in detail. They perform dilations using the slope triangle method and scaling method when given a scale factor and center of dilation. Students determine the scale factor and center of dilation of two figures that have been dilated and write a coordinate rule to describe the dilation. Students continue to refer back to their work in order to describe the properties of dilations. Students will then apply this knowledge to determine if two figures are similar, understanding that a two-dimensional figure is similar to another if there is a sequence of rigid motions and dilations that takes one figure onto the other. In addition, students will be given two figures that are similar and asked to describe a sequence of transformations that exhibits the similarity between them.

**Concepts and Skills to Master:**
*By the end of this section, students should be able to:*

1. Describe the properties of a figure that has been dilated.
2. Perform a dilation given a scale factor and center of dilation.
3. Describe a dilation in words and using a coordinate rule.
4. Determine the center of dilation using the properties of dilations.
5. Understand what it means for two figures to be similar.
6. Determine if two figures are similar.
7. Given two figures that are similar, describe the sequence of transformations that exhibits the similarity between them.


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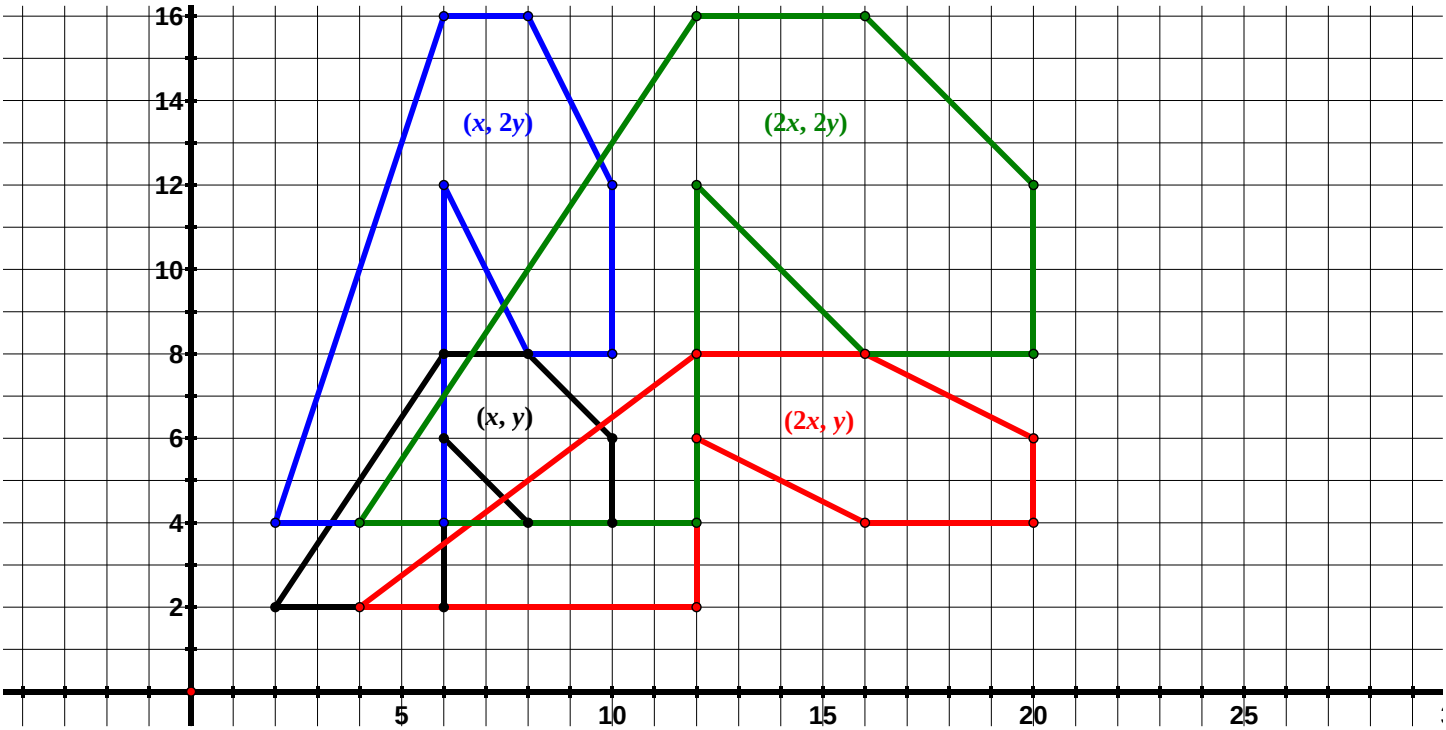
**9.2a Class Activity: Video Game Animation**

Computer animators are working on designing the head of a dragon for a new video game. The picture below shows the original shape and size of the dragon's head. However, when the dragon eats a plant, the lengths of the sides of the dragon head double in size. If the dragon eats a cricket, the lengths of the sides of the dragon head triple in size. When the dragon enters a cave, the lengths of the sides shrink to half their original size.




Four different animators submitted the following proposals for how to double the lengths of the sides of the dragon's head when it eats a plant:

- Animator 1 said to apply the following rule $(x, y) \rightarrow (2x, y)$
- Animator 2 said to apply the following rule $(x, y) \rightarrow (x, 2y)$
- Animator 3 said to apply the following rule $(x, y) \rightarrow (x + 2, y + 2)$
- Animator 4 said to apply the following rule $(x, y) \rightarrow (2x, 2y)$

The chart below shows the coordinates of the dragon's head when it is its original size. Write the new coordinates for the dragon's head for the coordinate rules proposed by each of the animators. Then graph each of the animator's new dragon heads. You may choose to let the students work in groups and assign one coordinate rule to each member of the group. If students do all of the dragons, have extra graph paper available – they will overlap on the grid and it may become confusing. Three dragons are shown above.

| **Original Size** | $(2x, y)$ | $(x, 2y)$ | $(x + 2, y + 2)$ | $(2x, 2y)$ |
|---|---|---|---|---|
| (2, 2) | (4, 2) | (2, 4) | (4, 4) | (4, 4) |
| (6, 8) | (12, 8) | (6, 16) | (8, 10) | (12, 16) |
| (8, 8) | (16, 8) | (8, 16) | (10, 10) | (16, 16) |
| (10, 6) | (20, 6) | (10, 12) | (12, 8) | (20, 12) |
| (10, 4) | (20, 4) | (10, 8) | (12, 6) | (20, 8) |
| (8, 4) | (16, 4) | (8, 8) | (10, 6) | (16, 8) |
| (6, 6) | (12, 6) | (6, 12) | (8, 8) | (12, 12) |
| (6, 2) | (12, 2) | (6, 4) | (8, 4) | (12, 4) |



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4. Which of the animator's rules results in a dragon that is the same shape as the original but whose side lengths are twice the size? What is the same about this dragon compared to the original dragon? What is different?

$(2x, 2y)$ This dragon is the same shape as the original dragon just a different size. Students may observe that each segment has doubled in length. You may also have students trace the original figure on tracing paper or an overhead transparency and then have them line up the corresponding angles to observe that corresponding angles have the same measure.

Ask students what has happened to the area? This is a great connection back to 7th grade. Does doubling the side lengths also double the area? No, it quadruples the area.

5. Describe what the coordinate rules of the other three animators do to the dragon's head.

$(2x, y)$: stretches the dragon's head in the horizontal direction, the height of the dragon's head is the same because the $y$-coordinates have not been changed. Students can use the tracing of the original dragon to see that corresponding angles no longer have the same measure.

$(x, 2y)$: stretches the dragon's head in the vertical direction, the width of the dragon's head is the same because the $x$-coordinates have not been changed. Again, students can use the tracing of the original dragon to see that corresponding angles no longer have the same measure.

$(x + 2, y + 2)$: This is a translation of the dragon's head. Since it is a rigid motion of the dragon's head, this figure is congruent to the original figure. Corresponding sides have the same length and corresponding angles have the same measure.

6. The animators need your help coming up with the coordinate rule that would reduce the size of the dragon's head when it enters a cave (the lengths of the sides should be half their original size).
   a. Write your proposed coordinate rule in the table below.
   b. Write the new coordinates for your rule.
   c. Graph the new coordinates.

| **Original Size** | **Coordinate Rule:** $(\frac{1}{2}x, \frac{1}{2}y)$ |
|---|---|
| (2, 2) | (1, 1) |
| (6, 8) | (3, 4) |
| (8, 8) | (4, 4) |
| (10, 6) | (5, 3) |
| (10, 4) | (5, 2) |
| (8, 4) | (4, 2) |
| (6, 6) | (3, 3) |
| (6, 2) | (3, 1) |

   d. What is the same about this dragon compared to the original dragon? What is different?

See #1 only this time each of the corresponding segments are half the length of the original segments

   e. Write the coordinate rule that would triple the size of the dragon's head when it eats a cricket. $(3x, 3y)$



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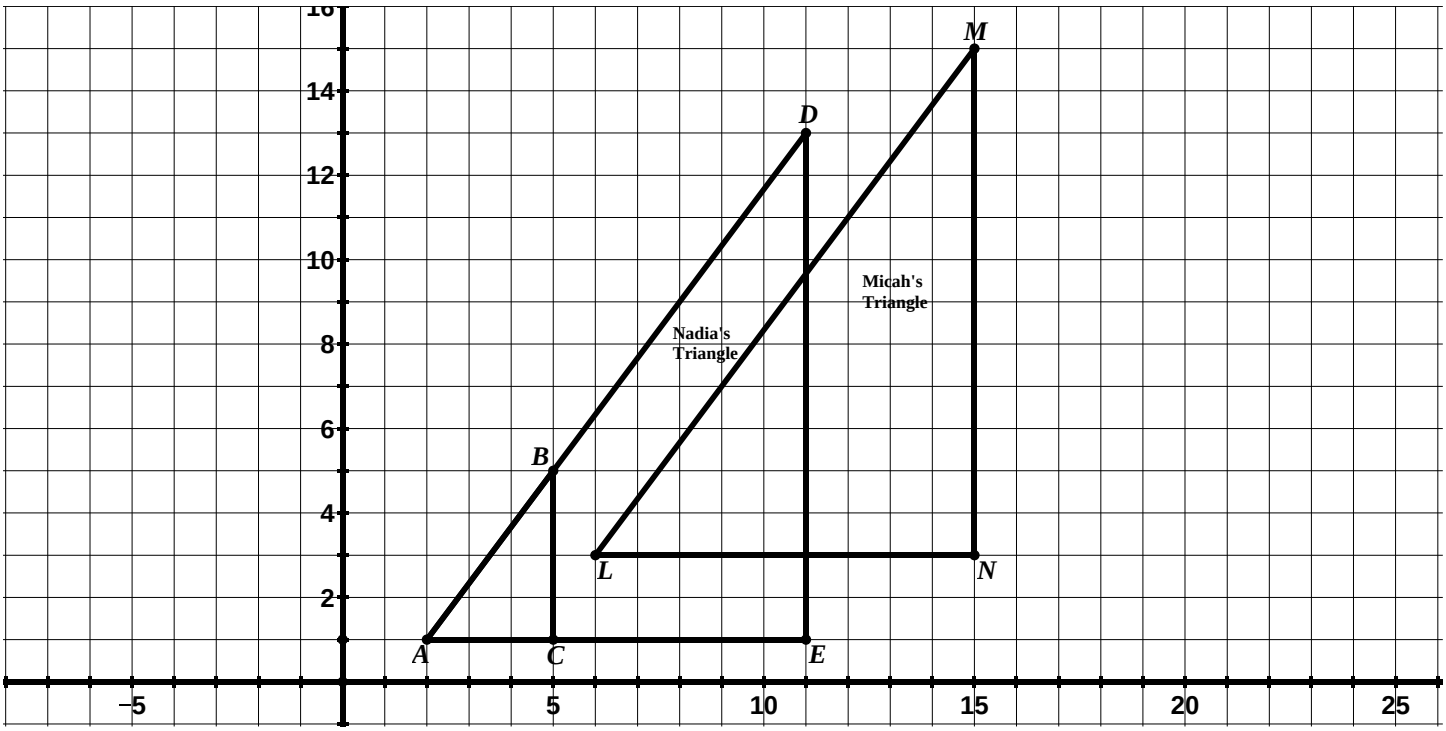
## 9.2b Class Activity: Properties of Dilations

1. Ms. Williams gave her students the grid shown below with $\Delta ABC$ graphed on it. She then asked her students to create a triangle that was the same shape as the original triangle but has sides lengths that are three times larger. Micah created $\Delta LMN$ shown below and Nadia created $\Delta ADE$ shown below.




a. The teacher asked the class who had done the assignment correctly, Nadia or Micah? Iya said they were both correct. Hendrix disagreed and said they could not both be correct because the triangles were not in the same location in the coordinate plane. Who do you agree with and why?

Have students share their thoughts. It is true that both triangles have sides lengths that are 3 times larger than the original so both students followed the teacher's instructions.

One way of looking at this is that Nadia used *A* as the center of dilation while Micah used the origin. Another way of looking at this is that Micah and Nadia may have both dilated their figures from *A* first and then Micah followed his dilation by a translation. Is a dilation followed by a translation always representable as a dilation, but with a different center? Students will explore this idea further in 9.2f when they have the opportunity to determine a sequence of rigid motions and dilations that carry one figure to another.



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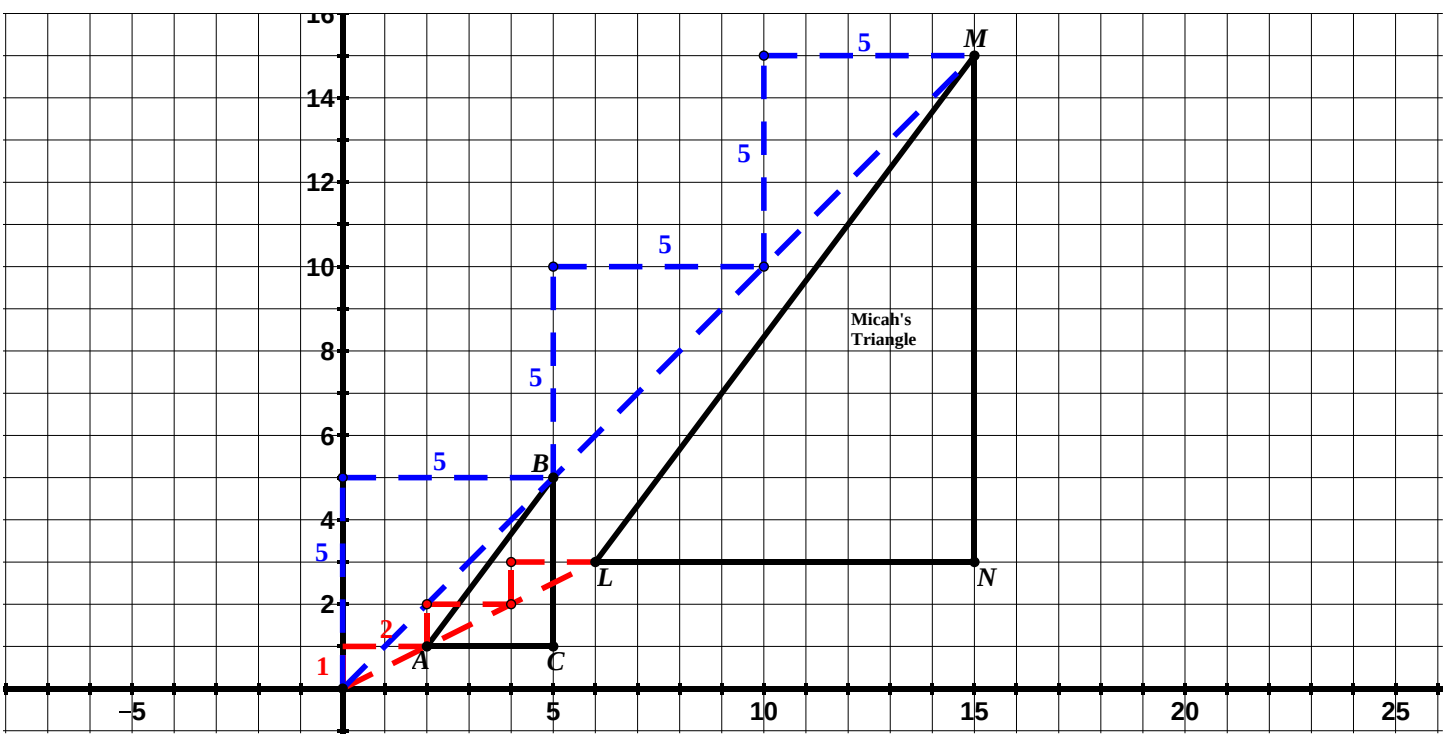
b. The teacher asked Micah and Nadia to explain the methods they used to create the triangles.

**Nadia's Method:** Using a ruler, I slid $C$ along the line containing the points $A$ and $C$ until my new segment was three times larger than $\overline{AC}$ and labeled the new point $E$. I then used my ruler to slide $B$ along the line containing the points $A$ and $B$ as shown below and labeled the new point $D$. Lastly, I checked to make sure $\overline{DE}$ was 3 times larger than $\overline{BC}$ and it was!
Make sure to use the diagram to ensure that students understand Nadia's method.

**Micah's Method:** I noticed that the slope of the line passing through the origin and $A$ had a rise of 1 and a run of 2. Since I wanted the image to be three times larger than $\Delta ABC$, I placed the point that corresponds to $A$ three slope triangles (with a rise of 1 and a run of 2) from the origin. I used the same method to plot the point that corresponds to $B$. The slope of the line passing through the origin and $B$ has a rise of 5 and a run of 5. Again, I moved three slope triangles with a rise of 5 and a run of 5 from the origin and plotted $M$. The slope of the line passing through $C$ and the origin has a rise of 1 and a run of 5. I moved three slope triangles with a rise of 1 and a run of 5 from the origin and plotted $N$.
Make sure students see the slope triangles that Micah used to plot the corresponding vertices. The slope triangles that Micah used to plot $N$ are not shown on the diagram. Consider having students draw these slope triangles.




c. Compare the two methods used. What is the same about the resulting triangles? What is different? What accounts for the differences in the triangles?

Again, the triangles are the same size (side lengths are 3 times larger than the original triangle). The difference in the triangles is the location in the plane. The reason for this difference is that Nadia expanded her points from $A$ while Micah expanded his points from the origin. They used a different center of dilation which will be defined on the following page.



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In the previous example, Micah and Nadia both **dilated** $\Delta ABC$. A **dilation** is a transformation that produces an image that is the same shape as the original figure but the image is a different size.

Every dilation has a center of dilation and a scale factor. The **center of dilation** is a fixed point in the plane from which all points are expanded or contracted. The **scale factor** describes the size change from the original figure to the image. We use the letter *r* to represent scale factor. The dilation is an enlargement if the scale factor is greater than 1 and a reduction if the scale factor is between 0 and 1.

In the example on the previous page, the scale factor for both Nadia and Micah was 3; however Nadia's center of dilation was *A*: (2, 1) while Micah's was the origin (0, 0).

2. We will use the example on the previous page to examine some of the **properties of dilations.**
   a. Find the following ratios for Nadia's triangle:

$\frac{AD}{AB} = \frac{15}{5} = 3$ $\frac{DE}{BC} = \frac{12}{4} = 3$ $\frac{AE}{AC} = \frac{9}{3} = 3$

   b. Find the following ratios for Micah's triangle:

$\frac{LM}{AB} = \frac{15}{5} = 3$ $\frac{MN}{BC} = \frac{12}{4} = 3$ $\frac{LN}{AC} = \frac{12}{4} = 3$

   c. Complete the following sentence. Under a dilation, **the ratios of the image segments to the corresponding pre-image segments** are…
   The same and equal to the scale factor

   d. Complete the following sentence. Under a dilation, **corresponding angles** are…
   Have the same measure or are congruent

   e. Complete the following sentence. Under a dilation, **corresponding segments** are…
   parallel

   f. Complete the following sentence. Under a dilation, **corresponding vertices**…
   Lie on the same line and they also lie on the same line as the center of dilation. Also, the distance of the image vertex from the center of dilation is equal to the scale factor multiplied by the distance of the pre-image vertex from the center of dilation. Use the example above to show this. The distance of *L* is three times the distance of *A* to the center of dilation. This holds true for the other vertices as well. It also holds true for Nadia's triangle. The distance of *E* is three times the distance of *C* to the center of dilation. This holds true for the other vertices as well. It is very important for students to understand these properties of dilations as they will use them throughout the remaining lessons.
   g. Complete the following sentence. Under a dilation, **segments connecting corresponding vertices**…

   Intersect at the center of dilation


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h. In the table below, list the coordinates of the corresponding vertices in Micah's dilation:

| | |
|---|---|
| A: $(2, 1)$ | L: $(6, 3)$ |
| B: $(5, 5)$ | M: $(15, 15)$ |
| C: $(5, 1)$ | N: $(15, 3)$ |

i. Write a coordinate rule for Micah's dilation using the information in the table above.
$(x, y) \rightarrow (3x, 3y)$

j. In the table below, list the coordinates of the corresponding vertices in Nadia's dilation:

| | |
|---|---|
| A: $(2, 1)$ | A: $(2, 1)$ |
| B: $(5, 5)$ | D: $(11, 13)$ |
| C: $(5, 1)$ | E: $(11, 1)$ |

k. Write a coordinate rule for Nadia's dilation using the information in the table above. Remember that Nadia's center of dilation is not at the origin. Think about how this shift off the origin will affect the coordinate rule.
$(x, y) \rightarrow ((3x - 4), (3y - 2))$ Compare Micah's triangle to Nadia's. If Nadia had dilated from the origin, her triangle would sit on top of Micah's. In order to move her triangle to where it should be, one would follow the dilation with center at the origin by a translation 4 units to the left and two units down.

