# Chapter 1
# Ratio Relations

Relations between quantities are often expressed as ratios, and these can take many forms, as we see in the following common expressions:

a) The ratio of males to females in the United States is 497: 501.

b) To make Fruit Jelly, use 5 cups of sugar for every 6 cups of fruit.

c) 67 out of 80 marathon runners wear flat soled shoes.

d) 3/5 of Americans today regularly drink carbonated sodas.

e) This bag contains 3 cookies per student.

f) In a survey, we found that 40% more females than males were well-disposed toward our product.

A ratio expresses a relationship between two quantities that may vary from observation to observation. We say that quantity $A$ is to quantity $B$ as $a$ is to $b$ (where $a$ and $b$ are positive numbers, written $A : B :: a : b$) if, for any observation of these quantities, the amount of $A$ is always the same multiple of the amount of $B$. As an example, we say that $A : B :: 5 : 3$, if every observation of the quantity $A$ is a multiple $m$ of 5, and the same observation of $B$ is the same multiple $m$ of $B$. As the list above shows, there are many ways of expressing a ratio; these will be discussed in the first section. Here let us illustrate in terms of the above list.

a) The ratio of men to women in the United States is 497: 501. This statement is meant to convey that, in any typical (but large) gathering of U. S. residents, we'd expect slightly more women than men. So, imagine that at some event, one has observed that there are 15,000 men, and now we ask: how many women? The answer would be close to 15,120 women. One word of caution: in an example like this, the ratio is an approximation. If an actual count produced 15,135 women, we would not say that this contradicted the estimate of the ratio.

b) To make Fruit Jelly, use 5 cups of sugar for every 6 cups of fruit. Now this is *not* an estimate; the recipe clearly requires the cook to be precise. However, it does say that, no matter how much jelly we want to make, or how much fruit we have, the amount of sugar to be used should be 5/6 of the amount of fruit available, where *amount* is measured in whatever units. So, if we have 15 cups of fruit, we'll need $12\frac{1}{2}$ cups of sugar. Or if we want to make 66 cups of jelly, we'll need 36 cups of fruit and 30 cups of sugar.

c) 67 out of 80 marathon runners wear flat soled shoes. Suppose we know that in the most recent Boston Marathon, there were 12,000 runners. Then we can estimate that about 10,050 of them were wearing flat soled shoes. We could also conclude that 83.8% of marathon runners wear flat soled shoes.

d) 3/5 of Americans today regularly drink carbonated sodas. Suppose we know that there are 300 million Americans. This datum tells us that 60% of them, or 180 million, regularly drink carbonated sodas. However, to make use of this fact, we need a numerical expression for the word *regularly* (i.e. 3 liters a week).

e) The word *per* is often used to represent a unit rate, as in this statement: the unit rate is 3 cookies per student. We may also use the word *for*, as in: Sale! 5 apples for 3 dollars!

f) In a survey, we found that 40% more females than males were well-disposed toward our product.This tells us that if 82 males were "well-disposed", then there were $0.40 \times 82 = 32.8$ more females "well disposed." Since there are no partial femaies we say that there were 33 more females than men "well-disposed." We conclude that, for the 92 well-disposed males, there were 115 well-disposed females in the survey. This statistic is misleading, for the reader of the survey may wonder: in all, how many of those surveyed were "well-disposed?" We have information only about those answering affirmatively, so we have no way of knowing how many did not. Maybe those well-disposed were only one-tenth of the number surveyed.

In the last example, we used percentage to express a ratio. When we say that 69% of New Hampshire apples are Macintosh, we are saying that the ratio of Macintosh apples to all apple (in New Hampshire) is 69:100. Percentages will be taken up in the next Chapter.

Sometimes the two quantities being compared are parts of a whole. So, in item a), the quantities in question are $A$: males in the U.S.; $B$: females in the U.S., and these are the component parts of the unmentioned $C$: people in the U.S. We call the ratio $A : B :: 497 : 501$ a *part to part* ratio. A *part to whole* ratio here would be males:people ($A : A + B$) or females:people ($B : A + B$). These ratios are, respectively, $497 : 998$ and $501 : 998$. These can be expressed as percentages: the percentage of males in the U.S. is 49.8%, and that of females is 50.2%.

Statement b) is also a part to part ratio, while c), d) and e) are part to whole ratios. Statement f) is a little different: again we have these quantities: and those who responded to the survey, those males respondents who were "well-disposed," those female respondents who were "well-disposed." We are told that the number of females is 40% greater than the number of males. This can to be interpreted as "the number of females is 1.4 times the number of males," or that the ratio of females to males among those well-disposed was 14:10.

A large part of the study of mathematics from Grade 6 forward is that of working to understanding how to express a relation, if it exists, between two variables. The first, and most important (for all other relations are typically described in terms of it) of these is the statement that the two quantities, no matter how measured, are in a fixed ratio to each other. The work of this chapter is to understand this relation (for example, if one quantity doubles, so does the other - no matter what the ratio). This relation is described by ratio, fraction or percentage. The point of these first two chapters is to create understanding of these representations and how to move from one to the other.

The last part of this chapter is devoted to illustrating the various ways of displaying a relation among two variables: algebraically (i.e. as a ratio), in a table (for sample quantities), or in a graph.

This theme is to be continually developed throughout the rest of the students' mathematics. In grade 6 we begin to grapple with the distinction between unknowns and variables; in this chapter we make the first plunge and the development in the subsection "Two variable equations" is designed to explore these ways of displaying how, of two variables, one *varies* with respect to the other.

Now $x$ (or $y$ or $s$ or $Z$) is not to be thought of as an *unknown*, but as a *variable*. This is a word that has been used up to now in a loose, intuitive form, but now we start thinking of the letters in an algebraic expression as representing quantities that *vary*, and now our interest is in how they vary with respect to one another.

For example, suppose a ball is dropped from a tower 256 feet high. If, in our minds, we stop the motion at a particular instant, we may denote by $t$ the time (in seconds) since the ball was released, and $x$ as the distance it has fallen. Now, $t$ and $x$ are variables, for they both vary during this experiment. How do they vary? Well, they both increase (so to speak, as time goes by), and if we made many experiments and many measurements, we would find that $x$ increases much more rapidly than time. This particular study was made by Galileo in the late 16th century, and his findings were fundamental to the invention of the Calculus almost a century later. In fact, Calculus is so named because its inventors thought of it as a technique to calculate the precise formulas that describe the variation of $x$ and $t$ in this experiment (the result was that $x = 16t^2$).

Simply put, $x$ and $t$ are variables, and they are related by the equation $x = 16t^2$. Similarly, suppose that $L$ and $W$ are the length and width (in inches) of a rectangle with area 360 sq. in. Then the *variables* $L$ and $W$ are related by the equation $LW = 360$.

Although this terminology is introduced here, in the last chapter for grade 6, it is studied in detail only for these types of relations of two variables: $y = x + a$ and $y = xa$, for $a$ a positive number. The description of the relation of the two variables $y$ and $x$ is this:

Suppose $y$ and $x$ are related by the equation $y = x + a$. If an amount $b$ is added to $x$, the same amount ($b$) is added to $y$.

Suppose $y$ and $x$ are related by the equation $y = ax$, then, if $x$ is multiplied by a factor $b$, then $y$ is also multiplied by the same factor ($b$).

In the class discussion of this section, many issues may come up. That is to be expected, and the teacher will have to decide to which to respond substantively, and which are to be deferred until a later class. The flow of discussion can follow any path, and in going down that path, it is best to remember that it is the concept of variation that is being introduced as a basis for further study, and, for the 6th graders, it suffices to understand the additive nature of the relation $y = x + a$, and the multiplicative nature of the relation $y = ax$.

## Section 1. Representing Ratios

*Understand the concept of a ratio and use ratio language to describe a ratio relationship between two quantities. For example, "the ratio of wings to beaks in the bird house at the zoo was 2:1, because for every 2 wings there was 1 beak." "For every vote candidate A received, candidate C received nearly three votes." 6.RP.1*

Ratios are an important part of mathematics and daily life. Whether it is cooking, exercising, or computing a tip, we often compare two (or more) related quantities. A ratio is commonly described as a pair of positive numbers, written $a : b$ and read as "$a$ to $b$." For example, in a certain cookie recipe, it may call for 3 cups of flour for every 2 cups of sugar. More generally, a ratio may involve more than two numbers, for we may be comparing more than 2 quantities. A recipe for walnut cookies may call for 3 cups of flour to 2 cups of sugar to every cup of walnuts. In this case we say that "the ratio of flour to sugar to nuts is 3:2:1." Sometimes we may speak in terms of *odds*, as in "the odds are 2:1 that it will rain tomorrow." This tells us: the records show that, for days that have (more or less) the same characteristics as today, rain occurred on the following day twice as often as not. Or, the weatherman might say, "The odds for rain or snow or no precipitation tomorrow are 7:4:3," telling us that, according to the records, of days with the same weather profile as today, for every 3 times the following day had no precipitation, there were 4 days of snow and and 7 days of rain.