Again, encourage students to compare the area of the original triangle to one whose side lengths are three times larger. What happens to the area? It is also tripled? No, it is increased by a factor of 9.

In the following section, students will be finding the center of dilation. One way to do this is to use a straightedge to draw lines that connect corresponding vertices. The intersection of these lines is the center of dilation. This strategy will work because the center of dilation lies on every line joining a point and its image. Students should also be able to use logic to determine an approximate location for the center of dilation. For example in #6 on the following page, they know that all of the pre-image points have to be closer to the center of dilation than the image points because the scale factor is greater than 1. This will put the center of dilation somewhere in quadrant I.

To find the scale factor, students can find the ratio of corresponding sides. Alternatively, they can find the ratio of the distance of an image vertex to the center of dilation to the distance of a pre-image vertex to the center of dilation.



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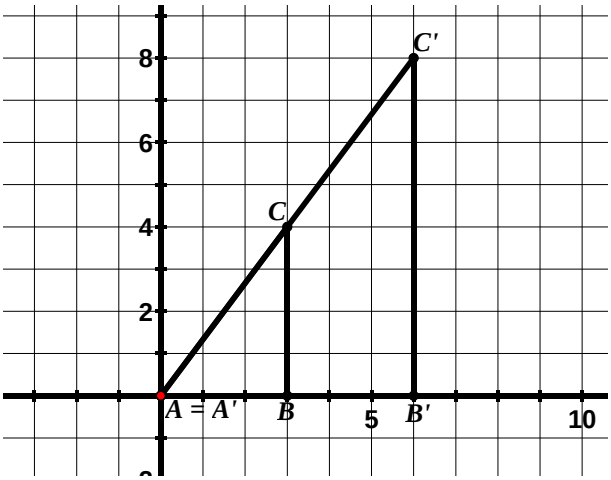
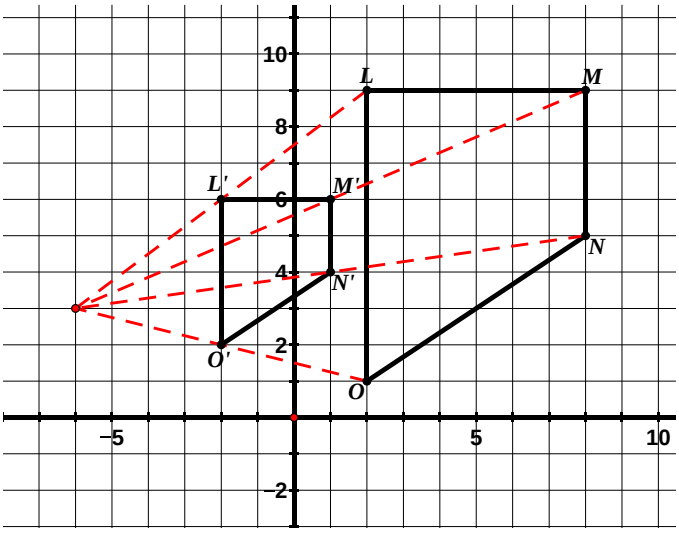
**Directions:** In the following problems, one figure has been dilated to obtain a new figure. Determine the scale factor and center of dilation. Determine the coordinate rule for the dilation when asked. Be sure to discuss with students the strategies mentioned on the previous page for how to find the scale factor and center of dilation for a given dilation. If students draw the lines connecting corresponding vertices, the lines should intersect at the center of dilation. For example in #4, the lines passing through corresponding vertices intersect at $(-6, 3)$. Also, have students use slope triangles to verify for this problem that the distance from the center of dilation to an image vertex is $\frac{1}{2}$ the distance from the center of dilation to the corresponding pre-image vertex. Encourage students to investigate and verify as many properties of dilations as they can for each problem.

3. In the picture below, $\Delta ABC$ has been dilated to obtain $\Delta A'B'C'$.




Scale Factor: ____2_____
Center of Dilation: _____ $A$ or origin _____
Coordinate Rule: ____$(x, y) \rightarrow (2x, 2y)$_____

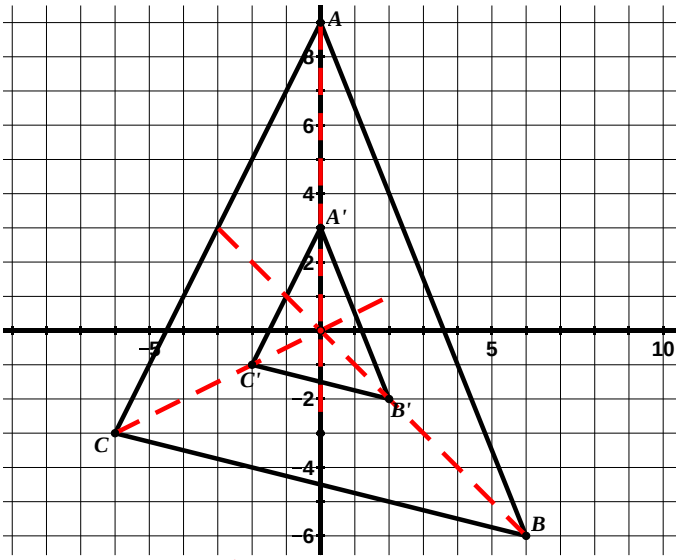
4. $LMNO$ has been dilated to obtain $L'M'N'O'$.




Scale Factor: ___$\frac{1}{2}$_____
Center of Dilation: _____$(-6, 3)$_____

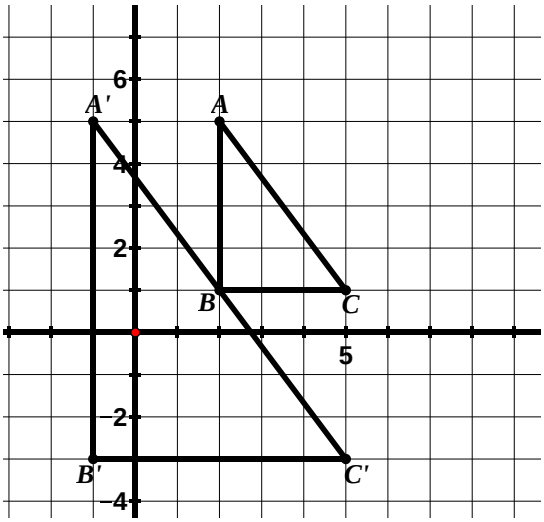
For #3 and 4, have students find the ratio of the image segment to its corresponding pre-image segment to see that they are all the same and equal to the scale factor.

5. In the picture below, $\Delta ABC$ has been dilated to obtain $\Delta A'B'C'$.




Scale Factor: ___$\frac{1}{3}$_____
Center of Dilation: __ origin _____
Coordinate Rule: _____$(x, y) \rightarrow (\frac{1}{3}x, \frac{1}{3}y)$____

6. $\Delta ABC$ has been dilated to obtain $\Delta A'B'C'$.




Scale Factor: ___2_____
Center of Dilation: _____(5, 5)_____
Honor's Extension: Coordinate Rule: $(x, y) \rightarrow ((5x - 5), (5y - 5))$



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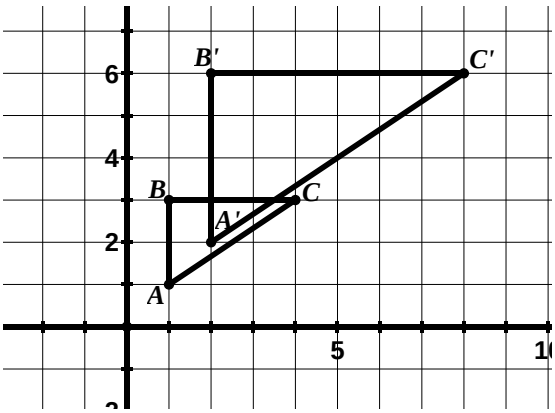
**9.2b Homework: Properties of Dilations**

**Directions:** In the following problems, one figure has been dilated to obtain a new figure. Determine the scale factor and center of dilation. Determine the coordinate rule for the dilation when asked.

1. In the picture below, $\Delta ABC$ has been dilated to obtain $\Delta A'B'C'$.

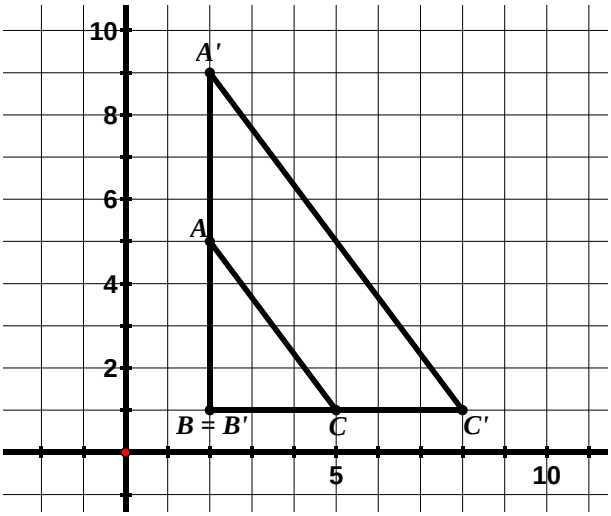



Scale Factor: __2_____

Center of Dilation: _____ origin _____

Coordinate Rule: _____$(x, y) \rightarrow (2x, 2y)$__

2. In the picture below, $\Delta ABC$ has been dilated to obtain $\Delta A'B'C'$.

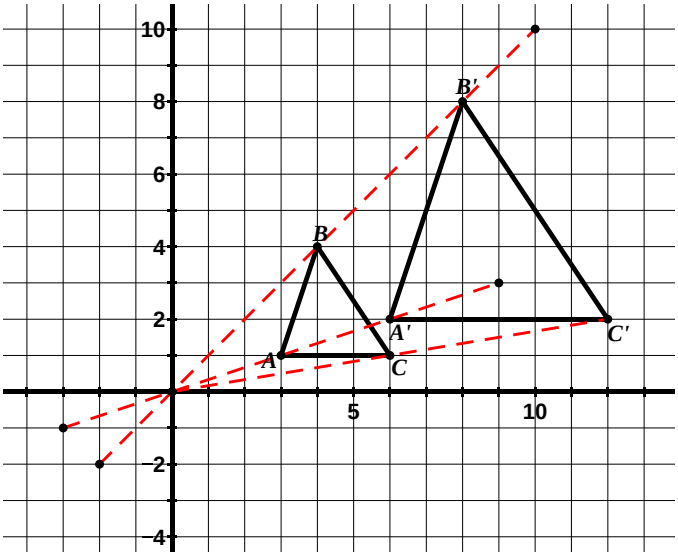



Scale Factor: ___2____

Center of Dilation: _____ *B* _____

Honor's Extension: Coordinate Rule: _$(x, y) \rightarrow ((2x - 2), (2x - 1))$

3. In the picture below, $\Delta ABC$ has been dilated to obtain $\Delta A'B'C'$.

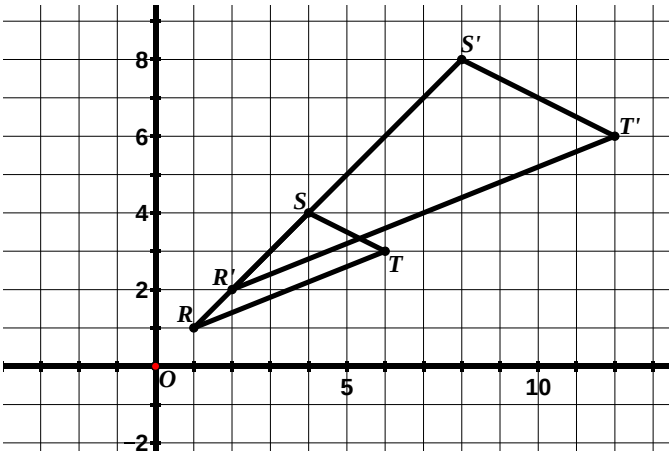



Scale Factor: ___2_____

Center of Dilation: ____ origin _____

Coordinate Rule: _____$(x, y) \rightarrow (2x, 2y)$

4. In the picture below, $\Delta ABC$ has been dilated to obtain $\Delta A'B'C'$.




Scale Factor: ___2_____

Center of Dilation: ____ origin _____

Coordinate Rule: ___$(x, y) \rightarrow (2x, 2y)$



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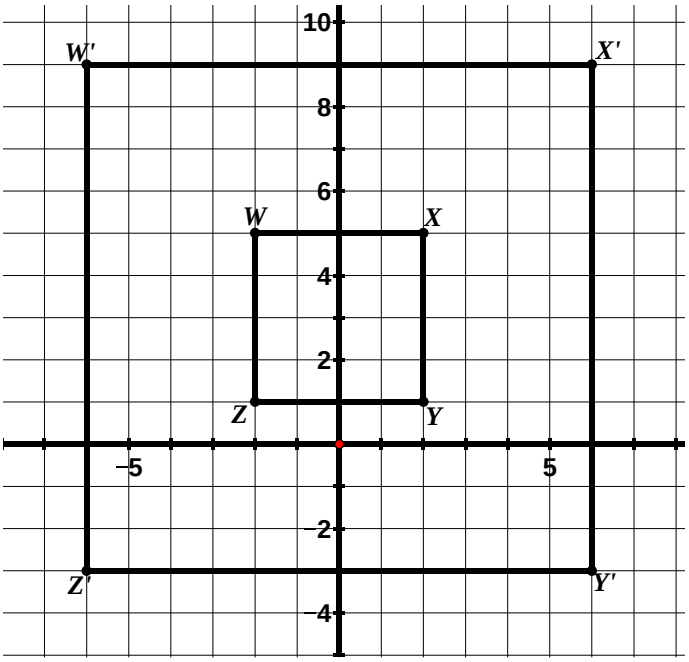
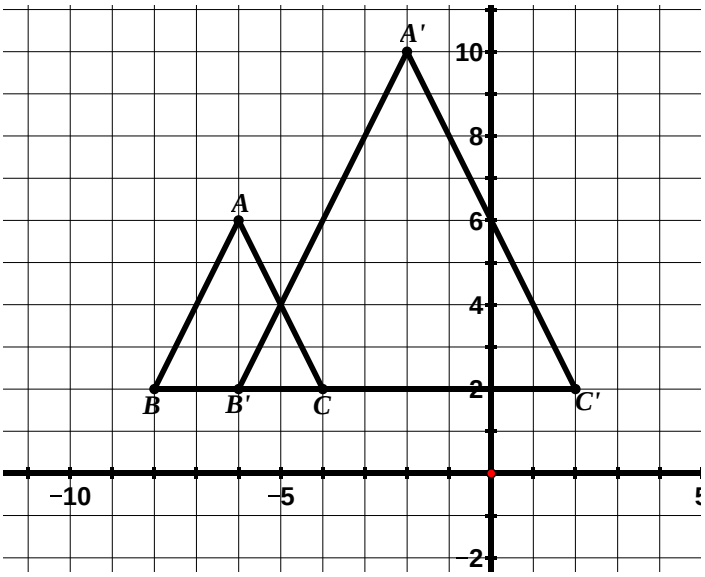
5. $WXYZ$ has been dilated to obtain $W'X'Y'Z'$.




Scale Factor: ___3_____

Center of Dilation: _____$(0, 3)$_____

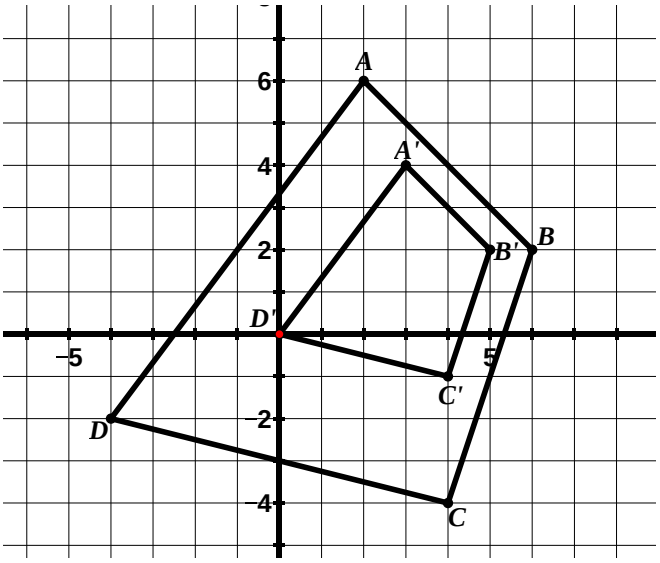
6. $\Delta ABC$ has been dilated to obtain $\Delta A'B'C'$.




Scale Factor: ____2_____

Center of Dilation: _____$(-10, 2)$_____

7. $ABCD$ has been dilated to obtain $A'B'C'D'$.




Scale Factor: ___$\frac{1}{2}$_____

Center of Dilation: ___(4, 2)_____



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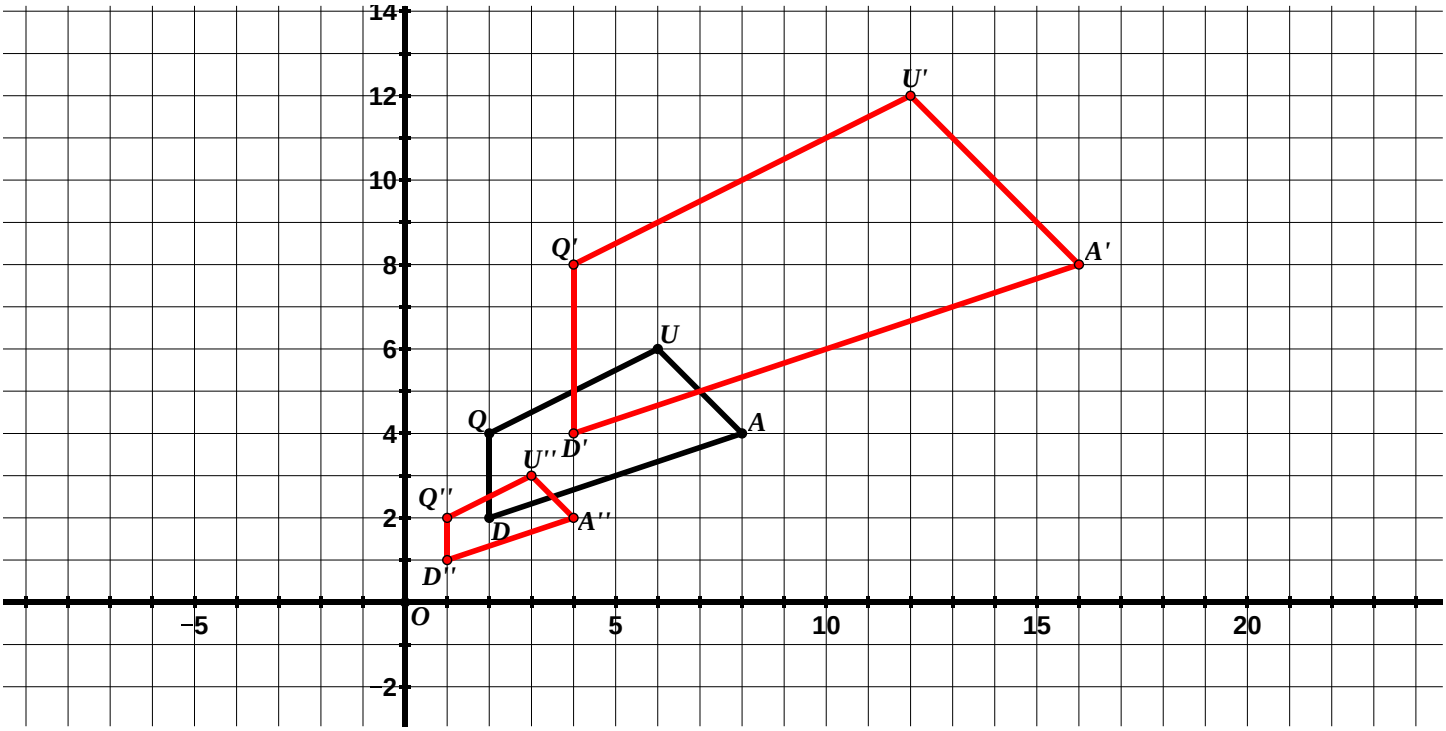
**9.2c Class Activity: Dilations cont.**

1. $QUAD$ is graphed below.




a. Create a new quadrilateral whose side lengths are two times larger than the side lengths of $QUAD$ with the center of dilation at the origin and label the image $Q'U'A'D'$. In the space below, describe the method you used to create your new quadrilateral.
Have students share their strategies. Some may use the slope triangle method. Others may have realized that since the center of dilation is the origin, they can double the coordinates of the pre-image vertices and graph these new vertices.

b. Based on what we have learned so far about dilations, what are some different ways you can verify that the side lengths of your new quadrilateral are in fact two times larger than the side lengths of the original?

Discuss with students. Are the image points twice the distance from the center of dilation as compared to the pre-image points? Are the segments in the image twice the length as the corresponding segments in the pre-image? You may also want to ask them how they can verify in another way that they used the origin as the center of dilation. All lines connecting corresponding vertices should intersect at the origin.

c. This time, create a quadrilateral whose sides lengths are $\frac{1}{2}$ the size of the side lengths of $QUAD$ with the center of dilation at the origin and label the image $Q''U''A''D''$.
You may consider asking students questions similar to the ones above.



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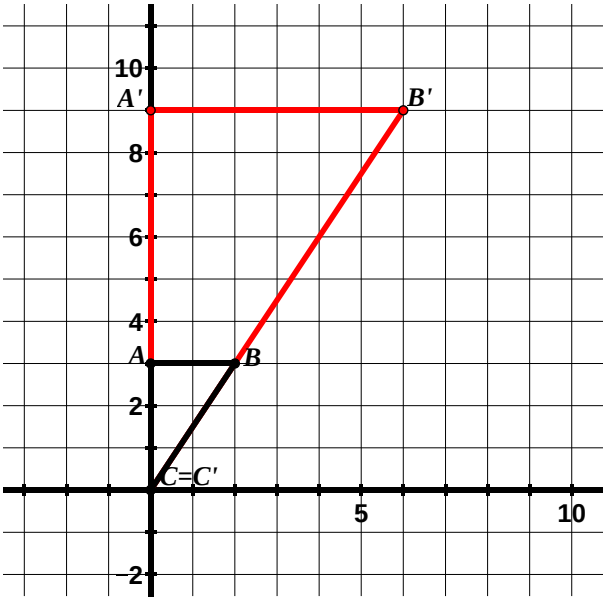
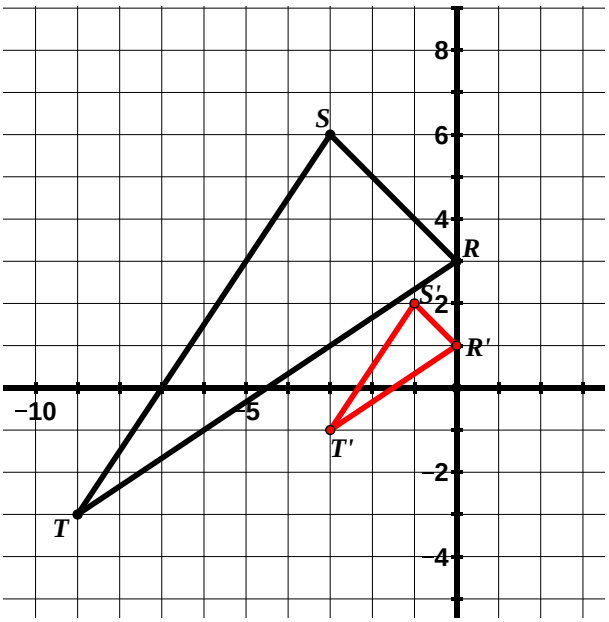
**Directions:** Find the image of each figure for a dilation with the given center and scale factor. Draw and label the image.

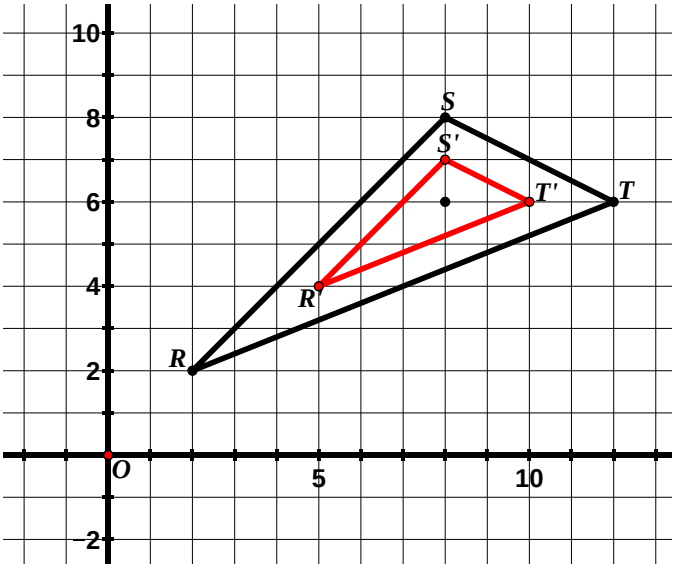
2. $r = 3$ Center of Dilation: $C$



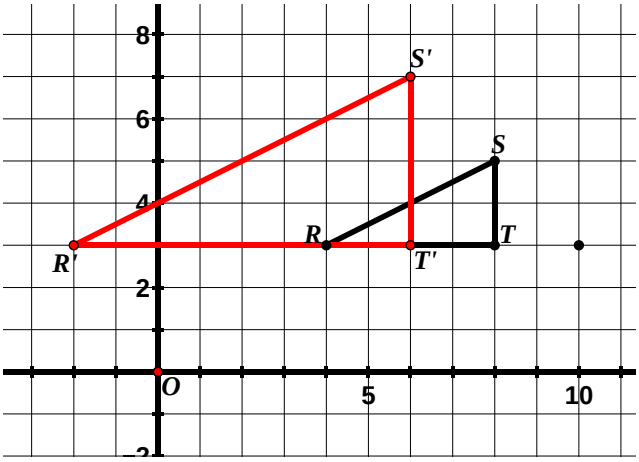

3. $r = \frac{1}{3}$ Center of Dilation: origin




4. $r = \frac{1}{2}$ Center of Dilation: $(8, 6)$.




5. $r = 2$ Center of Dilation: (10, 3)



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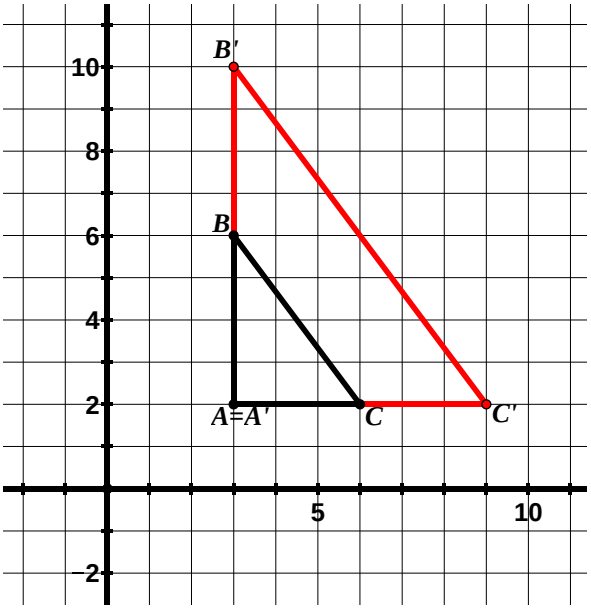
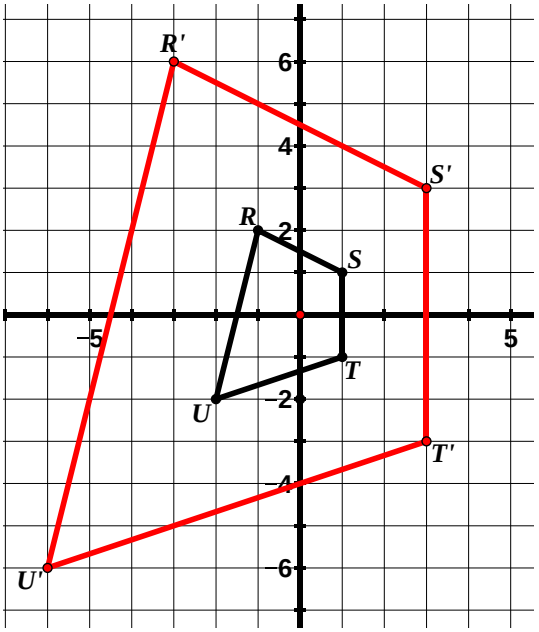
**9.2c Homework: Dilations cont.**

**Directions:** Find the image of each figure for a dilation with the given center and scale factor. Draw and label the image.

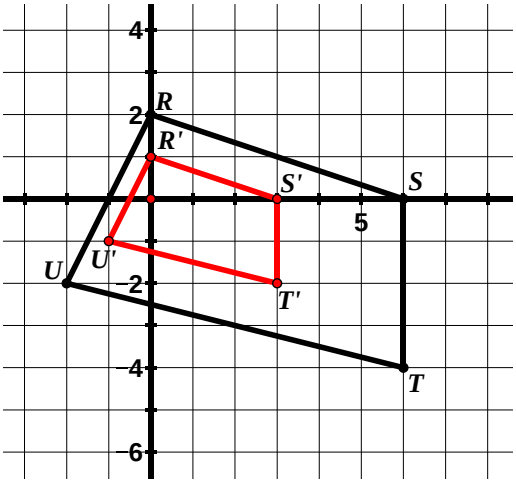
1. $r = 2$ Center of Dilation: $A$



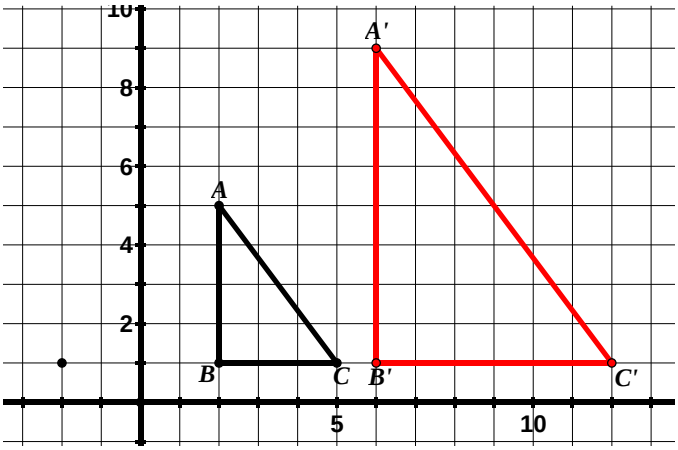

2. $r = 3$ Center of Dilation: origin




3. $r = \frac{1}{2}$ Center of Dilation: origin




4. $r = 2$ Center of Dilation: $(-2, 1)$



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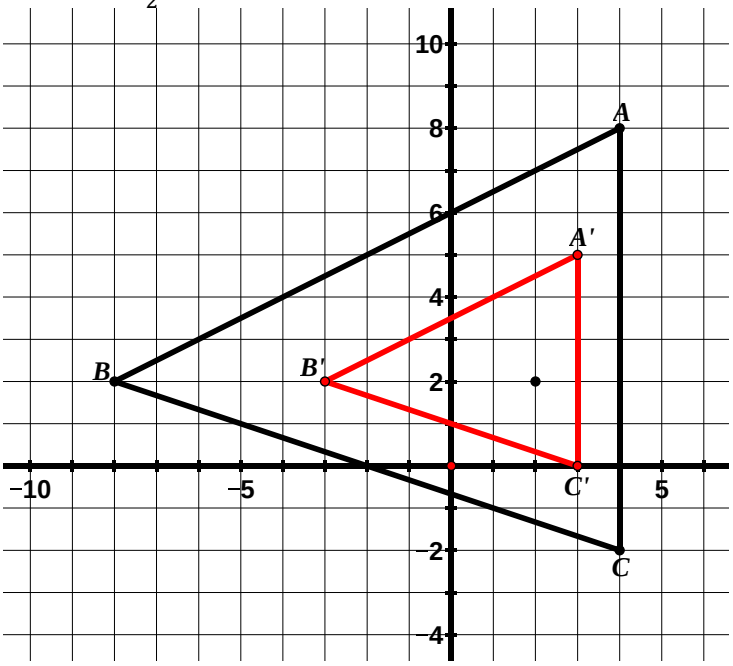
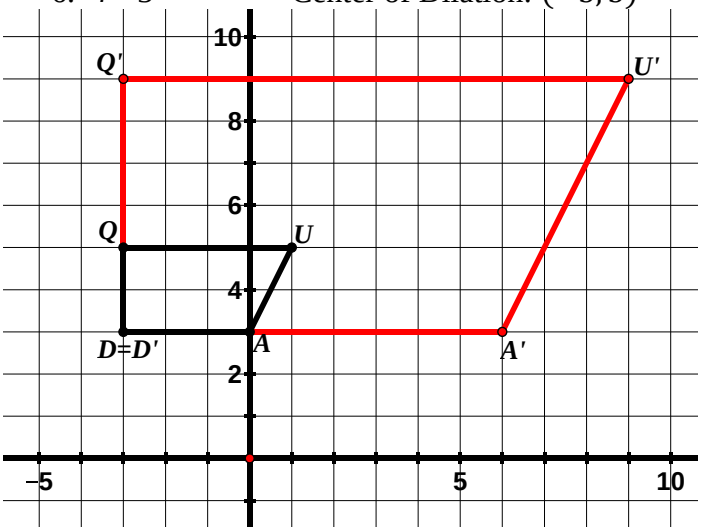
5. $r = \frac{1}{2}$ Center of Dilation: (2, 2)




6. $r = 3$ Center of Dilation: $(-3, 3)$



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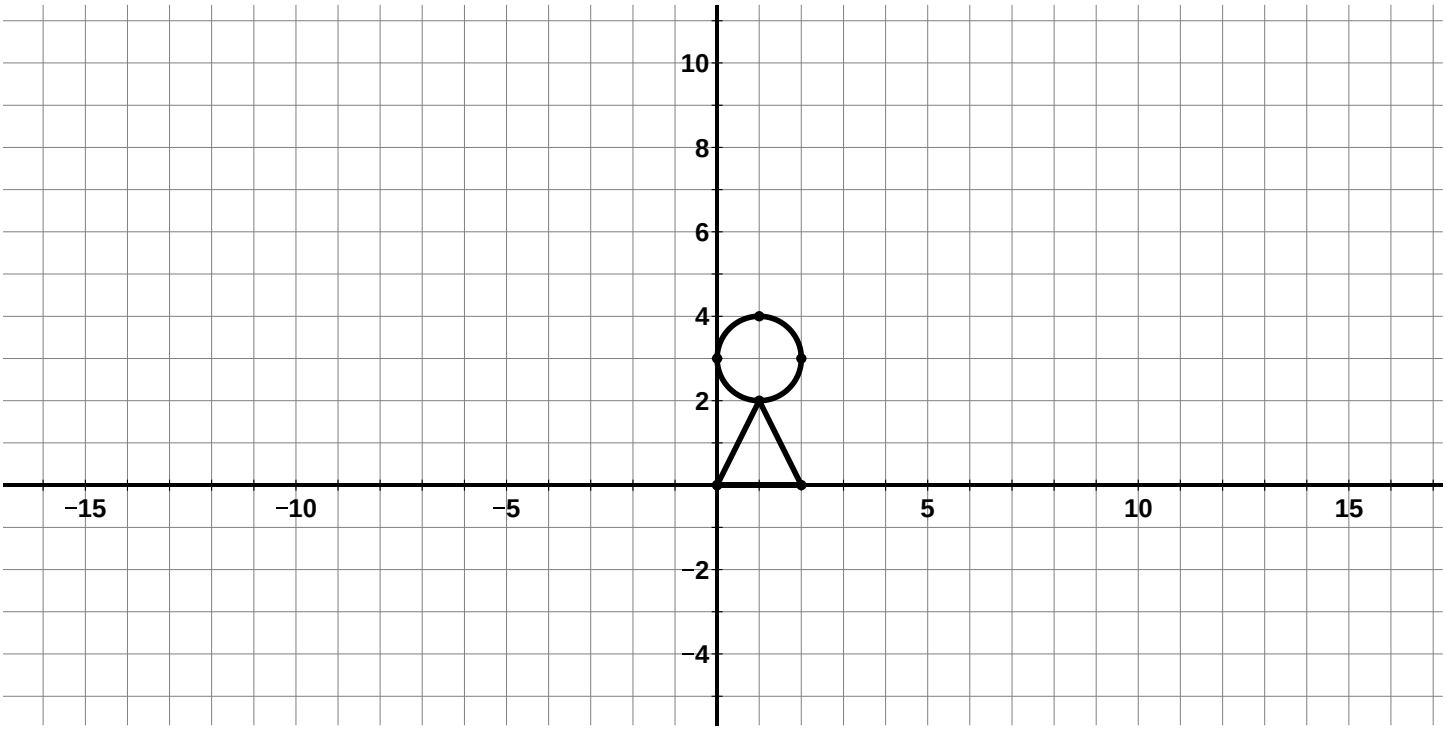
## 9.2d Class Activity: Problem Solving with Dilations

1. Where would you put your center of dilation to enlarge the picture below by a factor of 2 and put the image entirely in the second quadrant? Mark your center of dilation with an *O* and then do the dilation.




Answers will vary. One possible answer is $(6, -2)$. Have students share the strategies they use to solve this problem. Students may also choose to work out this problem on a separate piece of graph paper in order to give them more room.