But let us return to the simple cookie recipe with a 3:2 ratio of flour to sugar. We could also say that the ratio of sugar to flour is 2:3, but we *must* be sure that the order of the ingredients are *the same* as the order of numbers. A batch made with 3 cups of flour to 2 cups of sugar will turn out very different from a batch made with 2 cups of flour and 3 cups of sugar. Initially, students will usually draw pictures to represent ratios. They may be very explicit or representative as the five measuring cups pictured below.

As students progress, their pictures often become more abstract. For example, a student may draw 5 circles (of the same size) to represent the three cups of flour and two cups of sugar.

Another way to illustrate a ratio is to draw a *tape diagram*. A tape diagram is simply a rectangle composed of smaller, equal-sized pieces to represent each of the smaller parts of the ratio. For the ratio of 3 cups of flour to 2 cups of sugar, a tape diagram would look like Figure 1:

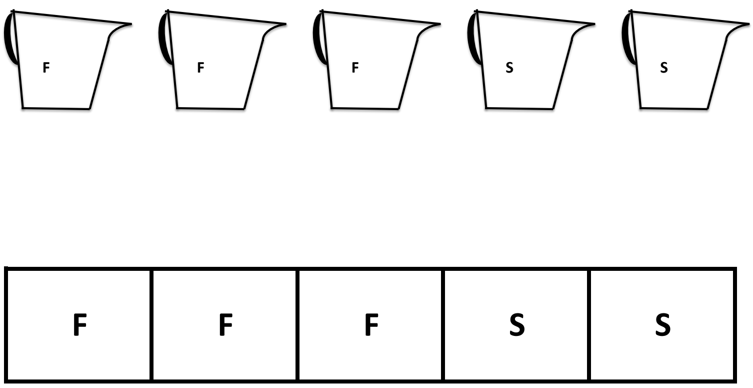



Figure 1. Flour is to Sugar as 3 is to 2

Note that each small rectangle represents one cup of ingredient and that there are three cups of flour for the two cups of sugar. More significantly, the fact that the boxes do not identify a quantity signals to us that the essential fact is the ratio 3:2; these could be cups or pounds or tablespoons - no matter what amount the box stands for, there are always 3 flours to 2 sugars.

When dealing with a ratio situation, we can often express the same relationship in a variety of ways. If there are 5 boys and 6 girls on the playground, we say that the ratio of boys to girls is 5:6. Since the boys and the girls each constitute different parts of the children on the playground, we say that this is a *part to part ratio*. Noting that 5 boys plus 6 girls equals 11 children, we could also express the same information in a *part to total ratio* (also called a *part to whole ratio*) of 5 boys (part) to 11 children (total), 5:11. We can also convey the same information in another part to total ratio by giving the number of girls to the number of children as 6:11. Finally, we sometimes state the total (or whole) first and the part second, as in 11 children to 6 girls, 11:6, and call this a *total to part ratio*. Back to the cookies, we can say that the ratio of flour to batter is 3:5, and the ratio of sugar to batter is 2:5.

If another group of children come in and the ratio of boys to girls remains the same, then the new group must also have 5 boys to 6 girls, or consist of several subgroups of 5 boys and 6 girls. That is, the combined group of 10 boys and 12 girls or 15 boys and 18girls is still in the ratio 6:5, and the total number of children is always a multiple of 5+6 = 11.

Back to cookies: if we are cooking for a large group, we may double or even triple a recipe. So that the food tastes right, we have to double (or triple) the amount of each ingredient. If we multiply one component of the recipe by some number, the other component has to multiply by the same number. This illustrates the concept of *equivalent ratios*. A tape diagram may be iterated (copied) to demonstrate the equivalent ratios. In the cookie recipe, if we double the batch, we will need twice as much flour and twice as much sugar (as well as any other ingredient). As you can extend a tape measure, you can extend the tape diagram by making copies. To show the doubling of the cookie recipe, see Figure 2. We first draw the box diagram below, and then rearrange it to get the second row, illustrating the equivalent ratio 6:4 more clearly.

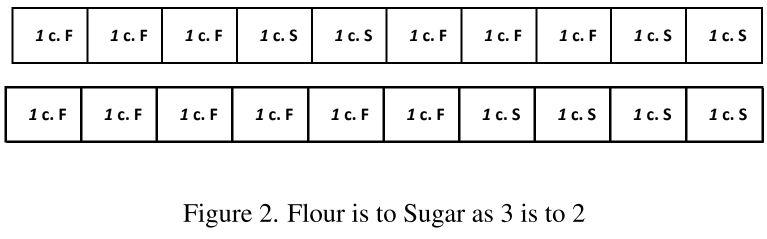



Figure 2. Flour is to Sugar as 3 is to 2

Alternatively we could double the recipe using the box diagram in Figure 1 above, by declaring that now, each box represents 2 cups (and triple by declaring that each box represents 3 cups).

All these diagrams show the same relative relationship between ingredients, namely of 6 cups of flour to 4 cups of sugar, or of 9 cups of flour to 6 cups of sugar. These ratios are related to the original ratio of 3 cups of flour to 2 cups of sugar because we simply multiplied each number in the original ratio by the same positive number (2, to double the recipe, and 3 to triple the recipe). 3:2 is equivalent to $2 \times 3 : 2 \times 2$ or 6:4; $3 \times 3 : 3 \times 2$ or 9:6. Similar results would be obtained by quadrupling, halving, or using any fixed multiple of the recipe. This leads to saying:

Two ratios, $a : b$ and $c : d$, are **equivalent ratios** if there is a positive number $p$ such that

$$c = p \times a, \ d = p \times b \, .$$

We can illustrate this equivalence simply by labeling the boxes as shown in Figure 1:

| | | | | |
|---|---|---|---|---|
| **p cups F** | **p cups F** | **p cups F** | **p cups S** | **p cups S** |

Figure 3. $3p : 2p$ is equivalent to $3 : 2$

Because of their similar mathematical behavior, ratios are sometimes written as fractions. For instance, the ratio of 3 cups of flour to 2 cups of sugar can be expressed as the quotient $\dfrac{\text{3 c. flour}}{\text{2 c. sugar}}$.

As expressed above, the ratios $3 : 2$ and $6 : 4$ are equivalent because we multiplied both parts of the smaller ratio by the same positive constant, 2, to obtain $2 \cdot 3 : 2 \cdot 2$. In a like manner, drawing on equivalent techniques learned previously, the quotient (3 c. flour)/(2 c. sugar) is equivalent to the quotient (6 c. flour)/(4 c. sugar) because we simply multiplied the numerator and denominator by the same positive number, 2. Similarly, the ratio 9:6 is equivalent to 3:2 because we multiply both numerator and denominator by the same positive number, 3. As with the other notation, the ratio 3:2 is equivalent to the ratio $p \cdot 3 : p \cdot 2$ because of the multiplication by the same positive number, $p$. Note that $p$ needs to be positive and not zero. If we have ratio $a : b$ and multiply both by zero, we get $0 : 0$, which is not a ratio because a ratio is a relation between a pair of *positive* numbers. Working with ratios in quotient form will be expanded on as students learn about unit rates in the second part of this chapter.

This similarity of finding equivalent quotients and equivalent ratios is often used to change a ratio into an equivalent form with smaller numbers (sometimes referred to as "simplifying"). To illustrate: if there are 24 children and 8 adult chaperones on a class field trip, we could say that the ratio of children to adults is 24:8. This ratio may be modeled using concrete objects or a diagram and students will find that for every three children there is one adult. After some practice, students will recognize that they can use their knowledge of division (factoring) to arrive at the answer more quickly. Noticing that 2 is a factor of both 24 and 8, we can find that the ratio 24:8 is equivalent to the ratio 12:4 because $24 = 2 \cdot 12$ and $8 = 2 \cdot 4$. However, this ratio can be further reduced. Noting that $12 = 4 \cdot 3$ and $4 = 4 \cdot 1$, the ratio 12:4 is equivalent to the ratio 3:1. Students will soon recognize that the process could have taken fewer steps if they had first found the greatest common factor, GCF, of the two numbers, 24 and 8. The GCF of 24 and 8 is 8 so $24 = 8 \cdot 3$ and $8 = 8 \cdot 1$, therefore, 24:8 is equivalent to the "easier" ratio of 3:1.