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2. A dilation with the center of dilation at the origin maps $\Delta ABC$ to $\Delta A'B'C'$.
   a. If $AB = 3$ and $A'B' = 6$, what is the scale factor of the dilation? 2

   b. If $B'C' = 8$, what is the length of $BC$? 4

   c. If $AC = 5$, what is the length of $A'C'$? 10

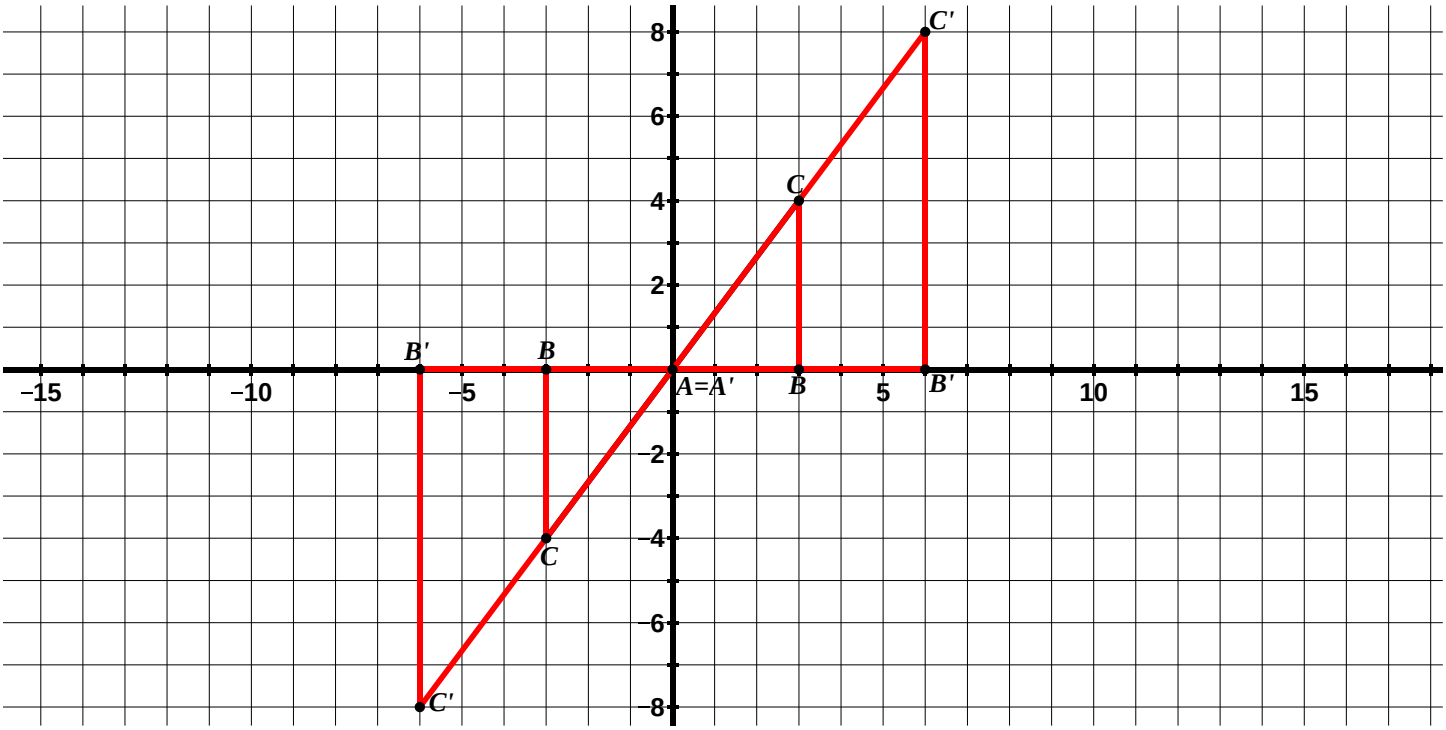
   d. If the slope of $\overline{AB}$ is 0, what is the slope of $\overline{A'B'}$? 0

   e. If the slope of $\overline{A'C'}$ is $\frac{4}{3}$, what is the slope of $\overline{AC}$? $\frac{4}{3}$

   f. Create a picture of this dilation on the grid below using the information from parts *a* – *e* and the additional pieces of information below. *Remember that the center of dilation is the origin.*

      - The slope of $CB$ is undefined
      - *A* is at the origin

There are two possible answers shown below. For students who finish early, encourage them to find the second answer.



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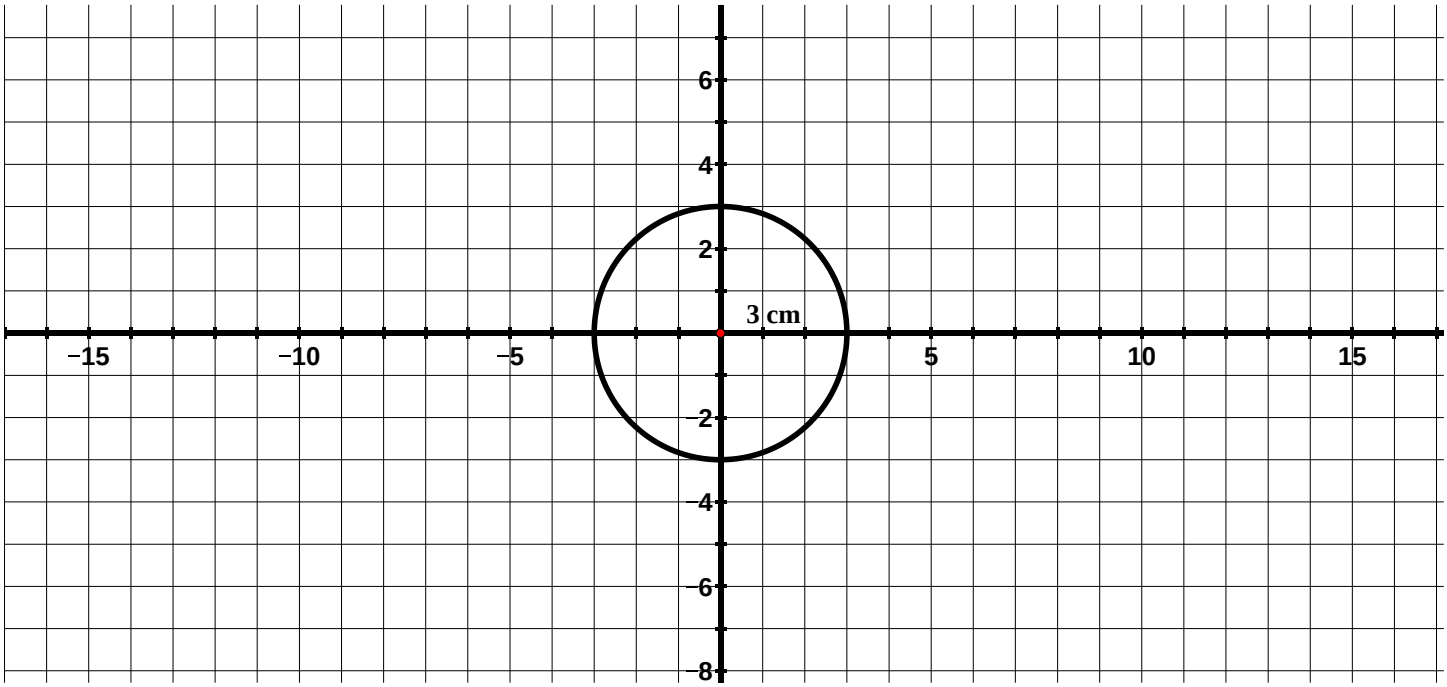
3. A circle with a radius of 3 cm is shown below.




a. Determine the length of the radius of a circle whose circumference would be twice as large as the circle pictured above. 6 cm

b. Determine the length of the radius of a circle whose area would be twice as large as the circle pictured above. $3\sqrt{2}$ cm



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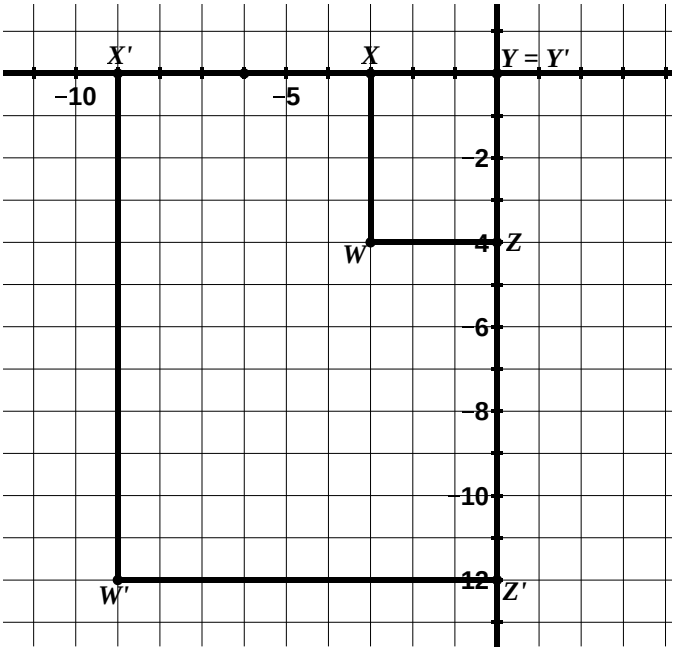
**9.2d Homework: Review of Dilations**

**Directions:** In the following problems, one figure has been dilated to obtain a new figure. Determine the scale factor and center of dilation. Determine the coordinate rule for the dilation when asked.

1. In the picture below, $WXYZ$ has been dilated to obtain $W'X'Y'Z'$.

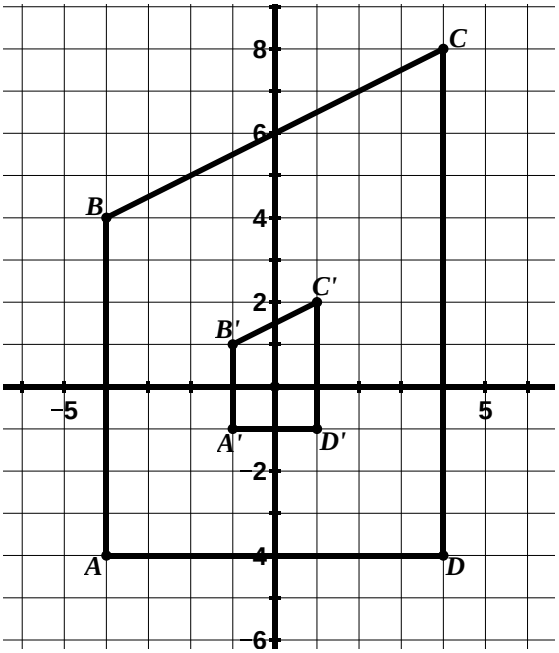



Scale Factor: ___3_____

Center of Dilation: _____ $Y$ _____

Coordinate Rule: ____$(x, y) \rightarrow (3x, 3y)$

2. In the picture below, $ABCD$ has been dilated to obtain $A'B'C'D'$.

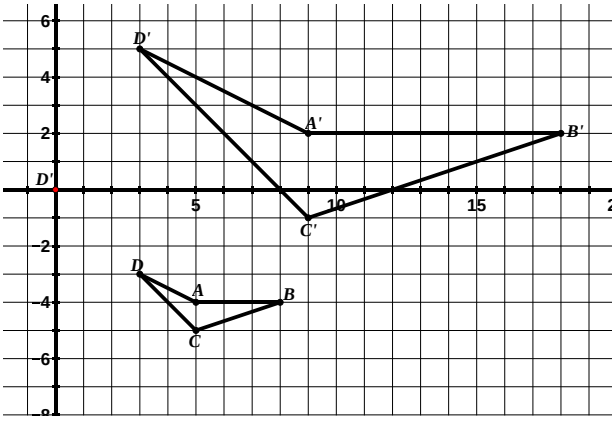



Scale Factor: ___$\frac{1}{4}$_____

Center of Dilation: __ origin _____

Coordinate Rule: $(x, y) \rightarrow (\frac{1}{4}x, \frac{1}{4}y)$

3. $ABCD$ has been dilated to obtain $A'B'C'D'$.




Scale Factor: __3_____

Center of Dilation: _$(3, -7)$_____

Honors: Coordinate Rule: $(x, y) \rightarrow (3x - 6, 3y + 14)$



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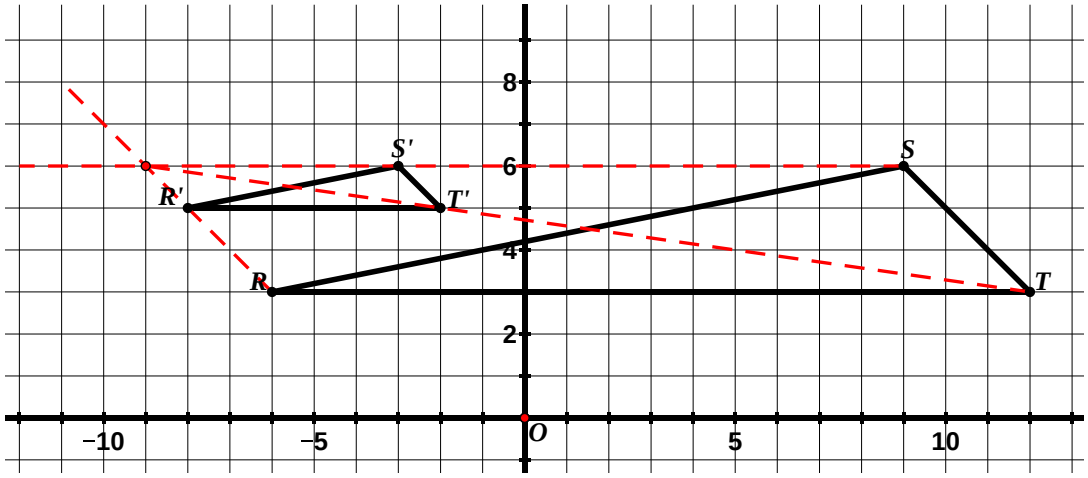
4. $\Delta RST$ has been dilated to obtain $\Delta R'S'T'$.

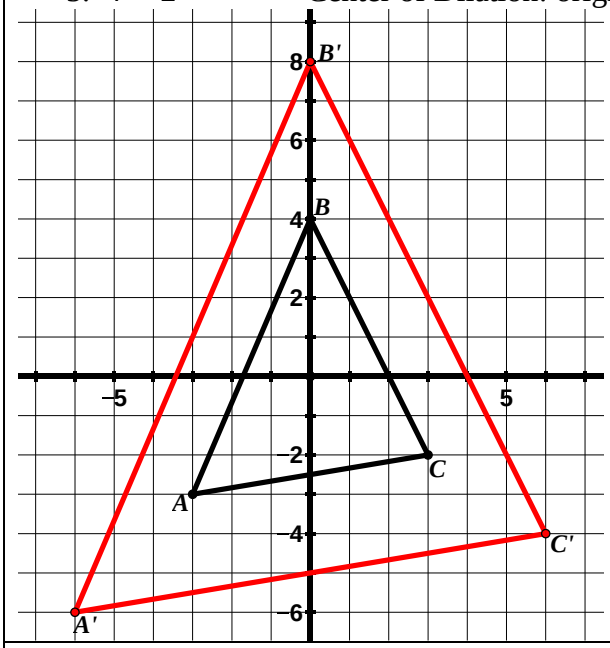
Scale Factor: $\frac{1}{3}$

Center of Dilation: $(-9, 6)$

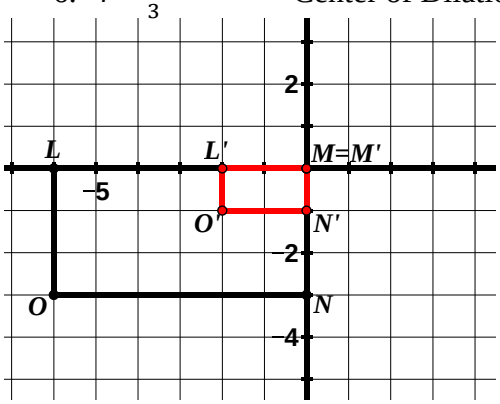



**Directions:** For #5 – 7, find the image of each figure for a dilation with the given center and scale factor. Draw and label the image.

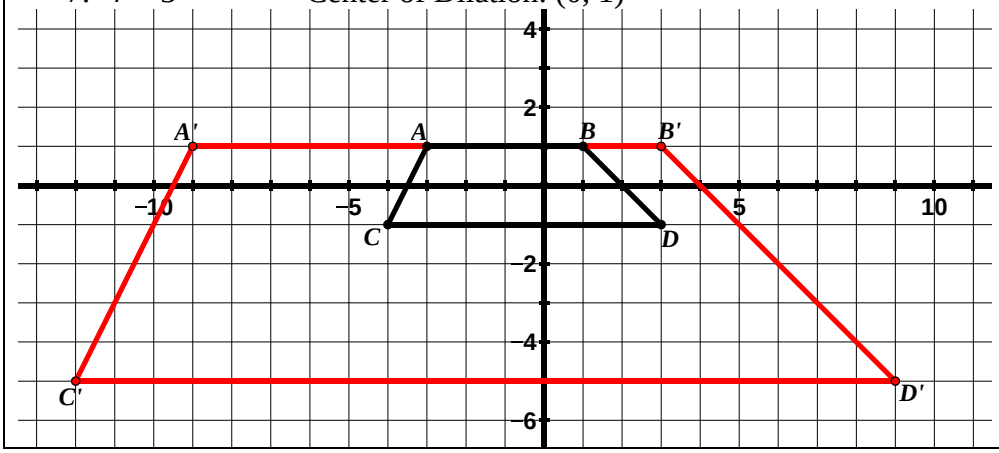
5. $r = 2$ Center of Dilation: origin




6. $r = \frac{1}{3}$ Center of Dilation: origin




7. $r = 3$ Center of Dilation: (0, 1)



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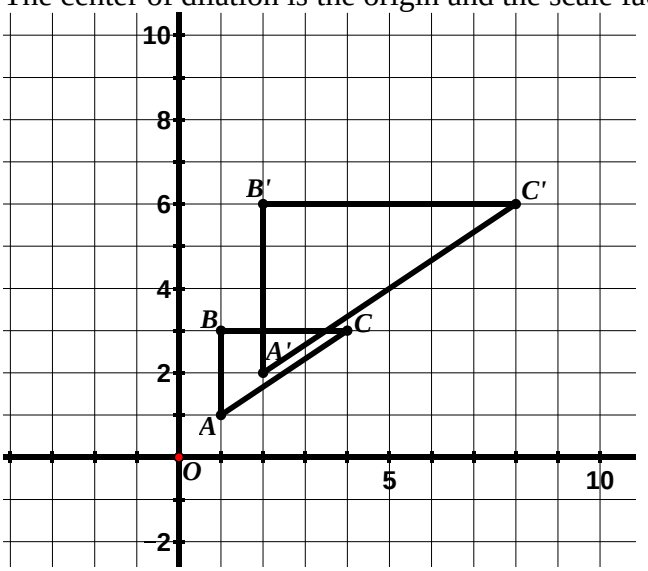
**9.2e Class Activity: Similarity**

In the first part of the chapter, we discussed congruence. Two figures are **congruent** if one can be obtained from the other by a rigid motion (rotation, reflection, or translation) or a sequence of rigid motions. If you can move one of the figures using one of these transformation or a series of these transformations so that it fits exactly on the other one, the two figures are congruent.

In this section we have seen problems where two figures are similar. In 7th grade, you learned that two figures are similar if they have the **same shape** – they may or may not be the same size. In 8th grade, we define similarity in terms of transformations. Two figures are said to be **similar** if there is a sequence of rigid motions and dilations that take one figure onto the other.

While studying dilations, we have learned that (1) a dilation creates a figure that is the same shape as the original figure but a different size, (2) the measure of corresponding angles is the same and (3) the ratios of corresponding sides are all the same. Since similar figures are produced by a dilation, these properties, as well as some others we observed, also hold true for similar figures.

Let's revisit a problem we have seen before. In the picture below, $\Delta ABC$ has been dilated to obtain $\Delta A'B'C'$. The center of dilation is the origin and the scale factor is 2.




Because $\Delta A'B'C'$ was produced by a dilation of $\Delta ABC$, the two triangles are similar. We can write a similarity statement for the two triangles. You can denote that two figures are similar by using the symbol $\sim$ and listing their vertices in corresponding order.

Write a similarity statement for the two triangles. $\Delta ABC \sim \Delta A'B'C'$
The order the vertices is written tells us which segments and angles are corresponding in the figures.

When two figures are similar, corresponding angles are congruent and corresponding sides are proportional. Write the congruent statements to represent this.

$\angle A \cong \angle A'$
$\angle B \cong \angle B'$
$\angle C \cong \angle C'$

The ratio of the lengths of the corresponding sides is a similarity ratio. Write the similarity ratio for these two triangles.

$$\frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{A'C'}{AC} = 3$$



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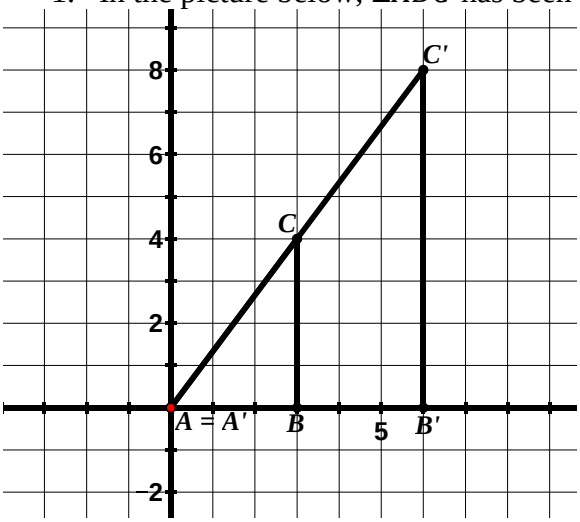
1. In the picture below, $\Delta ABC$ has been dilated to obtain $\Delta A'B'C'$. The center of dilation is the origin.




a. Write a similarity statement for the triangles.
$\Delta ABC \sim \Delta A'B'C'$

b. Complete each statement:

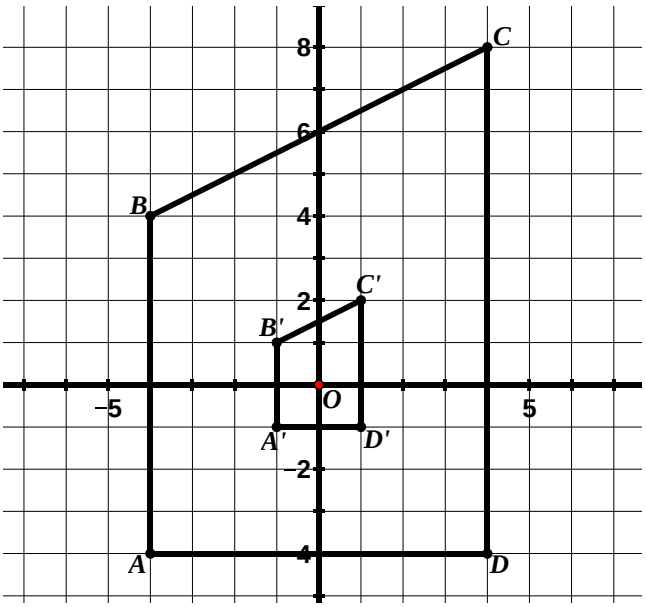
$m\angle C \cong m\angle C'$

If $m\angle B = 90°$, then $m\angle B' = 90°$

$\frac{A'C'}{AC} = 2$

$\frac{A'B'}{AB} = 2$

2. In the picture below, $ABCD$ has been dilated to obtain $A'B'C'D'$. The center of dilation is the origin.




a. Write a similarity statement for the trapezoids.
$ABCD \sim \Delta A'B'C'D'$

b. Complete each statement.

$m\angle C \cong m\angle C'$

If $m\angle A = 90°$, then $m\angle A' = m\angle A' = 90°$

$\frac{A'D'}{AD} = \frac{1}{4}$

$\frac{A'B'}{AB} = \frac{1}{4}$

Make it a point that the similarity statement does not always start with "A".
Denote the difference between = and ≅ and the measure of an angle versus naming the angle.



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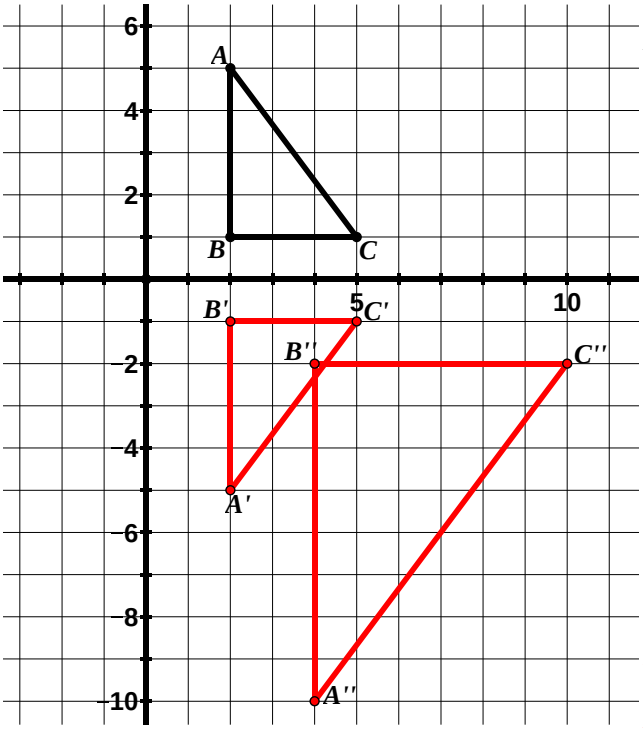
3. $\Delta ABC$ is graphed on the grid below.




a. Reflect $\Delta ABC$ over the $x$-axis. Label the new triangle $\Delta A'B'C'$.

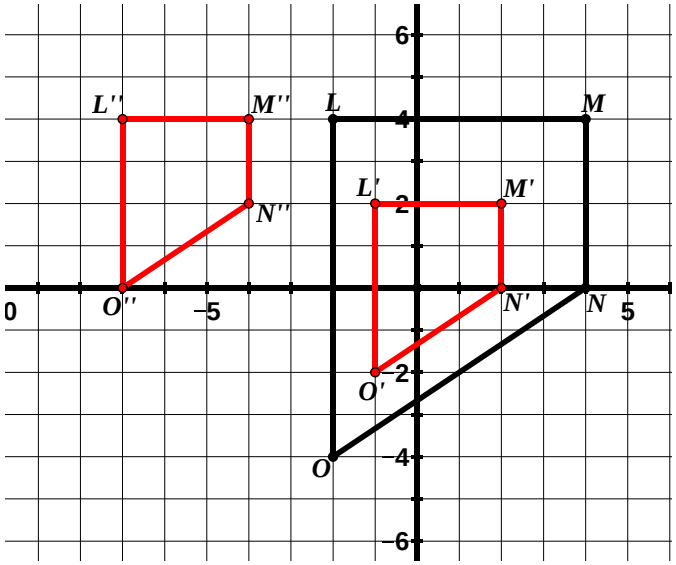
b. Dilate $\Delta A'B'C'$ by a scale factor of 2 with the center of dilation at the origin. Label the new triangle $\Delta A''B''C''$.

c. Write a statement that shows the relationship between $\Delta ABC$ and $\Delta A'B'C'$.
$\Delta ABC \cong \Delta A'B'C'$
It is also true that $\Delta ABC \sim \Delta A'B'C'$ (the sequence of rigid motions and dilations need not include dilations)

d. Write a statement that shows the relationship between $\Delta A'B'C'$ and $\Delta A''B'C''$.
$\Delta A'B'C' \sim \Delta A''B''C''$

e. List five other relationships between corresponding angles, sides, etc. that you know to be true based on the relationships in parts c and d.
Answers are vast and may vary. Here are a few sample answers
$\frac{A''C''}{A'C'} = 2$ $\quad m\angle C \cong m\angle C' \cong m\angle C''$ $\quad B'C' \cong BC$

4. $LMNO$ is graphed on the grid below.




a. Dilate $LMNO$ by a scale factor of $\frac{1}{2}$ with the center of dilation at the origin. Label the new quadrilateral $L'M'N'O'$.

b. Translate $L'M'N'O'$ according to the rule $(x, y) \rightarrow (x - 6, y + 2)$. Label the new quadrilateral $L''M''N''O''$.

c. Write a statement that shows the relationship between $LMNO$ and $L'M'N'O'$.
$LMNO \sim L'M'N'O'$.

d. Write a statement that shows the relationship between $L'M'N'O'$ and $L''M''N''O''$.
$L'M'N'O' \cong L''M''N''O''$ and
$L'M'N'O' \sim L''M''N''O''$ (see part c. in previous problem)

e. List five other relationships between corresponding angles, sides, etc. that you know to be true based on the relationships in parts c and d.
Answers may vary. See part e. in previous

You may have to emphasize here that relationships are similarity and congruence statements.



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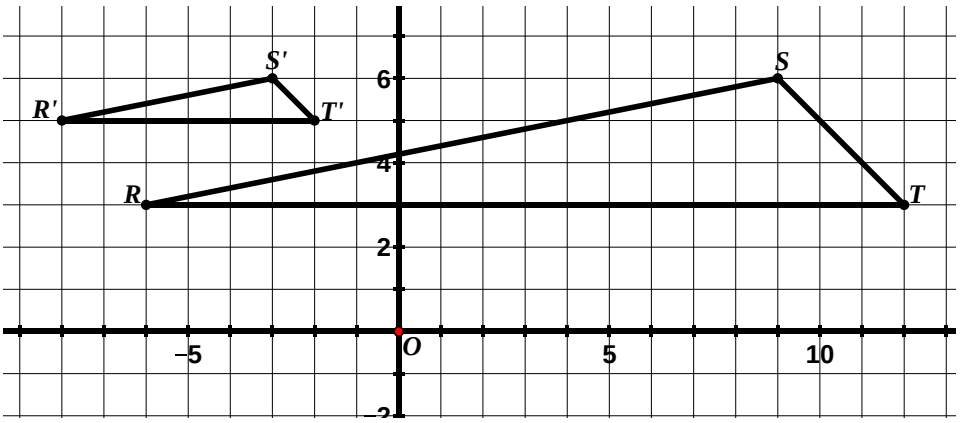
## 9.2e Homework: Similarity

1. In the picture below ΔRST has been dilated to obtain ΔR′S′T′.




a. Write a similarity statement for the triangles.
$\Delta RST \sim \Delta R'S'T'$

b. Complete each statement:

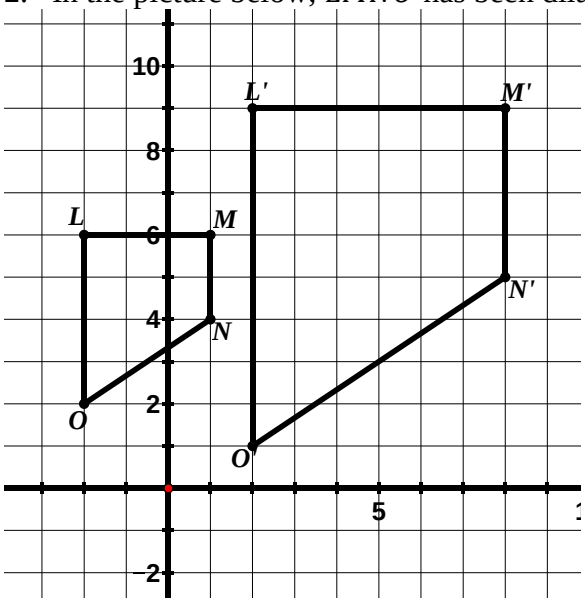
$m\angle S \cong m\angle S'$

If $m\angle T = 45^\circ$, then $m\angle T' = 45^\circ$

$\frac{R'T'}{RT} = \frac{1}{3}$

$\frac{R'S'}{RS} = \frac{1}{3}$

2. In the picture below, $LMNO$ has been dilated to obtain $L'M'N'O'$. The center of dilation is the origin.




a. Write a similarity statement for the triangles.
$LMNO \sim L'M'N'O'$.

b. Complete each statement.

$m\angle O \cong m\angle O'$

If $m\angle L = 90^\circ$, then $m\angle L' = 90^\circ$

$\frac{O'N'}{ON} = 2$

$\frac{L'O'}{LO} = 2$



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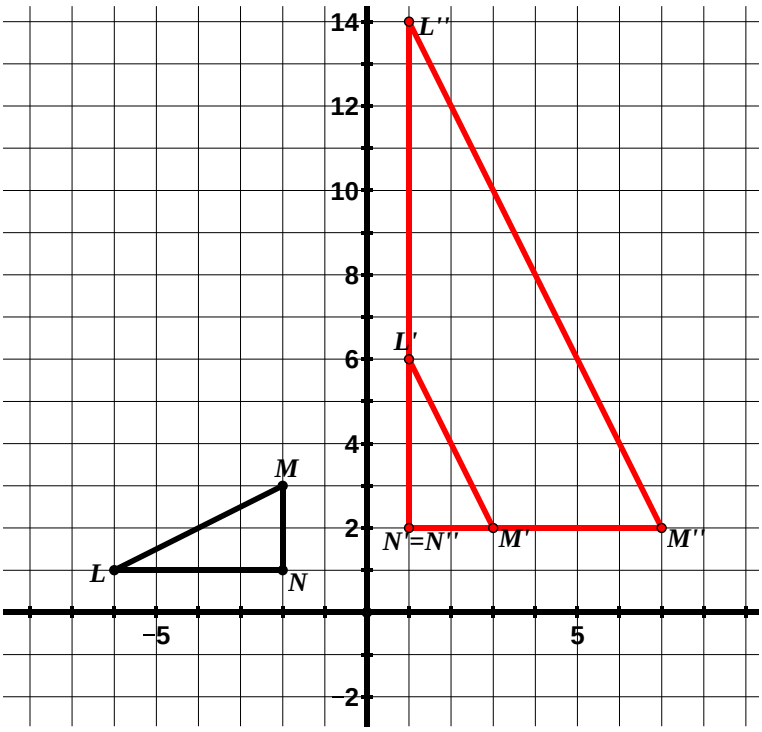
3. $\Delta LMN$ is graphed on the grid below.




a. Rotate $\Delta LMN$ 90˚clockwise about the origin. Label the new triangle $\Delta L'M'N'$.

b. Dilate $\Delta L'M'N'$ by a scale factor of 3 with the center of dilation at (1, 2). Label the new triangle $\Delta L''M''N''$.

c. Write a statement that shows the relationship between $\Delta LMN$ and $\Delta L'M'N'$.

$\Delta LMN \cong \Delta L'M'N'$

It is also true that $\Delta LMN \sim \Delta L'M'N'$ (the sequence of rigid motions and dilations need not include dilations)

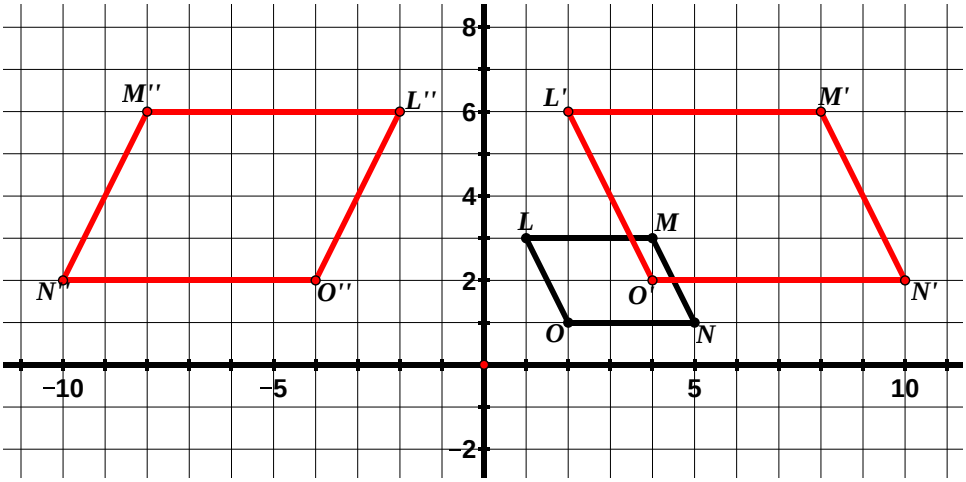
d. Write a statement that shows the relationship between $\Delta L'M'N'$ and $\Delta L''M''N''$.

$\Delta L'M'N' \sim \Delta L''M''N''$

e. List five other relationships between corresponding angles, sides, etc. that you know to be true based on the relationships in parts c and d.

Answers will vary

4. $LMNO$ is graphed on the grid below.




a. Dilate $LMNO$ by a scale factor of 2 with the center of dilation at the origin. Label the new quadrilateral $L'M'N'O'$.

b. Reflect $L'M'N'O'$ across the $y$-axis. Label the new quadrilateral $L''M''N''O''$.

c. Write a statement that shows the relationship between $LMNO$ and $L'M'N'O'$.

$LMNO \sim L'M'N'O'$

d. Write a statement that shows the relationship between $L'M'N'O'$ and $L''M''N''O''$.

$L'M'N'O' \cong L''M''N''O''$ or $L'M'N'O' \sim L''M''N''O''$



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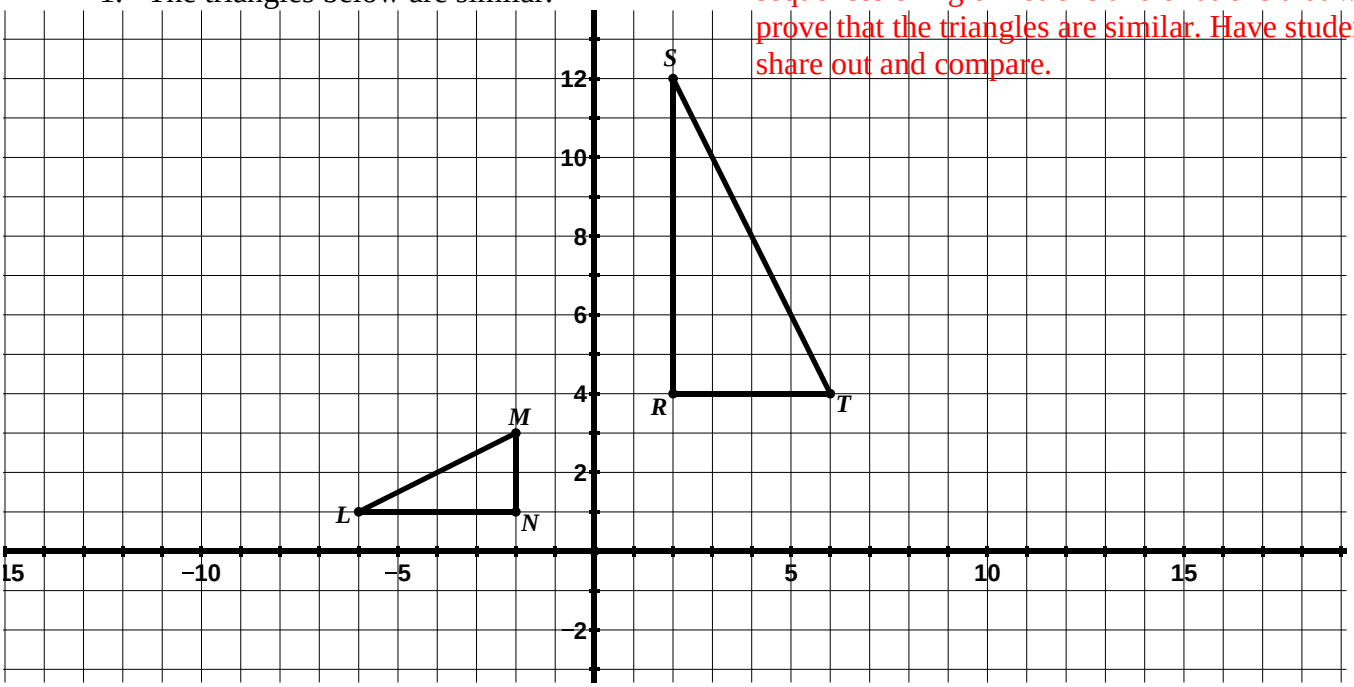
**9.2f Class Activity: Similarity cont.**

For the following section, there are many sequences of rigid motions and dilations that will prove that the triangles are similar. Have students share out and compare.

1. The triangles below are similar.



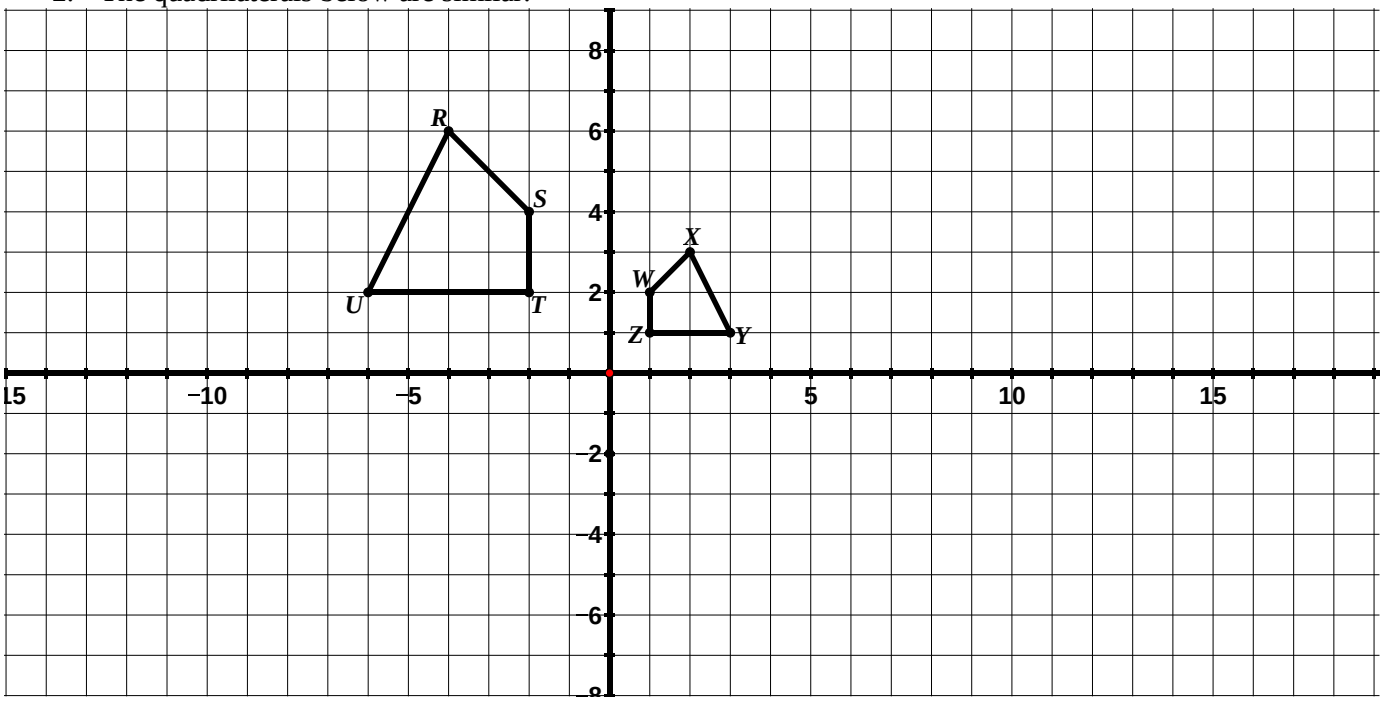

a. List the sequence of transformations that verifies the similarity of the two figures.