Again, quotient notation can also be used to simplify ratios. Drawing on work from previous grades (4.NF), students can simplify the fractional form of the ratio. For example, using the 24 children to 8 adults on the field trip, the student may create a quotient (24 children)/(8 adults). Recognizing that 24 is a integral multiple of 8 ($24 = 8 \cdot 3$) we can write this quotient as

$$\frac{8 \cdot 3 \text{ children}}{8 \cdot 1 \text{ adults}}.$$

The simplified ratio is 3:1, represented by the quotient 3/1.

This is a good place to discuss the understanding of the word *fraction* in the context of ratios. To begin with, colloquially, a fraction is taken to mean a portion of an assumed unit (as in "only a fraction of the wonderful array of goodies was consumed at the party"). In the primary grades this is the students' understanding of the word, and they learn that it can be written as a quotient $p \div q$, with both $p$ and $q$ counting numbers, with $p < q$. From there the understanding moves to *mixed fractions*, such as three and five eighths, written

$$3\frac{5}{8}.$$

Students learn that that can be written as a fraction, but now with numerator possibly larger than denominator. In this case

$$3\frac{5}{8} = \frac{29}{8}.$$

Then students learn of equivalence of fractions: two fractions $p/q$ and $p'/q'$ are equivalent if $p' = mp$ and $q' = mq$ for some positive integer $m$.

Now, we have defined the word ratio $p : q$ expressing the relative size of samples of certain quantities, and again, the ratios $p : q$ and $p' : q'$ are equivalent if $p' = mp$ and $q' = mq$ for some positive integer $m$. So, as a numeric quantity, a ratio is the same as a fraction. But, operationally, they are quite different. A *fraction* is a number - a point on the line. A *ratio* $a : b$ expresses a relationship between two quantities: given a particular sample of the quantities $A$ and $B$, the number of $A$ is the same multiple of $a$ as the number of $B$ is of $b$.

A ratio $a : b$ can be represented by the fraction $a/b$, and the representation is faithful. But the understanding of the two concepts is very different, and it is important to convey that difference.

Once students are familiar with the arithmetic of ratios, they can apply their knowledge to solve real world problems.

*Use ratio and rate reasoning to solve real-world and mathematical problems, e.g., by reasoning about tables of equivalent ratios, tape diagrams, double number line diagrams, or equations. 6.RP.3.*

**Example 1**. Amber and her younger brother Mateo are running laps in the school gym. They start at the same time. Amber runs 5 laps in the time it takes Mateo to run 3 laps. How many laps has Mateo completed when Amber finishes 20 laps? Use at least two strategies to solve your problem.

Solutio # 1: Create a tape diagram: Since Amber ran 20 laps and there are 5 boxes for Amber, each box represents

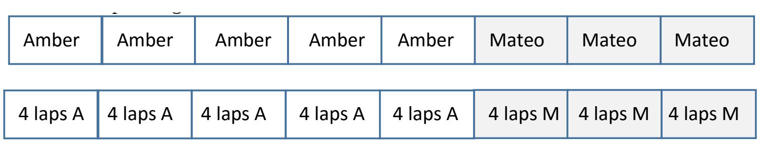



Figure 4. Each box represents 4 laps, whether those of Amber or those of Mateo

$20 \div 5 = 4$ laps, so we set each box equal to four laps. That means 4 laps go in each box for Mateo.

Counting the laps for Mateo, we get $4 + 4 + 4 = 3 \times 4 = 12$ laps.

Solution # 2: Use a table:

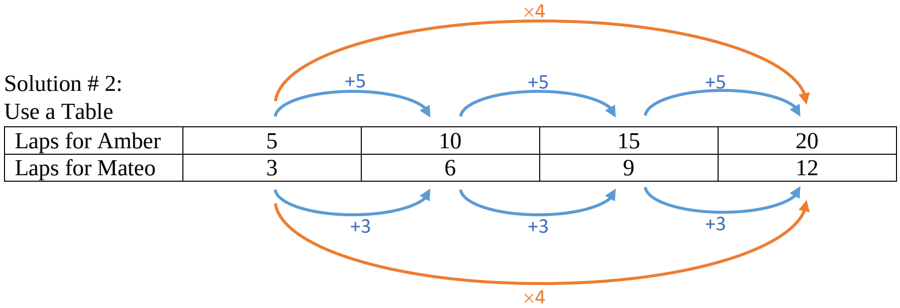



Figure 5, showing additive as well as multiplicative structure

As Amber runs 5 more laps, Mateo runs 3 more (see Figure 5). Recognizing that $20 = 4 \times 5$ we then need $4 \times 3 = 12$ laps for Mateo.

Solution # 3: Numeric: Create a fractional representation of the ratio. Since 20 laps for Amber is $4 \times 5$ laps, find an equivalent equivalent ratio:

$$\frac{\text{5 laps Amber}}{\text{3 laps Mateo}} = \frac{4 \cdot \text{5 laps Amber}}{4 \cdot \text{3 laps Mateo}} = \frac{\text{20 laps Amber}}{\text{12 laps Mateo}} .$$

From this we see that Mateo will run 12 laps in the time it takes Amber to run 20.

Before moving on to further develop the concept of the ratio of two quantities, we stop to mention that ratios can be established among more than two quantities. For example, we may say that, at the last Fourth of July banquet the ratio of seniors to adults to children was 4:6:3. That means that if we took a random sample of 13 participants, we'd expect 4 seniors, 6 adults and 3 children. Or if we picked any number of participants at random, we should expect the *ratio* of seniors:adults:children to be a multiple of 4:6:3.

**Example 2**. Jorge the tuna fisherman knows that the ratio of Yellowfin to Albacore to Bonito is about 4:6:3. On his latest trip through the North Banks he collected about 1800 tuna. How many yellowfin, albacore and bonito should he expect in the catch?

Solution. The precise answer consists of three numbers of the form $4p, 6p, 3p$, where $p$ is a whole number, that add up to 1800. Trying to solve $4p + 6p + 3p = 1800$, we get $p = 1800/13 = 138.46\ldots$, so there is not a precise answer. But Jorge just wants an idea of his catch, so he can take $p$ to be the closest integer, which is 138. So the answers are: there are about $138 \cdot 4 = 552$ Yellowfin, $138 \cdot 6 = 828$ Albacore, and $138 \cdot 3 = 414$ Bonito.

## Section 2. Rates, Graphs and Equations

*Understand the concept of a unit rate a/b associated with a ratio a:b with b ≠ 0, and use rate language in the context of a ratio relationship. For example, "This recipe has a ratio of 3 cups of flour to 4 cups of sugar, so there is 3/4 cup of flour for each cup of sugar." "We paid $75 for 15 hamburgers, which is a rate of $5 per hamburger." 6.RP.2*

*Make tables of equivalent ratios relating quantities with whole number measurements, find missing values in the tables, and plot the pairs of values on the coordinate plane. Use tables to compare ratios. 6.RP.3a*

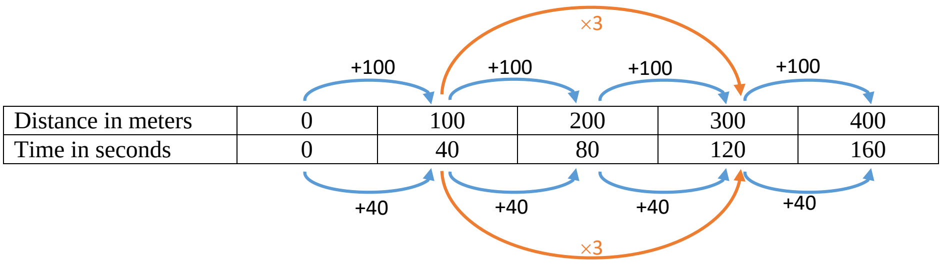
*Solve unit rate problems including those involving unit pricing and constant speed. For example, if it took 7 hours to mow 4 lawns, then at that rate, how many lawns could be mowed in 35 hours? At what rate were lawns being mowed? 6.RP.3b*

The preceding section deals with ratios of like units (cups of sugar to cups of flour, the unit being cups of ingredients or boys to girls, the unit being children). After learning how to find equivalent ratios with like units, students are ready to expand their view to ratios of unlike units. These arise in many common scenarios such as speed (miles per hour) or shopping (cost per item). As they learn, students model such ratios with double number lines or graphs and move to an understanding of *unit rate*. Then they use unit rates to solve problems (such as which is the better deal) and begin to write equations that model the ratio situations.