One possible answer: a rotation of $\Delta LMN$ 90˚ clockwise about the origin followed by a translation according to the rule $(x, y) \rightarrow (x + 1, y + 2)$ followed by a dilation with center of dilation at $R$ and scale factor of 2.

b. Write a similarity statement for the triangles. $\Delta NML \sim \Delta RTS$

Students may list vertices in a different order but they should be written in corresponding order

2. The quadrilaterals below are similar.




a. List the sequence of transformations that verifies the similarity of the two figures.

One possible answer: a reflection of $RSTU$ across the $y$-axis followed by a dilation with the center of dilation at the origin and a scale factor of $\frac{1}{2}$

b. Write a similarity statement for the quadrilaterals. $RSTU \sim XWZY$



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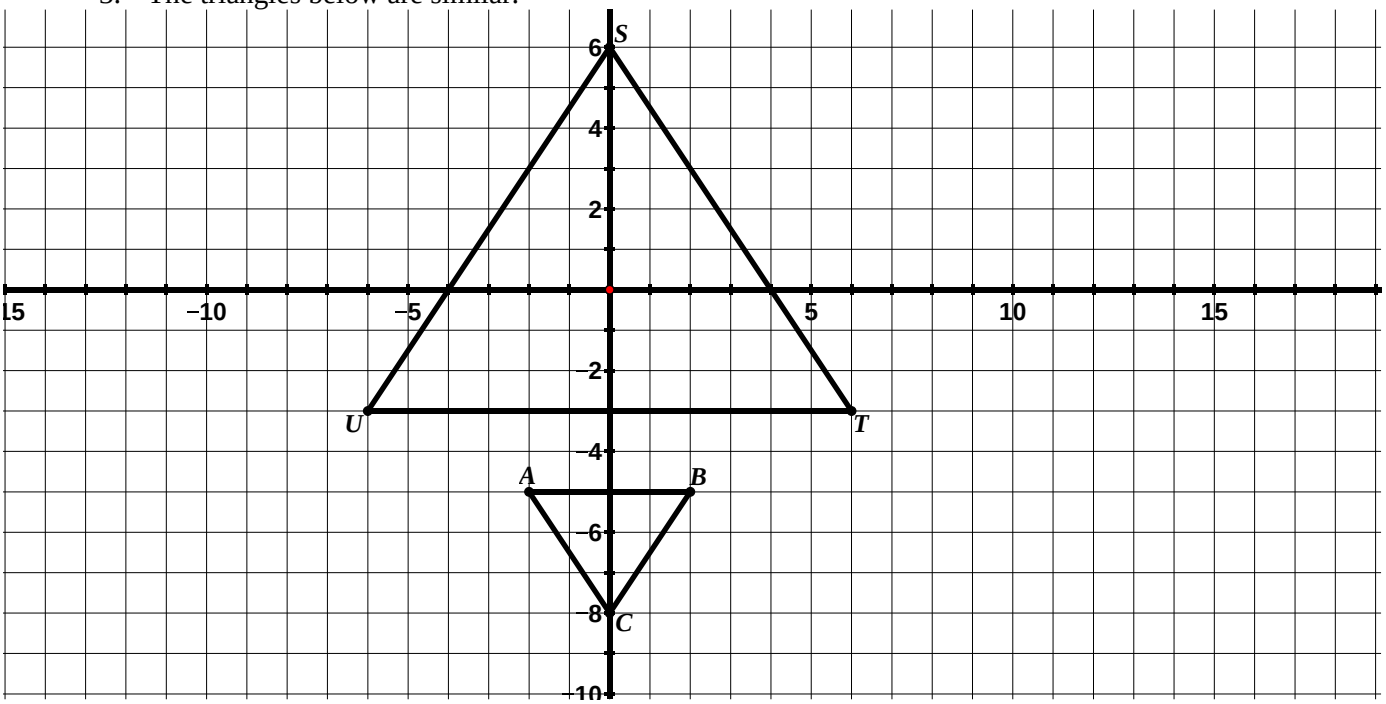
3. The triangles below are similar.



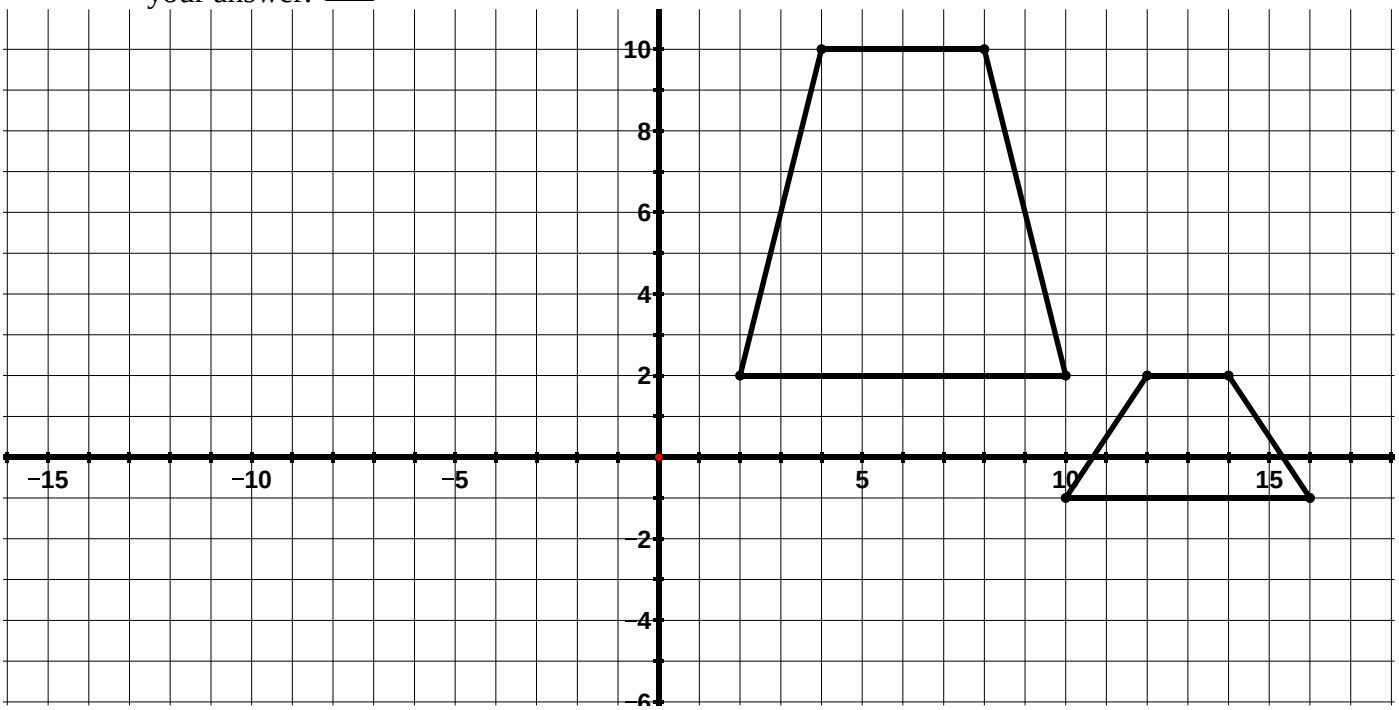

a. List the sequence of transformations that verifies the similarity of the two figures.

One possible answer. A reflection of $\Delta ABC$ across the line $y = -4$ followed by a translation according to the rule $(x, y) \rightarrow (x - 4, y)$ followed by a dilation with center at $U$ and scale factor of 3

b. Write a similarity statement for the triangles.

$\Delta ABC \sim \Delta UTS$

4. Determine whether the figures below are congruent, similar, or neither. Provide a justification for your answer.




Students may reason that these figures are not similar because the ratio of sides is not the same. They may also line up angles and see that they are not congruent.



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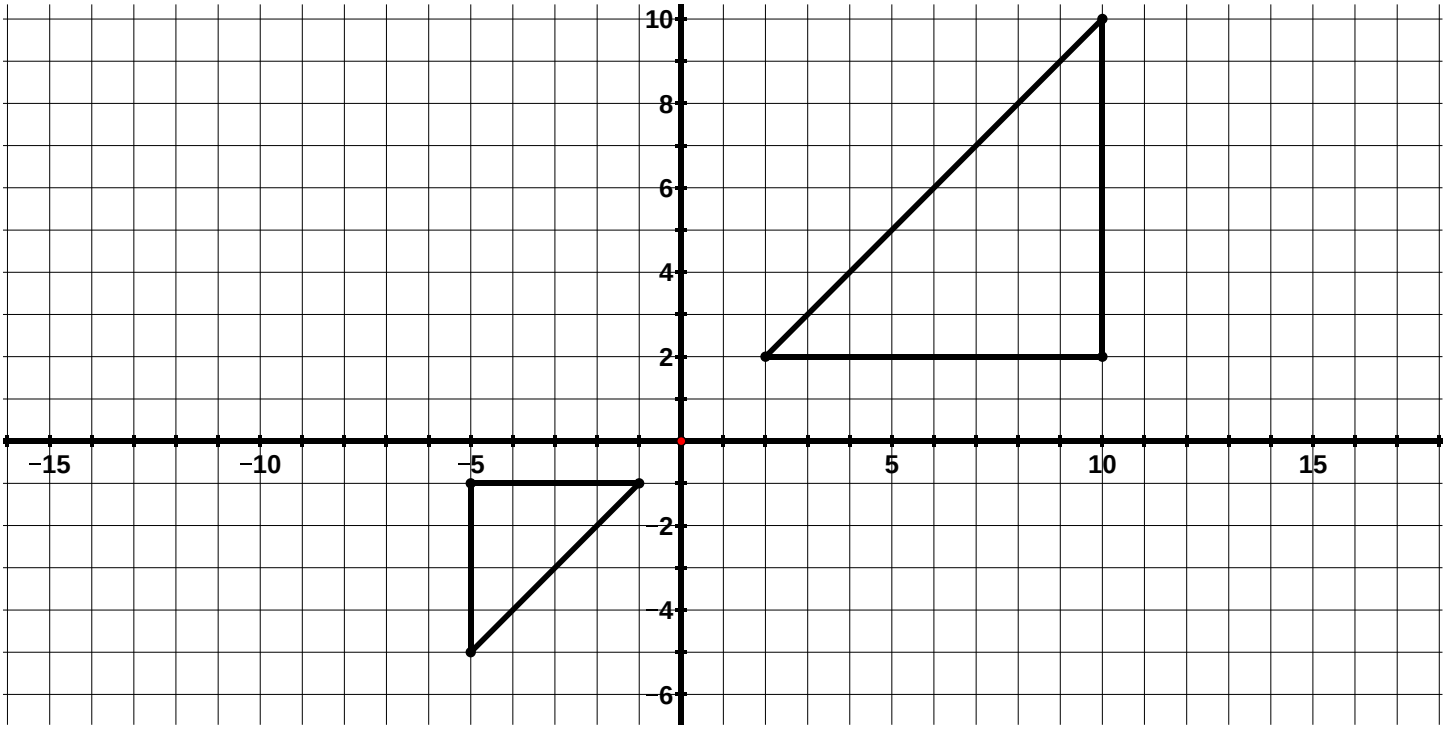
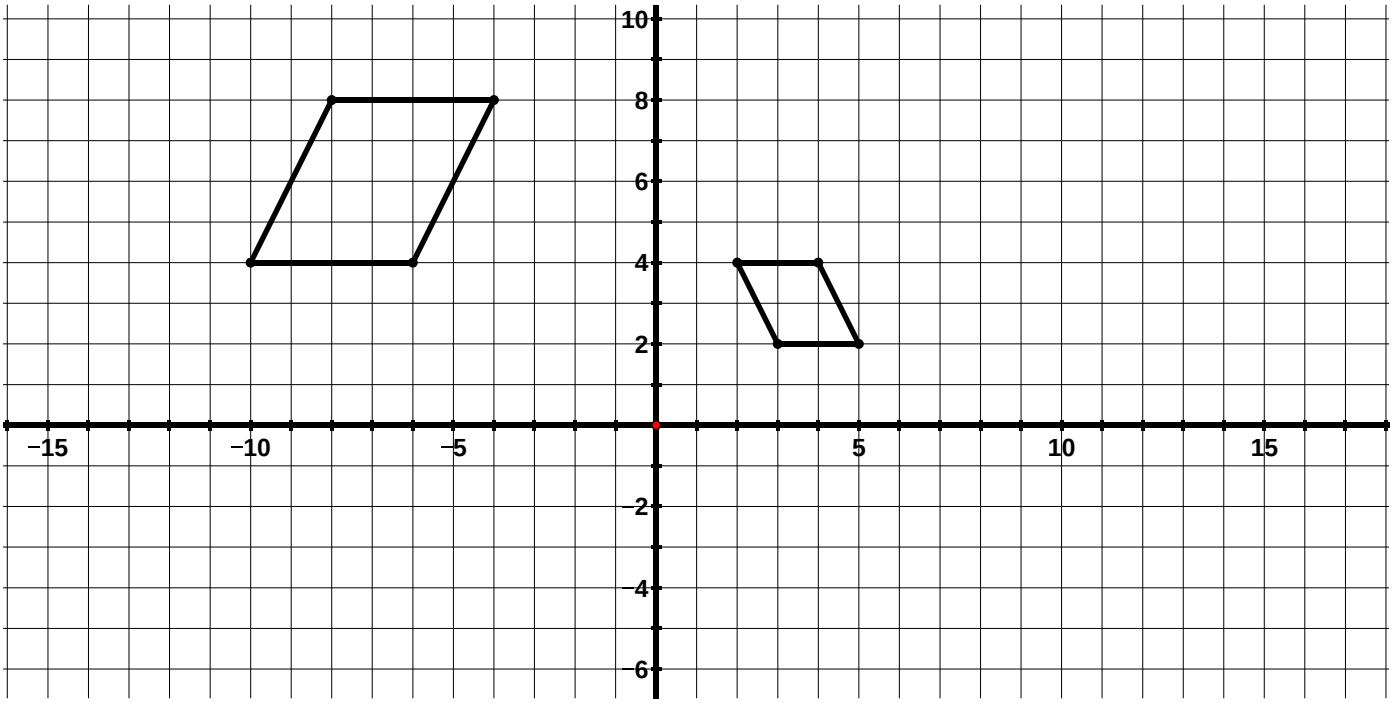
5. Determine whether the figures below are congruent, similar, or neither. Provide a justification for your answer.




The two figures are similar. The smaller triangle can be mapped to the larger triangle through a 180˚ rotation followed by a dilation with center at the origin and scale factor of 2 (justifications may vary)

6. Determine whether the figures below are congruent, similar, or neither. Provide a justification for your answer.




The figures are similar. The larger parallelogram can be mapped to the smaller one by a reflection across the $y$-axis followed by a dilation with center at $(6, 4)$ and scale factor $= \frac{1}{2}$ followed by a translation according to the rule $(x, y) \rightarrow ((x-3), (y-2))$. Justifications may vary



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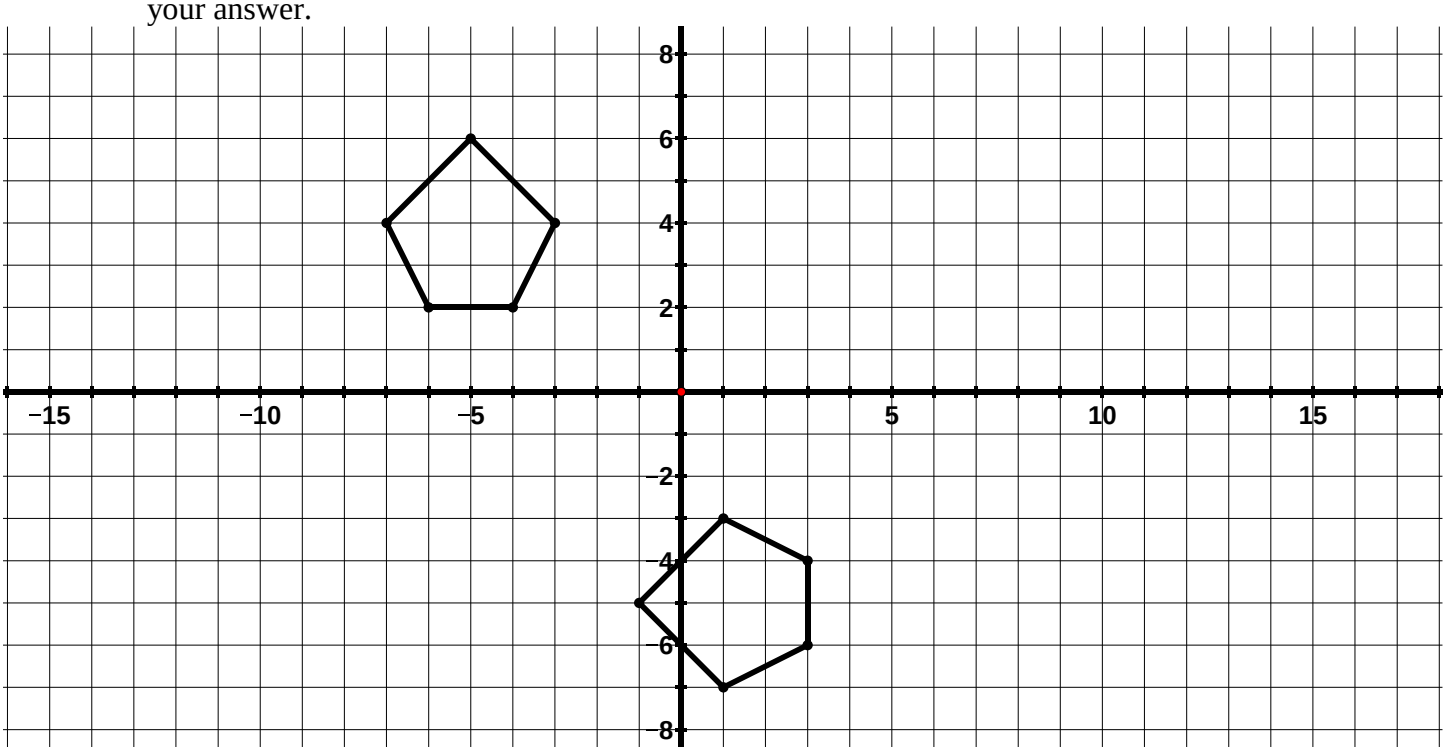
7. Determine whether the figures below are congruent, similar, or neither. Provide a justification for your answer.