One of the most familiar ratios, *speed*, relates two different types of units: *distance* and *time*. Although these measures are dissimilar, in the context of a moving object, they both serve to describe the event of traveling, say from Salt Lake City to Moab. In order to establish this (contextual) relation between dissimilar measures, it is good for students to learn another tool for modeling: the *double number line*. A double number line is simply two number lines "aligned" so that they start at the same location and the first marks represent the ratio in question. On each number line, the given amount for that portion of the ratio is iterated (copied), creating marks of equal measure.

**Example 3**. Jordan ran 100 meters in 40 seconds. How far can Jordan run in 80 seconds? 120 seconds? In 60 seconds? Solve using at least two different strategies.

SOLUTION. # 1: Construct a table.




| | | | | | |
|---|---|---|---|---|---|
| Distance in meters | 0 | 100 | 200 | 300 | 400 |
| Time in seconds | 0 | 40 | 80 | 120 | 160 |

Figure 6, showing additive as well as multiplicative structure

At the start (time = 0), Jordan has run 0 meters. Notice the patterns of repeated addition and multiplication for each line in the table. Jordan will run 200 meters in 80 seconds, 300 meters in 120 seconds. Note that we can find the distance run in 60 seconds by finding half the distance at time 120 seconds. Therefore Jordan will run 150 meters in 60 seconds.

SOLUTION. # 2: Construct a double number line.

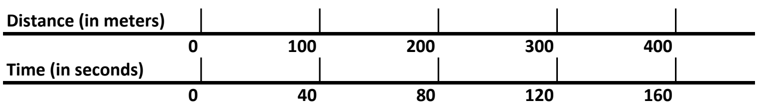



Figure 7

Finding the corresponding pairs of distance and time, we see that Jordan will run 200 meters in 80 seconds, 300 meters in 120 seconds. Since 60 seconds is halfway between 40 seconds (100 meters) and 80 seconds (200 meters), we find that Jordan ran 150 meters in 60 seconds.

Once students are proficient at representing ratios with unlike units by tables and double number lines, they move to a graphical representation, as in Figure 8. Using a table and selecting one part of the ratio to represent the horizontal measure ($x$) and the other piece of the ratio to represent the vertical measure ($y$), we can plot each pair of numbers on a grid with an appropriately chosen scale. In Figure 8, we have chosen the first column to be the horizontal measure, and the second column as the vertical measure. In general, this will be how the axes are chosen, unless specifically mentioned otherwise.

| Time (sec) | Dist. (m) | Point $(x, y)$ |
|---|---|---|
| 40 | 100 | (40, 100) |
| 80 | 200 | (80, 200) |
| 120 | 300 | (120, 300) |
| 160 | 400 | (160, 400) |

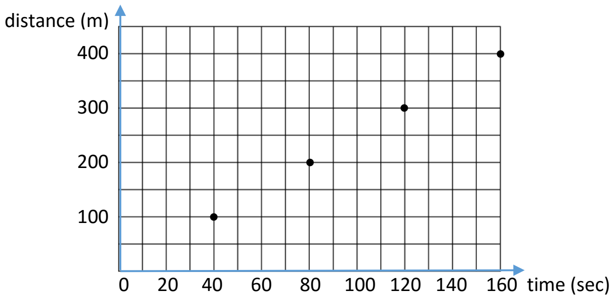



Figure 8. Graphical representation

One of the practice standards is: *Look for and express regularity in repeated reasoning*. Examining the table and the graph reveals patterns inherent in ratios. In the example of Jordan running, we see that each time the number of seconds increases by 40, the distance increases by 100. We can also see multiplicative patterns in moving from the point (40, 100) to (120, 300) as the 40 and the 100 both tripled (multiplied by 3). When graphing equivalent ratios, students may notice that the points lie on a line in the first quadrant that starts at (0, 0). They will build on that observation as they write equations for relationships later in this chapter. This theme will be developed throughout grades 7 and 8, ending with a complete integration of linear graphs, slopes and unit rates. Here in 6th grade we expect students to notice that the graphs they construct are straight lines and to become familiar with this on an intuitive level.

| Time (sec) | | Dist. (m) | | Point $(x, y)$ | |
|---|---|---|---|---|---|
| 40 | +40 | 100 | +100 | (40, 100) | |
| 80 | +40 | 200 | +100 | (80, 200) | •3 |
| 120 | +40 | 300 | +100 | (120, 300) | |
| 160 | | 400 | | (160, 400) | |

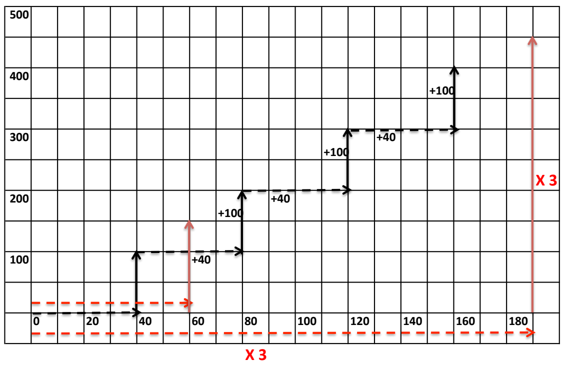



Figure 9. The preceding Figure 8, now explicitly exhibiting the additive and multiplicative structure

Using the patterns and techniques learned, students next apply knowledge of ratios to solving problems involving rates. Rates, like ratios can be expressed in variety of ways (10 kilometers in half an hour, 6 cups of flour to 4 cups of sugar, and so forth). Eventually it comes down to the idea of *unit rate*: the amount of one quantity that corresponds to 1 unit of the other quantity. So 10 km in half an hour leads to the unit rate of 20 km per hour (or 1/3 km per minute); the unit rate for cookies is 1.5 cups of flour per cup of sugar (or 2/3 cup of sugar per cup of flour). In short, the designation of unit rate *must* be clear about the choice and order of the units. We illustrate this in the next example.

**Example 4**. Taylor is buying candy for a class party, paying $5 for a 2-pound bag of candy. At that price, how much would one pound of candy cost? And how much candy could Taylor buy for $1?

Students may model the situation in a variety of ways; here we focus on the development of thinking. First, we look for the unit rate (see Figure 10, applying the relation 2 = 1+1).

| |
|---|
| 5 Dollars |
| 2 pounds |

| | |
|---|---|
| $2.50 | $2.50 |
| 1 pound | 1 pound |

Figure 10, exhibiting the unit rate for dollars per pound.

Or, one could argue: 1 pound is half of 2 pounds, so if we divide each bar in half, we will find out how much it would cost for 1 pound of candy. From Figure 10, we see that the unit rate of dollars to 1 pound of candy is $2.50 to 1.

To find the other unit rate (pounds of candy for $1), we recognize that $5 needs to be split into 5 equal pieces, and thus the weight also needs to be split into 5 equal pieces (see Figure 11), giving 2/5 = 0.4 pounds in each piece. Since 2/5 = 0.4 as decimal, this gives a unit rate of pounds to dollars of 0.4:1.

| | | | | |
|---|---|---|---|---|
| **$1.00** | **$1.00** | **$1.00** | **$1.00** | **$1.00** |
| **0.4 lb** | **0.4 lb** | **0.4 lb** | **0.4 lb** | **0.4 lb** |

Figure 11

We could also have modeled this problem using the double number line learned earlier. In a similar way, we cut the distance (from 0 to 2 pounds) into two equal pieces and then do the same for the interval from 0 to 5 (dollars) to find that the unit rate is 2.50 dollars to 1 pound of candy (see Figure 12).

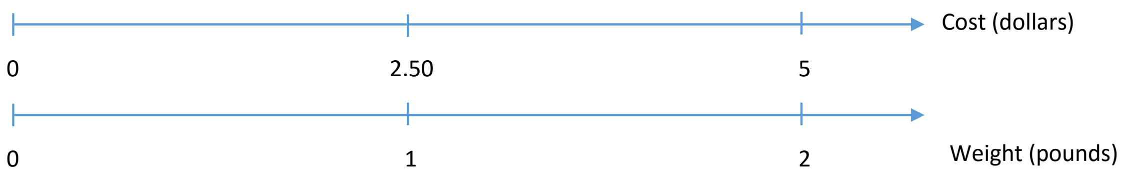



Figure 12

Cutting the interval from 0 to 5 into five equal pieces and then doing the same to the interval of 0 to 2 gives a unit rate of 2/5 = 0.4 pounds of candy for every dollar (see Figure 13).

Recalling the multiplicative patterns in the tables for ratios, students may use a partial table to find the unit rates.

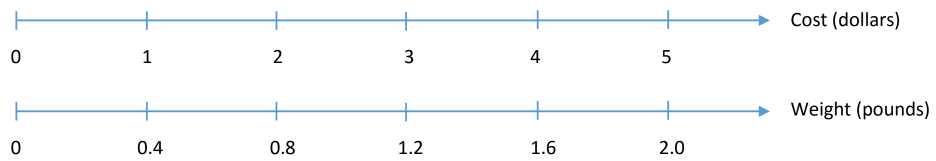
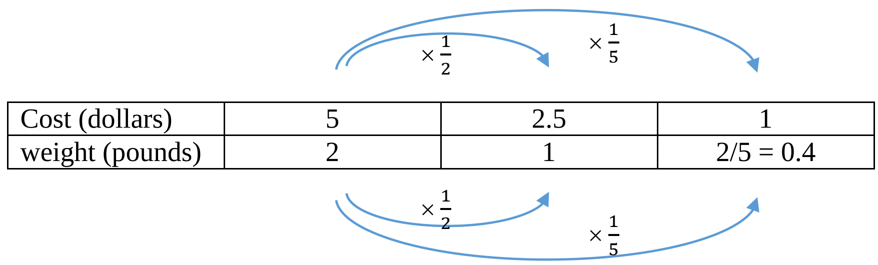



Figure 13

| Cost (dollars) | 5 | | 1 |
|---|---|---|---|
| weight (pounds) | 2 | 1 | |




| Cost (dollars) | 5 | 2.5 | 1 |
|---|---|---|---|
| weight (pounds) | 2 | 1 | 2/5 = 0.4 |

Figure 14

There are also numeric methods that students may choose to use. Writing the ratio in quotient form

$$\frac{5 \text{ dollars}}{2 \text{ pounds}},$$

students can use techniques to simplify:

$$\frac{5 \text{ dollars}}{2 \text{ pounds}} = \frac{2 \cdot 2.5 \text{ dollars}}{2 \cdot 1 \text{ pounds}} = \frac{2.5 \text{ dollars}}{1 \text{ pound}} = \frac{5}{2}\frac{\text{dollars}}{\text{pound}},$$

which can also be expressed as \$2.5 per pound. Using the same method, the unit rate of pounds per dollar can be found:

$$\frac{2 \text{ pounds}}{5 \text{ dollars}} = \frac{5 \cdot 0.4 \text{ pounds}}{5 \cdot 1 \text{ dollars}} = \frac{0.4 \text{ pounds}}{1 \text{ dollar}} = \frac{2}{5}\frac{\text{pounds}}{\text{dollar}},$$

Students often note that the two unit rates are reciprocals of each other. Once again, it is important to attend to precision when identifying which unit rate you are referring to, pounds per dollar or dollars per pound.

Finding the unit rate is very helpful in solving real world problems.

**Example 5**. This problem continues the preceding: remember from that problem that the cost of a 2 pound bag of candy is \$5. Two other classes are going to join Taylor's class for a 6th grade party. Taylor is going to need more candy. With help from the teacher, Taylor estimates they will need 12 pounds of candy. How much will it cost to purchase this much candy? If the students raise \$40, how much candy can they buy? Use at least two different strategies.

SOLUTION. #1: A double-bar diagram can also be used. Since 12 lbs is $6 \times 2$ lbs, we write down six copies of the bar \$5 over 2 lbs, and see that 6 groups of 2 lbs corresponds to 6 groups of \$5 = \$30.

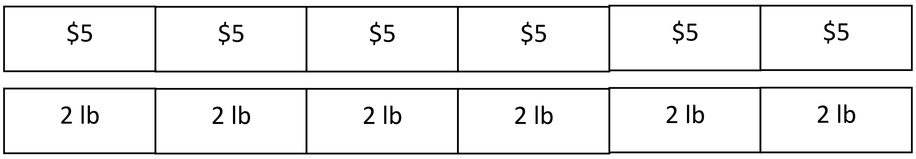



Figure 15

Similarly, \$40 is eight groups of \$5, so we write down eight copies of the bar \$5 over 2 lbs, and see that \$40 is 8 groups of \$5 dollars, so that corresponds to 8 groups of 2 lbs, or 16 lbs in total.

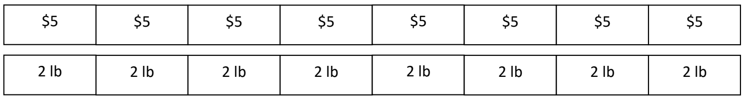



Figure 16

SOLUTION. #2: Double number line:

Using a double number line, and iterating the ratio, we can find the answers to these two questions: 12 pounds of candy corresponds to \$30 and \$40 corresponds to 16 pounds of candy.

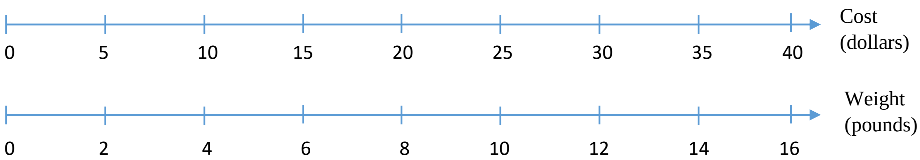



Figure 17

SOLUTION. #3: Students can also use repeated addition to solve the problems. $2 + 2 + 2 + 2 + 2 + 2 = 6 \times 2 = 12$ pounds of candy so it will cost $5 + 5 + 5 + 5 + 5 + 5 = 6 \times 5 = \$30$. Noting that $\$40 = 8 \times 5$, we see that \$40 will buy $8 \times 2 = 16$ pounds of candy.

SOLUTION. #4: Recalling the multiplicative patterns in the tables for ratios, student may use a partial table to find the unit rates.

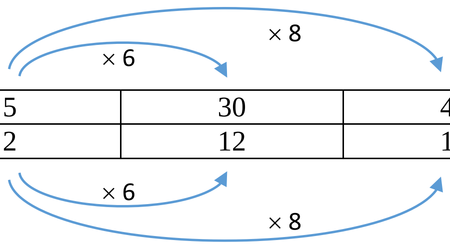
SOLUTION. # 5: Numerical methods can also be used. Again, using the quotient form for ratios we can solve these problems.

$$\frac{5 \text{ dollars}}{2 \text{ pounds}} = \frac{6 \times 5 \text{ dollars}}{6 \times 2 \text{ pounds}} = \frac{30 \text{ dollars}}{12 \text{ pounds}} \qquad \frac{5 \text{ dollars}}{2 \text{ pounds}} = \frac{8 \times 5 \text{ dollars}}{8 \times 2 \text{ pounds}} = \frac{40 \text{ dollars}}{16 \text{ pounds}}$$

*Use ratio reasoning to convert measurement units; manipulate and transform units appropriately when multiplying or dividing quantities. 6.RP.3d*

**Example 6**. When I ride my stationery bicycle, I have data presented to me. One datum is *pace*, another datum is *speed*. After many hours of riding the bike I have concluded that the product of these two numbers is always 60. Why?

| | | | |
|---|---|---|---|
| Cost (dollars) | 5 | ? | 40 |
| weight (pounds) | 2 | 12 | ? |




| | | | |
|---|---|---|---|
| Cost (dollars) | 5 | 30 | 40 |
| weight (pounds) | 2 | 12 | 16 |

Figure 18

SOLUTION. *Pace* refers to minutes per mile and *speed* refers to miles per hour. So, for example, my pace is 4 minutes per mile, and my speed is 15 miles per hour. Yes, the product is 60. Let's see why:

$$\frac{x \text{ minutes}}{\text{mile}} \times \frac{y \text{ miles}}{\text{hour}} = xy\frac{\text{minutes}}{\text{hour}} = 60\,.$$

Yes, so the product $xy$ is always 60 minutes per hour.