The two figures are congruent (and similar). You can map the figure in the second quadrant to the other figure by a rotation of 90˚counterclockwise about the origin followed by a translation according to the rule $(x, y) \rightarrow (x + 5, y)$

8. Determine whether the following transformations or sequence of transformations will result in similar figures, congruent figures, or neither. Provide a justification for your answer.

   a. $(x, y) \rightarrow (x - 6, y + 2)$ congruent and similar

   b. $(x, y) \rightarrow (-x, y)$ followed by $(x, y) \rightarrow (2x, 2y)$ similar

   c. $(x, y) \rightarrow (2x, 3y)$ followed by a reflection across the $x$-axis neither

   d. $(x, y) \rightarrow (x + 5, y + 5)$ followed by a 90˚ rotation counterclockwise about the origin congruent and similar

   e. $(x, y) \rightarrow \left(\frac{1}{4}x, \frac{1}{4}y\right)$ followed by $(x, y) \rightarrow (y, x)$ similar

   f. $(x, y) \rightarrow (x, y + 4)$ followed by a 180˚ rotation clockwise about the origin congruent and similar



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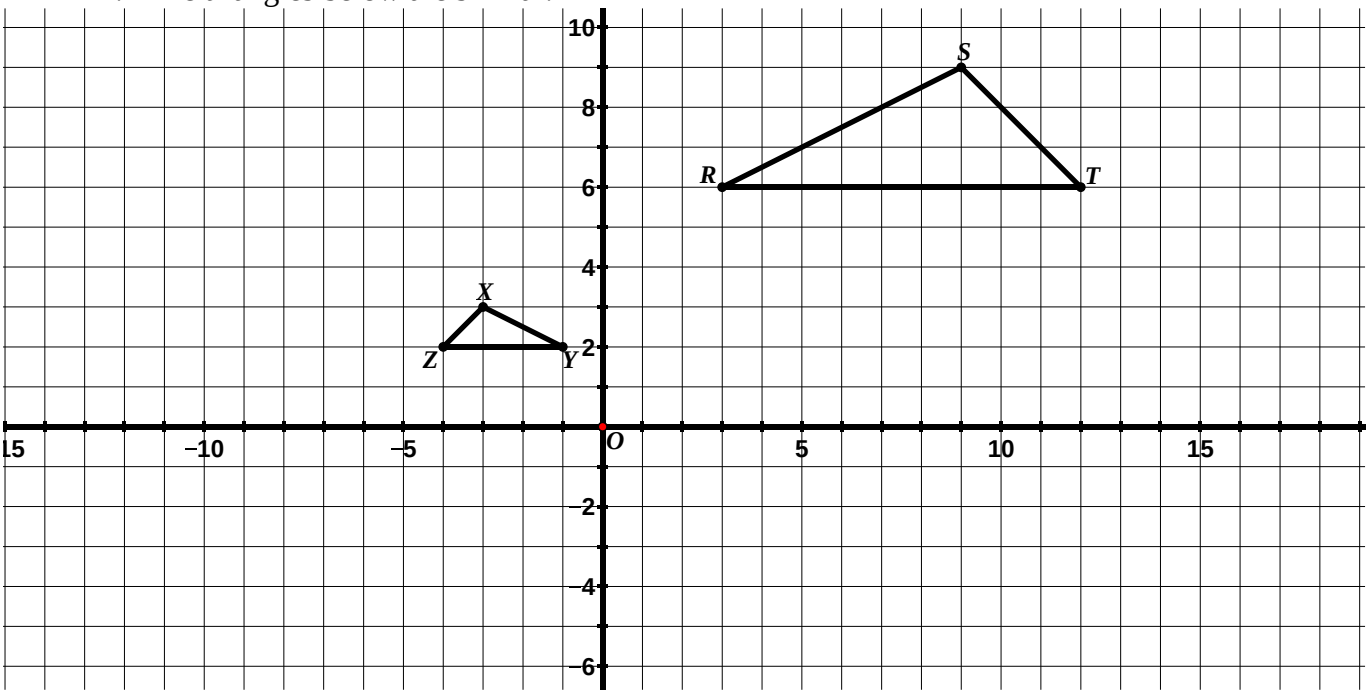
**9.2f Homework: Similarity cont.**

1. The triangles below are similar.

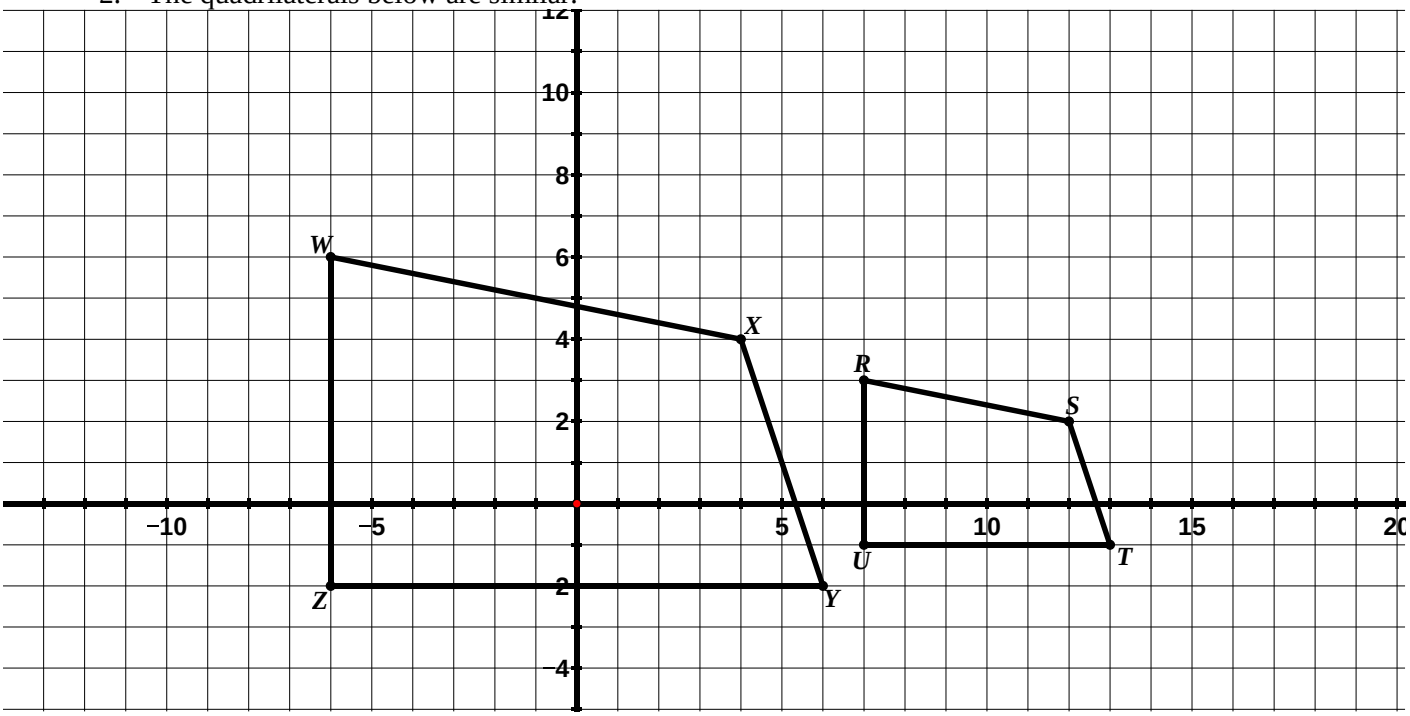



a. List the sequence of transformations that verifies the similarity of the two figures.

Many answers. One possible answer. A dilation of $\Delta XYZ$ with center of dilation at origin and scale factor of 3 followed by a reflection across the $y$-axis.

b. Write a similarity statement for the triangles. $\Delta XYZ \sim \Delta SRT$

2. The quadrilaterals below are similar.




a. List the sequence of transformations that verifies the similarity of the two figures.

A translation of $WXYZ$ according to the rule $(x, y) \rightarrow (x + 13, y + 1)$ followed by a dilation with center at $U$ and scale factor of $\frac{1}{2}$

b. Write a similarity statement for the quadrilaterals. $WXYZ \sim RSTU$



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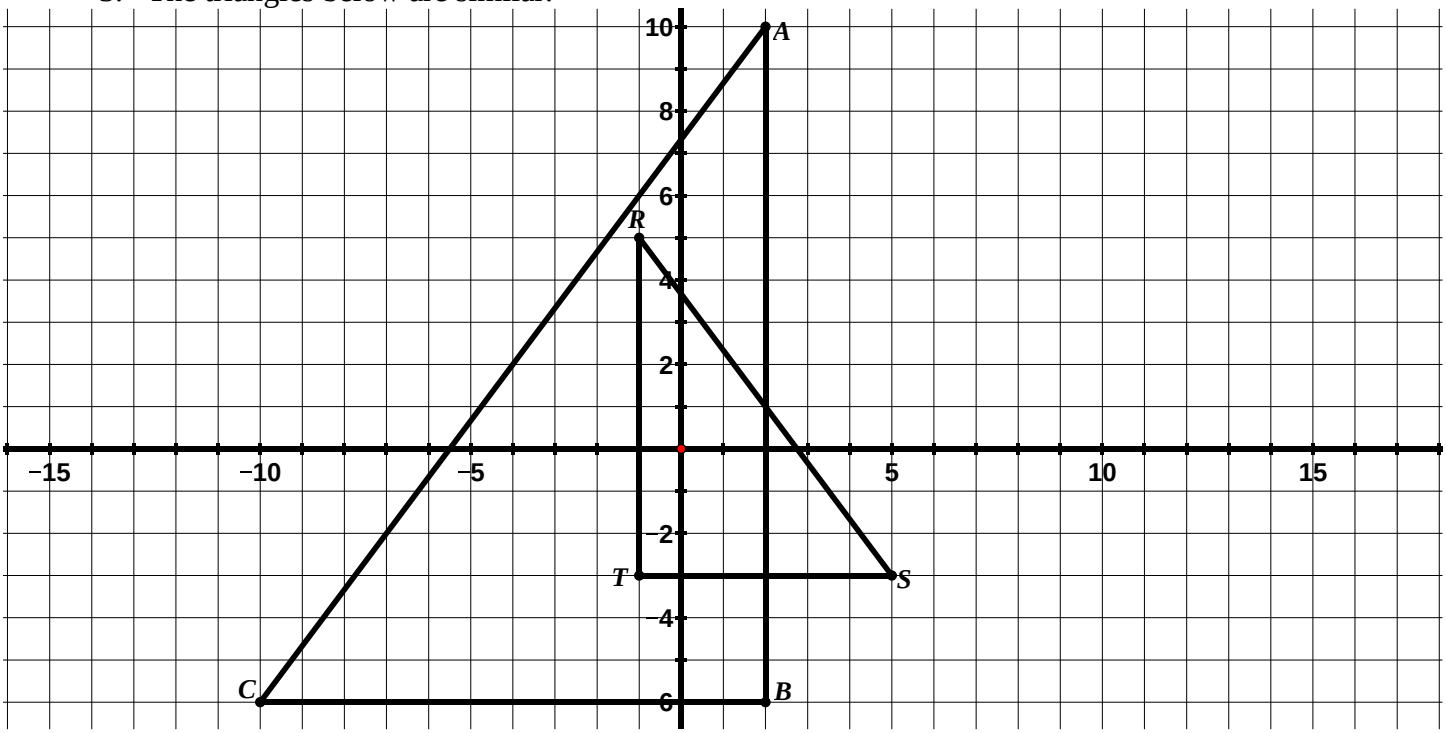
3. The triangles below are similar.



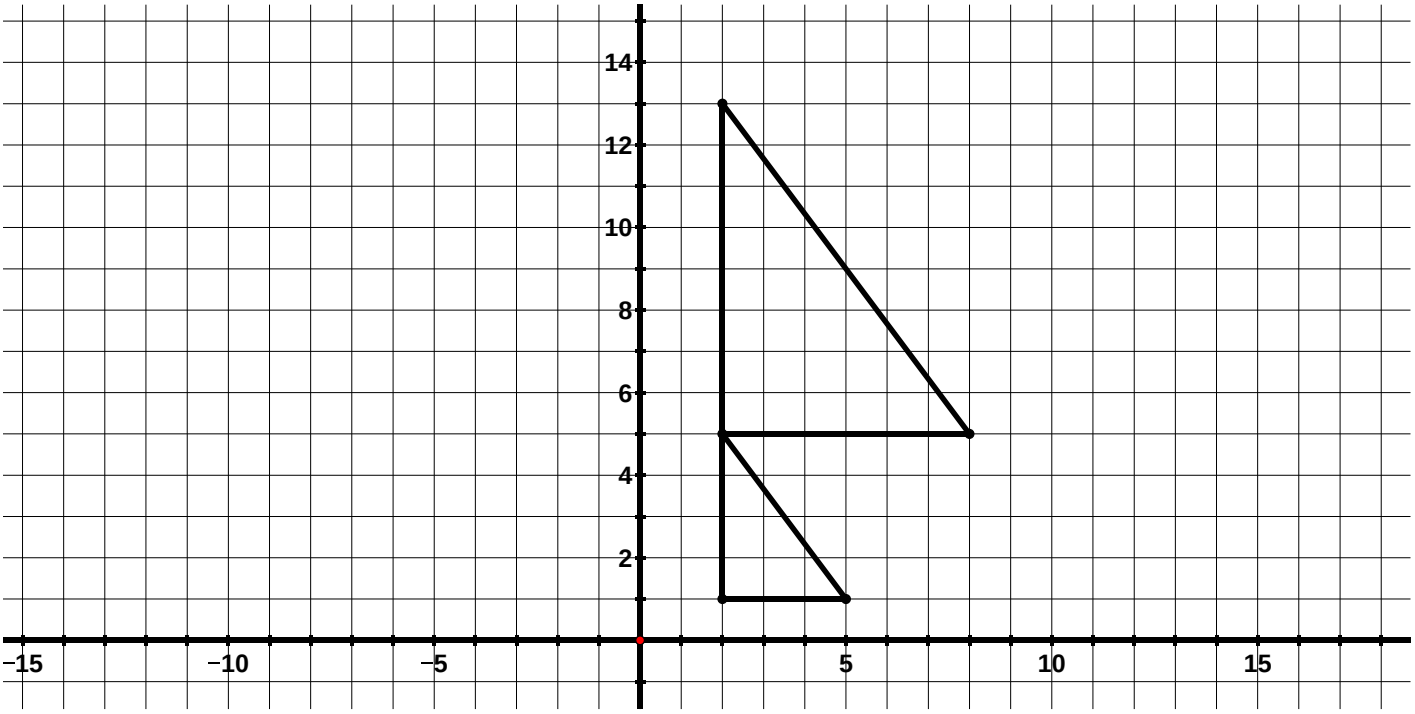

a. List the sequence of transformations that verifies the similarity of the two figures.

One possible answer: a reflection of $\Delta RST$ across the $y$-axis followed by a dilation with the center at the origin and a scale factor of 2.

b. Write a similarity statement for the triangles. $\Delta RST \sim \Delta ACB$

4. Determine whether the figures below are congruent, similar, or neither. Provide a justification for your answer.

The two triangles are similar. The smaller triangle can be mapped to the larger by a translation 4 units up and a dilation with center at (2, 5) and scale factor of 2.



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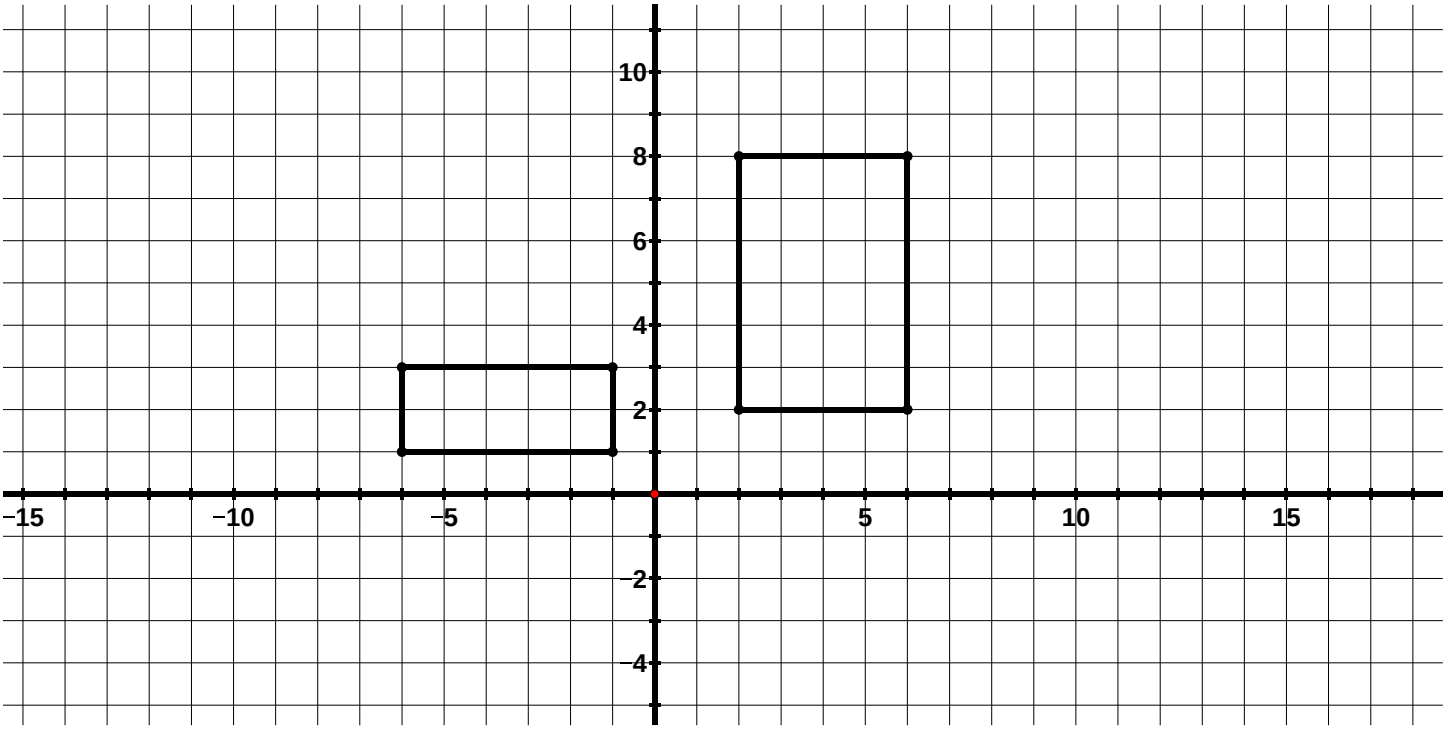
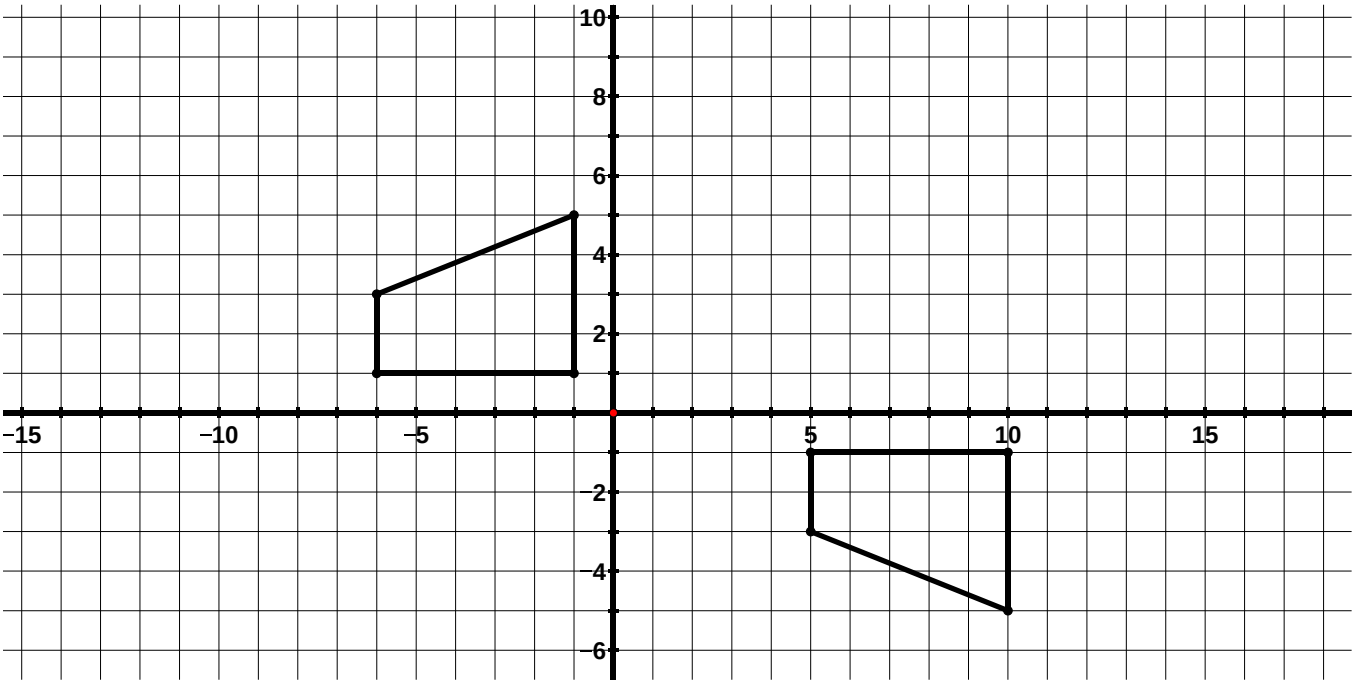
5. Determine whether the figures below are congruent, similar, or neither. Provide a justification for your answer.