**Example 7**. Three children are trying to decide who can bike the fastest. Caesar can bike 4 miles in 15 minutes, Jasmine can pedal 3 miles in 10 minutes, and Drew can cycle 2 miles in 8 minutes. Who can bike the fastest?

SOLUTION. #1a: One method to solve this problem is to find the unit rate per mile for each child. For example, Caesar's rate of 15 minutes in 4 miles means he takes

$$\frac{15 \text{ minutes}}{4 \text{ miles}} = \frac{4 \times 3.75 \text{ minutes}}{4 \times 1 \text{ miles}} = \frac{3.75 \text{ minutes}}{1 \text{ mile}} \qquad \text{or 3.75 minutes per mile.}$$

Similarly, Jasmine bikes

$$\frac{10 \text{ minutes}}{3 \text{ miles}} = \frac{3 \times \frac{10}{3} \text{ minutes}}{3 \times 1 \text{ miles}} = \frac{3.\overline{3} \text{ minutes}}{1 \text{ mile}} \qquad \text{or } 3.\overline{3} \text{ minutes per mile.}$$

Drew bikes

$$\frac{8 \text{ minutes}}{2 \text{ miles}} = \frac{2 \times 4 \text{ minutes}}{2 \times 1 \text{ miles}} = \frac{4 \text{ minutes}}{1 \text{ mile}} \qquad \text{or 4 minutes per mile.}$$

Jasmine takes the least amount of time to bike one mile, so Jasmine is the fastest, followed by Caesar, then Drew.

SOLUTION. # 1b: Instead of finding the unit per mile, we could find the other associated unit rate, miles per minute. In this case, the fastest is the one who does the most miles per minute. So, for Caesar, we calculate

$$\frac{4 \text{ miles}}{15 \text{ minutes}} = \frac{15 \times \frac{4}{15} \text{ miles}}{15 \times 1 \text{ minutes}} = \frac{\frac{4}{15} \text{ miles}}{1 \text{ minute}}\,,$$

telling us that Caesar bikes at a rate of 4/15 miles per minute, or $0.2\overline{6}$ miles /minute. Repeating the calculation for Jasmine and Drew, we find that Jasmine bikes 0.3 miles in a minute, and Drew bikes 0.25 miles a minute. Since Jasmine covers the greatest distance in one minute, she is the fastest, followed by Caesar and then, Drew.

SOLUTION. # 2: Another method to solve this problem would be to find common reference points in tables. Notice that from the table Caesar bikes 8 miles in 30 minutes (second row) while Jasmine bikes 9 miles in 30

| Caesar | |
|---|---|
| Distance (miles) | Time (minutes) |
| 4 | 15 |
| 8 | 30 |
| 12 | 45 |
| 16 | 60 |
| 20 | 75 |

| Jasmine | |
|---|---|
| Distance (miles) | Time (minutes) |
| 3 | 10 |
| 6 | 20 |
| 9 | 30 |
| 12 | 40 |
| 15 | 50 |

| Drew | |
|---|---|
| Distance (miles) | Time (minutes) |
| 2 | 8 |
| 4 | 16 |
| 6 | 24 |
| 8 | 32 |
| 10 | 40 |

Figure 19

minutes (third row). Since Jasmine covers more distance in the same amount of time, she is faster than Caesar. Jasmine covers 12 miles in 40 minutes (fourth row) while Drew bikes 10 miles in 40 minutes (fifth row). Because Jasmine bikes farther in the same amount of time, she is faster than Drew. Finally, we need to compare Caesar and Drew. Compare the second row of Caesar's table (8 miles in 30 minutes), to the fourth row of Drew's table (8 miles in 32 minutes) to find that Caesar bikes the same distance in less time so he is faster than Drew. Putting it all together, we see that Jasmine is the fastest, followed by Caesar, and then Drew.

**Two-variable Equations**

*Use variables to represent numbers and write expressions when solving a real-world or mathematical problem; understand that a variable can represent an unknown number, or, depending on the purpose at hand, any number in a specified set. 6.EE.6*

*Use variables to represent two quantities in a real-world problem that change in relationship to one another; write an equation to express one quantity, thought of as the dependent variable, in terms of the other quantity, thought of as the independent variable. Analyze the relationship between the dependent and independent variables using graphs and tables, and relate these to the equation. For example, in a problem involving motion at constant speed, list and graph ordered pairs of distances and times, and write the equation* $d = 65t$ *to represent the relationship between distance and time. 6.EE.9*

Before grade 6, students have been solving problems like this: Suppose the price of 2 pounds of peaches at the local grocery store is $2.50.

- If Jonathan bought 3.5 lbs of peaches, what was his cost?
- If Joanna spent $12.50 on peaches, how many pounds of peaches did she buy?

It is anticipated that students understand these questions operationally, and proceed as follows. First to find the cost of a pound of peaches, divide $2.50 by 2, to get $1.25. Now in the first problem, *multiply* the number of pounds by $1.25, and in the second case, divid the exenditure by $1.25

In this first chapter of grade 6, students begin to understand the problem in generality: the *ratio* "cost: weight of peaches" is "2.5:2." where the units are dollars for cost and pounds for weight, or in terms of the *unit rate* : $1.25 per pound.

This is the first step in a developmental progression leading to the understanding of the concept of a relation between two variables given by an equation (or inequality). At this point standards 6.EE.6 and 6.EE.9 come into play. The intent here is to introduce students to functional relations, illustrating it in the context of ratios, so as to prepare students for an eventual understanding of the abstract concept.

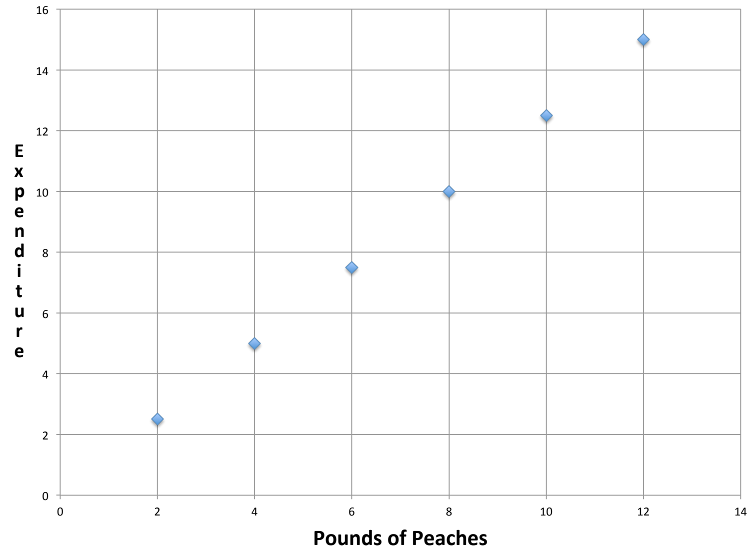
Although it may seem premature, we introduce this in the first chapter so that students have an opportunity to develop their understanding through the subjects of ratio, percentage and unit rate.

We begin with the transition from *unknown* to *variable* and from *equation* to *relation*; a transition that is central to the mathematics of the next few years. We have already, on occasion, used the terms "variable" and "relation" colloquially, without reference to their mathematical meaning; here we begin to explore that meaning.

Let's move from Jonathan's expenditure to any possible purchase of peaches. If Jonathan spends $E$ dollars to buy an amount $A$ (in pounds) of peaches at \$1.25 per pound, then we know that $E$ and $A$ are related by this price: specifically $E = (1.25) \times A$. This equation *relates* the *variables* $E$ and $A$ in this context. We can use the relation to solve these problems: the expenditure for $A$ pounds of peaches is $(1.25) \times A$; An expenditure of $E$ dollars on peaches, brings home $E/1.25$ pounds of peaches.

If we want to better understand the equation $E = (1.25)A$ as a relation between the *variables* $E$ and $A$, we can do so by creating a *table* of corresponding values, and then graphing those pairs of values. Let's illustrate with the peach purchase:

| | | | | | | |
|---|---|---|---|---|---|---|
| Pounds of Peaches | 2 | 4 | 6 | 8 | 10 | 12 |
| Expenditure | 2.5 | 5 | 7.5 | 10 | 12.5 | 15 |




Comparing the pairs of values in the table, we observe that every time we add 2 more pounds of peaches, the expenditure increases by \$2.50. We might also notice that if we double the amount of peaches, then the expenditure doubles. A more careful observation shows that if we multiply the amount of peaches by any number $m$, the expenditure also is multiplied by $m$.

The graph suggest strongly that the values lie on a line. If we use that observation as a fact, we can then use the graph to find other pairs of values; for example three and a quarter pounds of peaches will cost about \$4. We would also know that all we needed was one of the pairs in the table, since (0,0) is clearly on that line (no money, no peaches). This is where we are headed in middle school mathematics.

To recapitulate: when we want to visualize a relation among two variables, we calculate representative pairs of numbers that are in this relation, and then plot those pairs on a piece of graph paper. The variable corresponding to the first number of these pairs is plotted along the horizontal axis, and is typically called the *independent variable*. The variable corresponding to the second number of these pairs is plotted along the vertical axis, and is typically called the *dependent variable*. It is not always clear which is which and unless the text specifies a preference, the decision is a matter of choice.

For example, in the above example, one can buy peaches at \$2.50 for two pounds. We use the variables $E$ for expense, and $A$ for the number of pounds of peaches. Then, we can express the ratio ($E$:$A$:: 2.50:2) as an equality of quotients

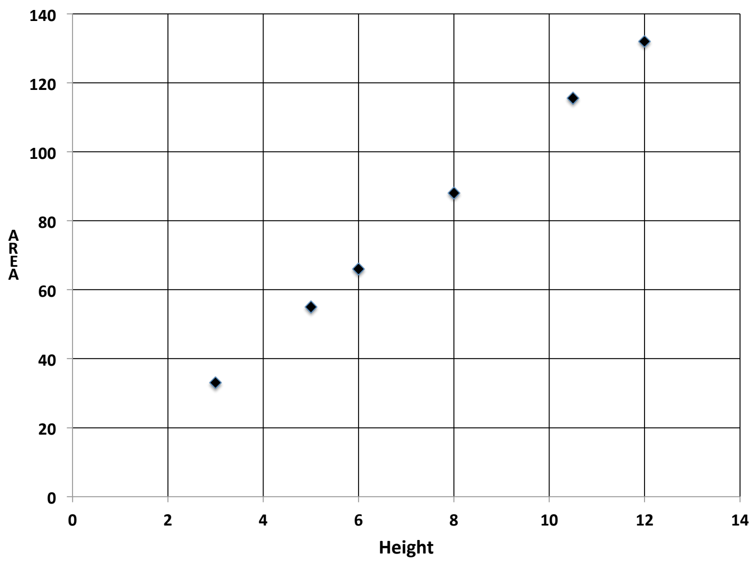
$$\frac{E}{A} = \frac{2.50}{2} = 1.25,$$

where "\$1.25 per pound." is the *unit rate*. If our thinking is oriented around the amount of peaches to be bought, we select (as we did) $A$ for the independent variable, and $E$ for the dependent variable, and write $E = 1.25 \cdot A$. However, if the amount of peaches is determined by the funds available to Jonathan, we would have made the other choice ($A = E/1.25$). In many cases it is clear which variable is the independent variable, because the problem is so stated.

**Example 8**. Linda wants to draw various images on a part of her study wall that is of 11 inches in width. She has in mind six images of height 3, 5, 6, 8, 10.5, 12 inches. To calculate the amount of paint (of each color) she will use, she needs to know the areas of these rectangular images. Make a table and a graph relating height to area.

SOLUTION. This uses the fact that the area of a rectangle of dimensions $W$ by $H$ is $A = WH$. We return to this fact in Chapter 5, as the beginning point of various area formulas. Since the width is fixed at 11 inches, the formula is $A = 11H$.

| | | | | | | |
|---|---|---|---|---|---|---|
| Height | 3 | 5 | 6 | 8 | 10.5 | 12 |
| Area | 33 | 55 | 66 | 88 | 115.5 | 132 |




The table and graph show the same features as in the preceding illustration, and that should not be surprising since the expression for the relation is of the same form. Note the *form* of the relation, and not the letters being used: the form is $y = px$, where $p$ is a positive number, and $x, y$ are replaced in each case by letters relevant to the context.

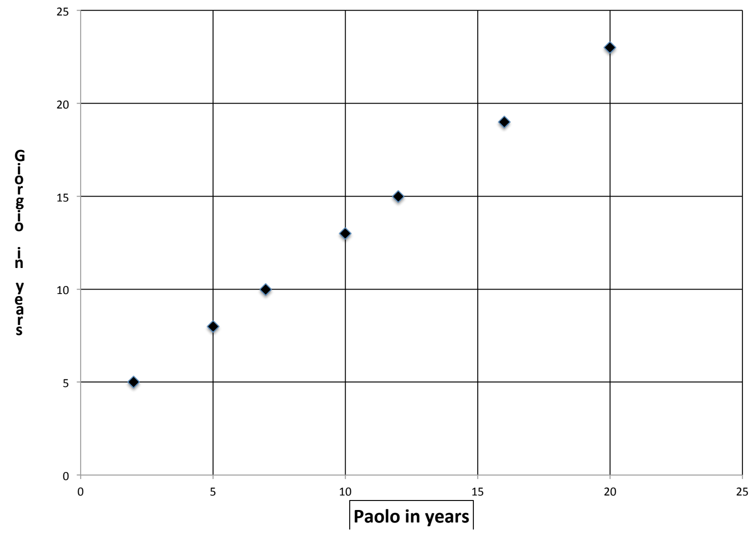
The preceding examples have illustrated the equation $y = px$ in a couple of contexts; in each case the *variables* $y$ and $x$ are in a constant ratio; that is, we could write this as $y : x :: p : 1$.

So far we have considered the case where the variables $y$ and $x$ are related by a positive factor: $y = px$. The other equation studied in grade 6 is the case where the variables $y$ and $x$ are related by a positive addend: $y = x + p$. Let's illustrate that as a relation among variables.

**Example 9**. When Giorgio was 5 years old, his younger brother, Paulo, was 2 years old. Make a table and graph of the age of Giorgio when Paulo is 5, 7, 10, 12, 1 6, 20 years old.

SOLUTION.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| Paulo | 2 | 5 | 7 | 10 | 12 | 16 | 20 |
| Giorgio | 5 | 8 | 10 | 13 | 15 | 19 | 23 |




Here the table shows us something different. It is *not* true that if we double one variable, the other automatically doubles. What is true is adding some number (say $N$) to one of the variables, the other is changed (additively) by the same amount. Of course, this is obvious: in $N$ years *everyone* will be $N$ years older. What is also true is that the difference of the variables is a constant (in this case 3). Compare this to the case of a ratio, where it is the quotient of the variables that is constant.

Nevertheless, the graph of these value pairs also seems to lie on a line. It will be useful to the students to compare the graphs of an equation of the form $y = ax$ and $y = x + b$. In both cases, the value pairs appear to lie on a line. However, there is this *significant* difference: In the case of $y = x + b$, if we add a certain amount to $x$, we add the same amount to $y$. However, in the case of $y = ax$, if we multiply $x$ by a certain amount, $y$ multiplies by the same amount. In both cases the graphs of value pairs for the equation produces a straight line. This similarity, despite the difference in "how the variables change" can give rise to some student consternation. IngGrade 7, students will study proportional relations (the variables always have the same quotient), and additive relations (the difference between two samples is the same for both variables) and, in grade 8, linear relations (the difference between the variables for any two samples has the same quotient). The purpose of this discussion in grade 6 is to set a good foundation for this development over the subsequent two years.

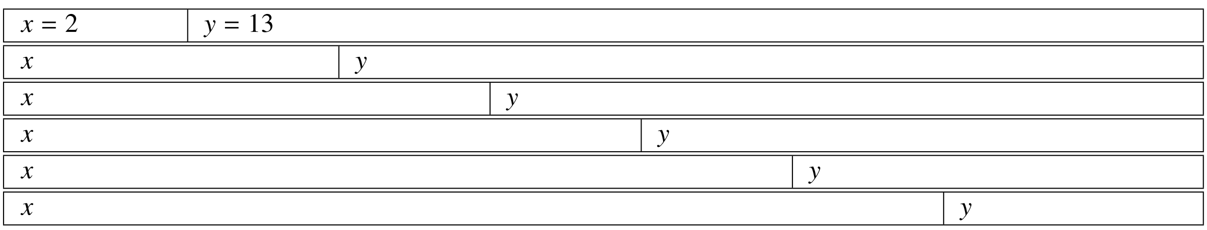
The following example goes beyond the goals for the course, but it is within reach of the students and thus come up. For example, what if the equation relating the variables is $x + y = 15$?