Students may reason that these figures are not similar because the ratio of sides is not the same. An alternative approach is that students may try to prove similarity through a series of rigid motion and dilation and see that they cannot map one image to the other.




6. Determine whether the figures below are congruent, similar, or neither. Provide a justification for your answer.

The two figures are congruent (and subsequently similar). You can map the quadrilateral in quadrant 2 to the quadrilateral in quadrant 4 by a translation of 11 units to the right followed by a reflection across the *x*-axis



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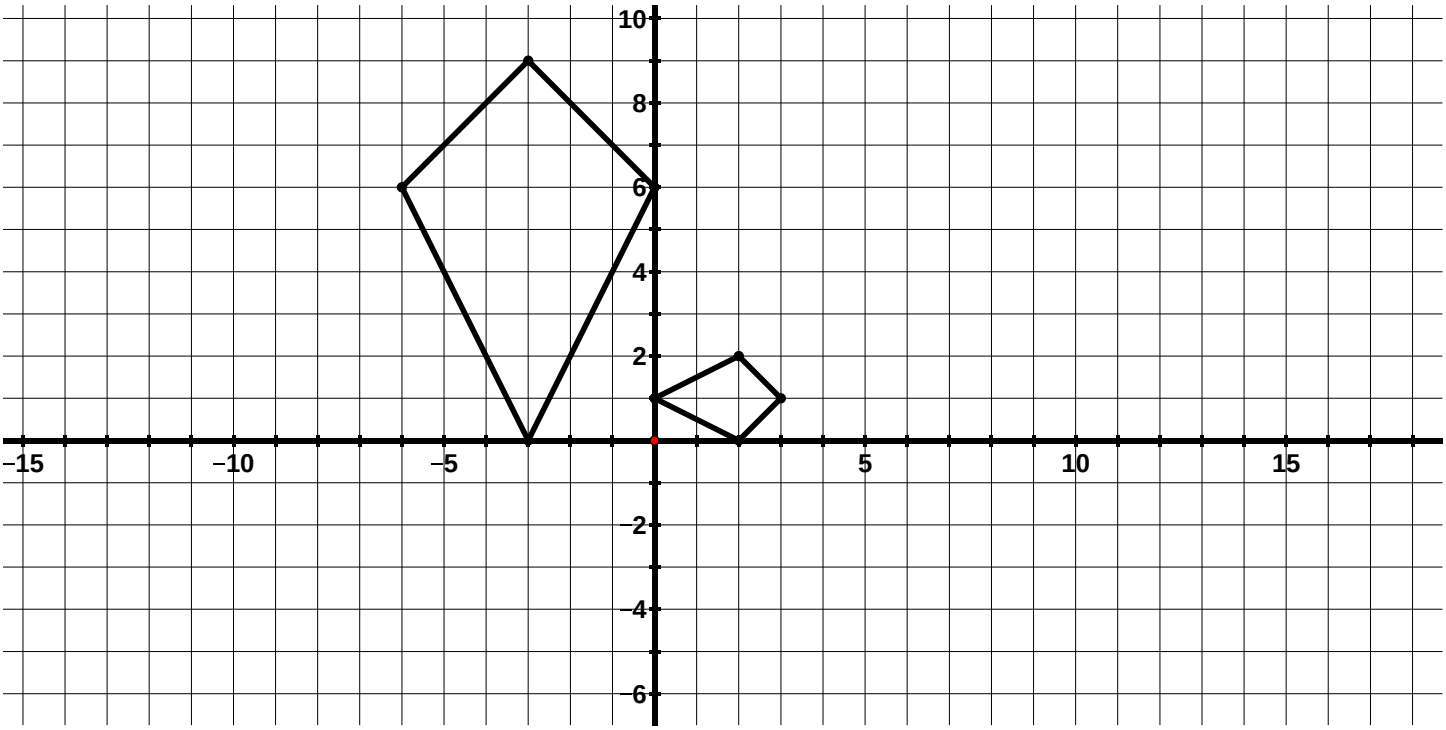
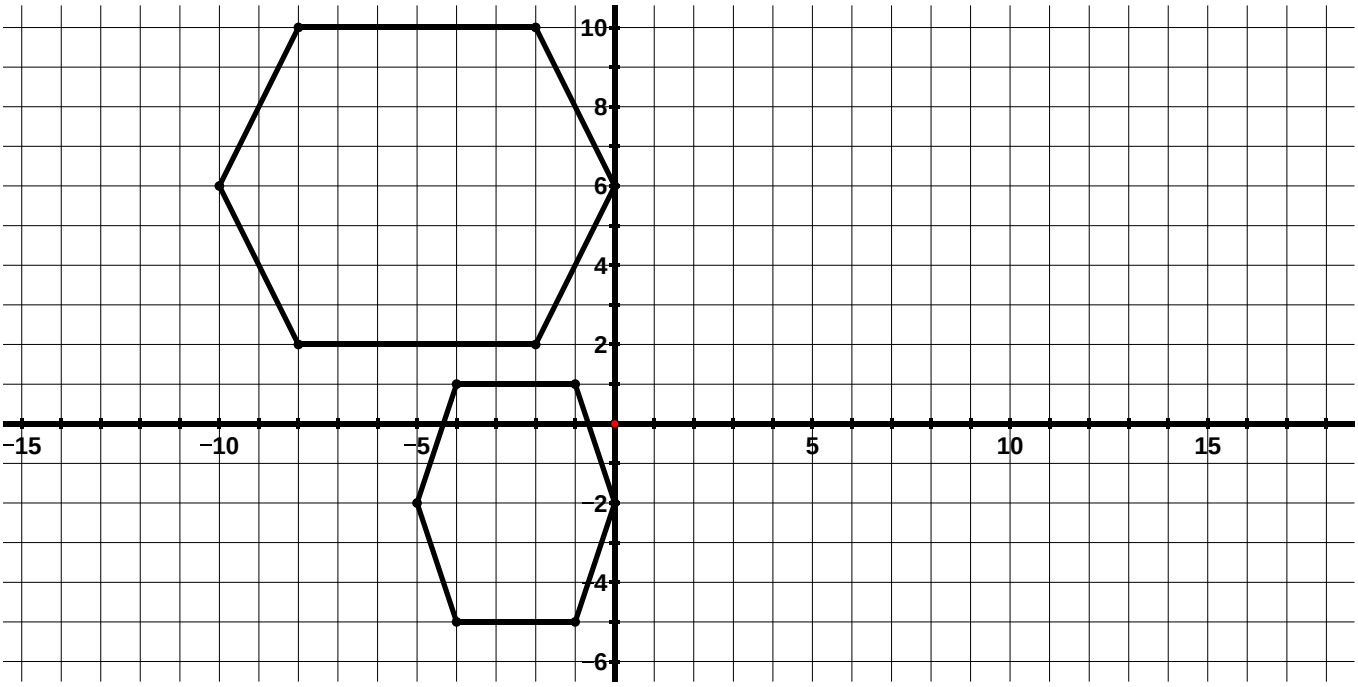
7. Determine whether the figures below are congruent, similar, or neither. Provide a justification for your answer.

Similar, a rotation of the larger figure 90˚ clockwise about the origin followed by a translation 2 units down followed by a dilation with center at (0, 1) and scale factor of $\frac{1}{3}$ (justifications may vary)




8. Determine whether the figures below are congruent, similar, or neither. Provide a justification for your answer.
Neither, angles are not congruent



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9. Determine whether the following transformations or sequence of transformations will result in similar figures, congruent figures, or neither.

   a. $(x, y) \rightarrow (x + 6, 6y)$ Neither

   b. $(x, y) \rightarrow (x, -y)$ followed by a $270^\circ$ rotation clockwise about the origin congruent (similar)

   c. $(x, y) \rightarrow (4x, 4y)$ followed by $(x, y - 2)$ similar

   d. A $180^\circ$ rotation counterclockwise about the origin followed by $(x + 2, y + 2)$ congruent (similar)

   e. $(x, y) \rightarrow (3x, x + y)$ neither

   f. $(x, y) \rightarrow \left(\frac{1}{3}x, \frac{1}{3}y\right)$ followed by a reflection across the $x$-axis similar

   g. Write your own transformation or sequence of transformations that will result in two figures that are congruent. Answers will vary

   h. Write your own transformation or sequence of transformations that will result in two figures that are similar. Answers will vary

   i. Write your own transformation or sequence of transformations that will result in two figures that are neither congruent nor similar. Answers will vary

Honor's Extension: Have students prove that all circles are similar. Some students will enjoy the challenge of this on their own, while others will work best in groups. This requires some time and may be considered an additional separate assignment.



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**9.2g Self-Assessment: Section 9.2**

Consider the following skills/concepts. Rate your comfort level with each skill/concept by checking the box that best describes your progress in mastering each skill/concept. Sample problems for each standard can be found on the following page(s).

| **Skill/Concept** | **Minimal Understanding 1** | **Partial Understanding 2** | **Sufficient Mastery 3** | **Substantial Mastery 4** |
|---|---|---|---|---|
| 1. Describe the properties of a figure that has been dilated. | | | | |
| 2. Perform a dilation given a scale factor and center of dilation. | | | | |
| 3. Describe a dilation in words and using a coordinate rule. | | | | |
| 4. Determine the center of dilation using the properties of dilations. | | | | |
| 5. Understand what it means for two figures to be similar. | | | | |
| 6. Determine if two figures are similar. | | | | |
| 7. Given two figures that are similar, describe the sequence of transformations that exhibits the similarity between them. | | | | |



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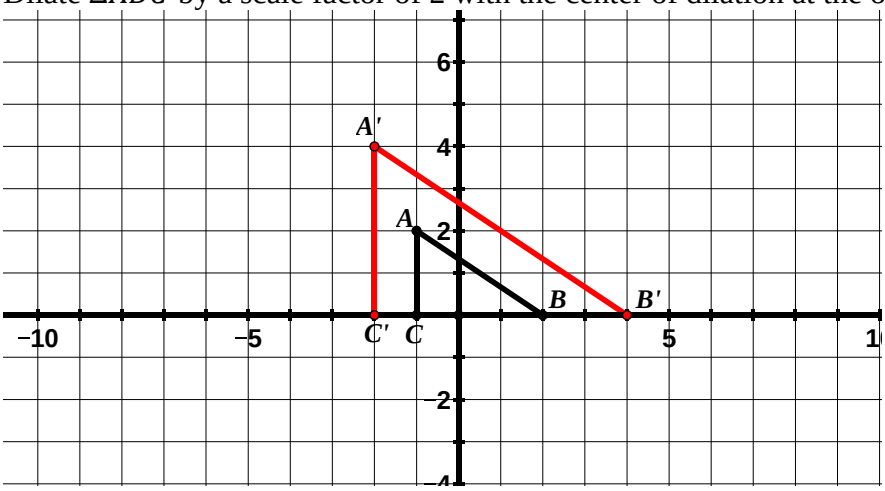
**Section 9.2 Sample Problems** (For use with self-assessment)

1. Which of the following properties are true about a figure that has only been dilated. Check all that apply.

   - [x] The ratios of the image segments to their corresponding pre-image segments are equal to the scale factor.
   - [ ] Corresponding vertices lie on the same circle as the center of dilation.
   - [x] Corresponding vertices lie on the same line as the center of dilation.
   - [x] Corresponding segments are parallel.
   - [ ] Corresponding segments are perpendicular.
   - [ ] The area of the image is always the same as the area of the pre-image.

   A good test-taking strategy to teach students for answering a question like the one above is to first perform a dilation so that they can consider the statements above while looking at an example of a dilation. Use the completed dilation below or another dilation to help them answer question 1 and to discuss additional properties of dilations (i.e. in the dilation below $B'$ is twice the distance from the center of dilation as compared to $B$.)

2. Dilate $\Delta ABC$ by a scale factor of 2 with the center of dilation at the origin.



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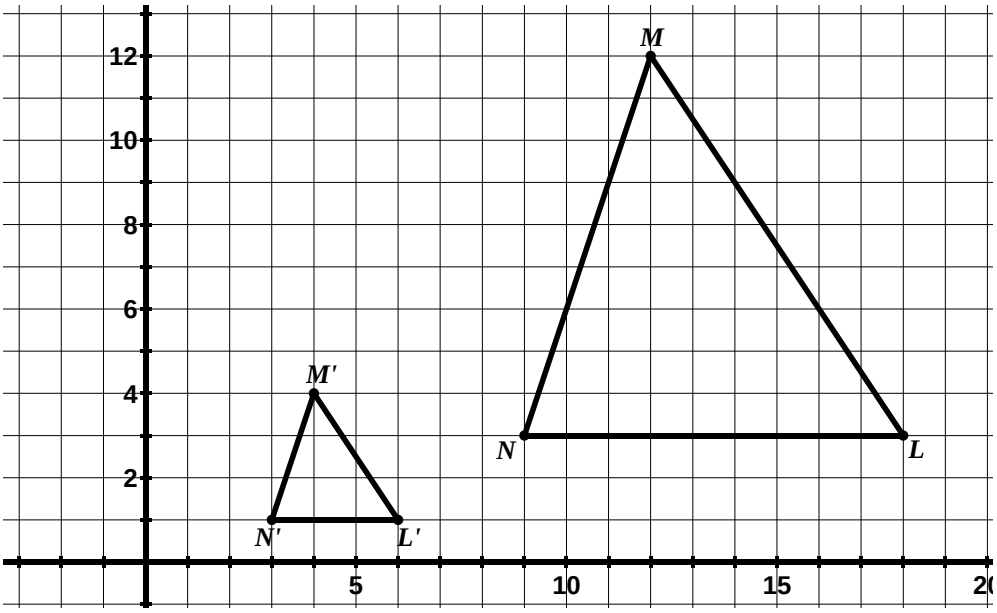
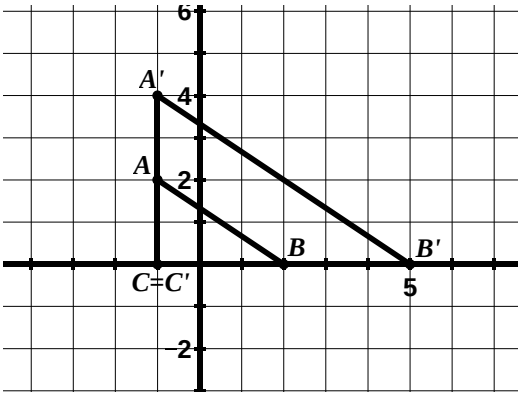
3. Describe the dilation below in words and with a coordinate rule. Be sure to specify the center of dilation and the scale factor.




Many answers, one possible answer: A dilation with a scale factor of $\frac{1}{3}$ and center of dilation at the origin. Coordinate rule: $(x, y) \rightarrow \left(\frac{1}{3}x, \frac{1}{3}y\right)$. Alternatively, students may first translate the pre-image 2 units down and 6 units to the left and then dilate from $N$. If the center of dilation is not on the origin, it may be challenging for students to write the coordinate rule.

4. $\Delta ABC$ was dilated to produce $\Delta A'B'C'$. Determine the scale factor and center of dilation.




The scale factor is 2 and the center of dilation is $C$: $(-1,0)$.

5. Describe in your own words what it means for two figures to be similar.
Two figures are said to be similar if there is a sequence of rigid motions and dilations that take one figure onto the other. Corresponding angles are congruent and corresponding sides are proportional (the ratios of corresponding sides are all the same and are equal to the scale factor).



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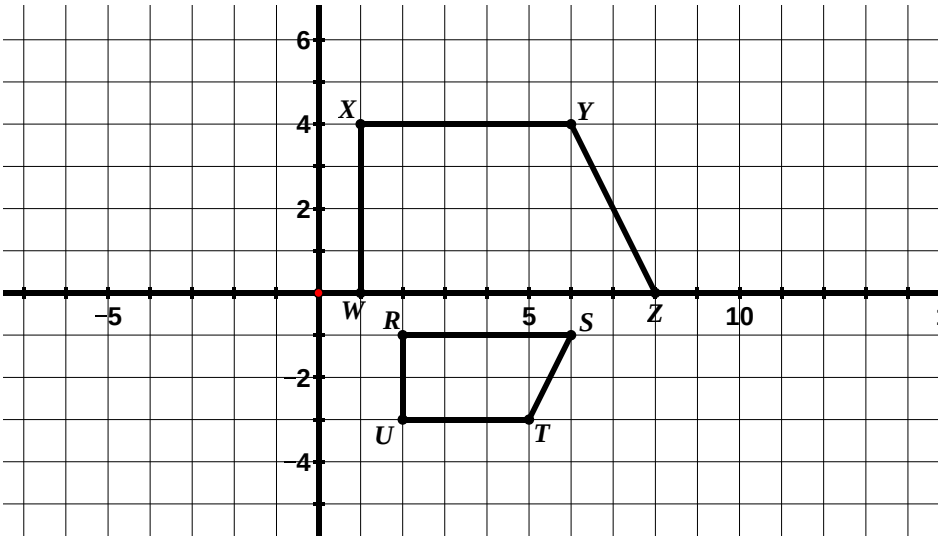
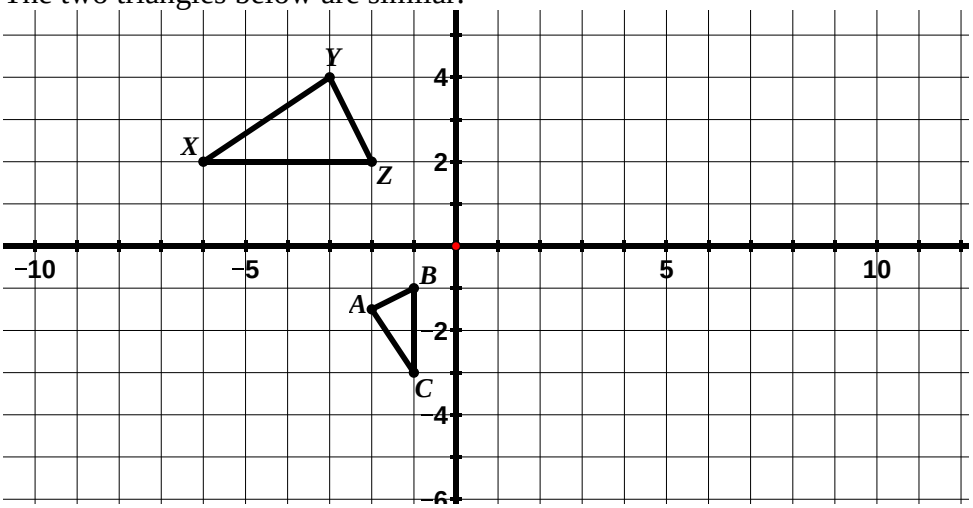
6. Are the parallelograms shown below similar? Provide a justification for your response.




No, the parallelograms are not similar. The ratios of corresponding sides are not the same (i.e. $\frac{UT}{XY} = \frac{3}{5}$ while $\frac{UR}{XW} = \frac{1}{2}$).

7. The two triangles below are similar.




a. Describe a sequence of transformations that verifies the similarity of the triangles.
Many possible answers. One possible sequence is a rotation of $\Delta XYZ$ 90˚counterclockwise about the origin, followed by a dilation with a scale factor of $\frac{1}{2}$ with the center of dilation at the origin.

b. Write a similarity statement for the triangles. $\Delta BAC \sim \Delta ZYX$

c. Determine which angles are congruent. $\angle B \cong \angle Z$; $\angle A \cong \angle Y$; $\angle C \cong \angle X$

d. Complete the following statements:

$\frac{BC}{ZX} = \frac{1}{2}$ $\qquad$ $\frac{ZY}{BA} = 2$



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