**Example 10**. In the image on the next page, a rod of length 15 cm has been partitioned into two pieces of length $x$ cm and $y$ cm in 6 different ways. We start with $x = 2$ and increase $x$ by 2 cm in each successive image.

a) Make a table of $x$ and $y$ values.

b) Make a graph of these values.

c) Write an equation relating $x$ and $y$.



SOLUTION. a)

| $x$ | 2 | 4 | 6 | 8 | 10 | 12 |
|---|---|---|---|---|---|---|
| $y$ | 13 | 11 | 9 | 7 | 5 | 3 |

b) The horizontal axis is that of the $x$ values, and the vertical is that of the $y$ values, as in Figure 20 below.

c) Since $x$ and $y$ are the lengths of the partition of the same rod, it must be the case that their sum is the total length, 15 cm. Thus, the equation is $x + y = 15$. Note that in order to do the calculation of the $y$ values, one used the equation $y = 15 - x$.

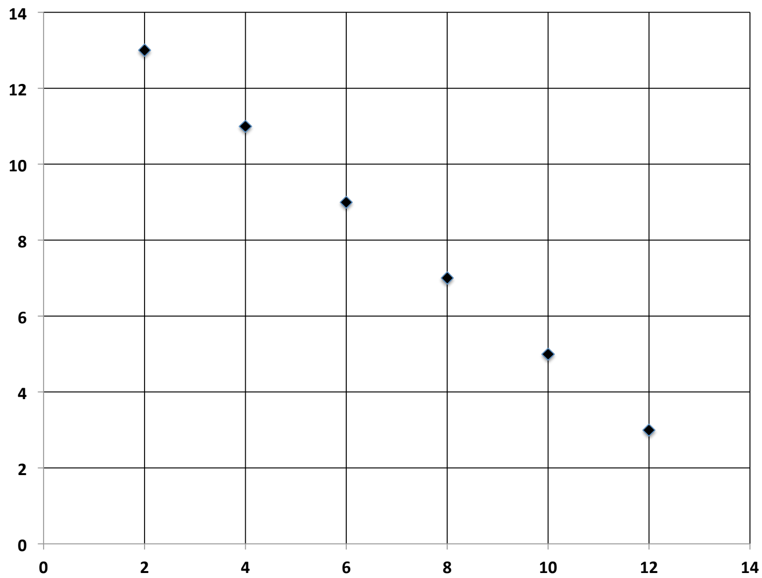



**Figure 20**

Notice that we have not labeled the axes. If the issue comes up, one answer is: "if the context doesn't suggest axis values, it is standard to put the $x$ values along the horizontal axis, and the $y$ values along the vertical axis. In this case, it doesn't matter: the graph will be the same."

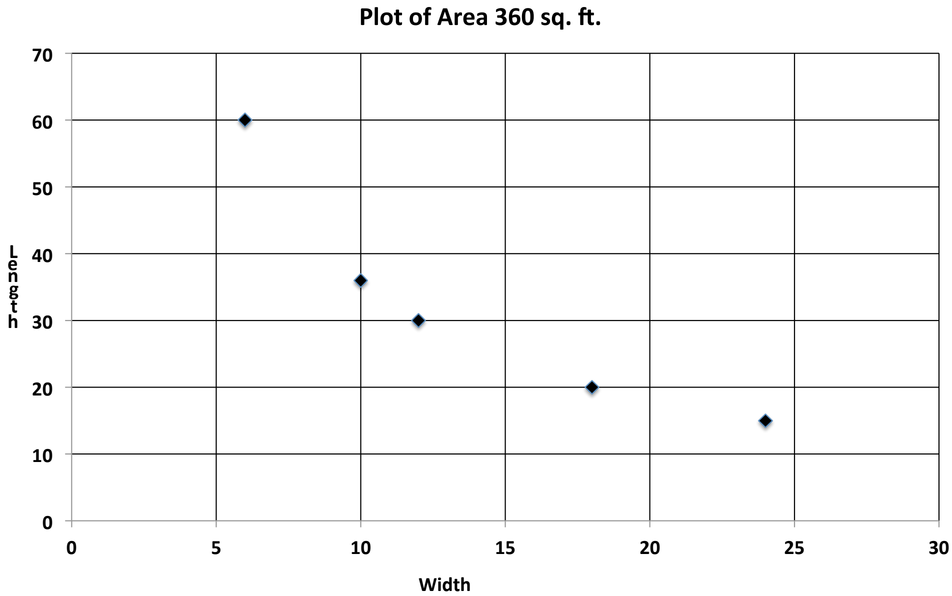
**Example 11**. Rodrigo will begin growing cucumbers on his farm, and has decided to set aside a rectangle of area 360 sq. ft for the cucumber patch. To decide on the shape, he has considered patches of the following widths (in feet): 6, 10, 12, 18, 24.

a) Draw the rectangles corresponding to these widths.

b) Graph the pairs (width, length) for these 5 samples.

SOLUTION. a)

| width | 6 | 10 | 12 | 18 | 24 |
|---|---|---|---|---|---|
| height | 60 | 36 | 30 | 20 | 15 |




The purpose of this example is simply to show that not all equations with which students have become familiar lead to a graph of values lying on a straight line.

Let us now recapitulate the content of this chapter. Often in life, we run into situations where we notice patterns in the numbers generated by the situation. More precisely, patterns in the pairs of numbers generated by a relation between two quantities. In modeling the situation by a table or a graph, the nature of the relation could just appear before our eyes. Going further, we try to represent the relation with an equation. To illustrate, let's return to Taylor's potential purchase of candy. Taylor was buying candy for a class party. 2 pounds of candy cost \$5. We then found the unit rate of \$2.50 for 1 pound of candy. We can calculate the amounts of candy we can buy with several different amounts of money; this we reproduce in Figure 20:

| Pounds of candy | Cost (in dollars) | Pattern |
|---|---|---|
| 1 | 2.50 | $2.50 = 2.50 \times 1$ |
| 2 | 5.00 | $5.00 = 2.50 \times 2$ |
| 3 | 7.50 | $7.50 = 2.50 \times 3$ |
| 4 | 10.00 | $10.00 = 2.50 \times 4$ |
| 5 | 12.50 | $12.50 = 2.50 \times 5$ |
| 6 | 15.00 | $15.00 = 2.50 \times 6$ |
| $p$ | | $c = 2.50 \times p$ |

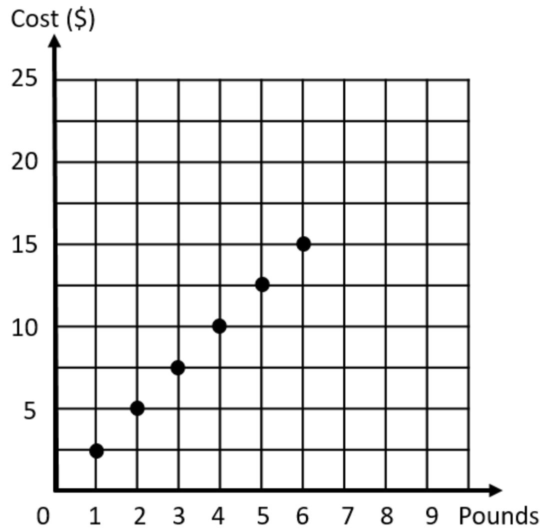



Figure 20. Taylor buys candy.

We could never list all possible pairs of points that represent the number of pounds of candy and the cost. However, we do notice a pattern in the table: the cost (in dollars) is always 2.5 times the number of pounds of candy. We can summarize this by letting a letter, $p$, stand for the number of pounds of candy and another letter, $c$, represent the cost. Looking at the last row of our table, we see that cost, $c = 2.50 \times p$. We can verify that this equation works by inserting into this equation values for $p$ and $c$ taken from any row, and finding a true statement (as has been done in the last column of the table). Looking at the graph, we may notice that the pairs (cost in dollars, pounds of candy) appear to lie on a line.

At this point, it suffices that students recognize patterns in these situations, draw a graph and write an equation to represent the pattern. In later grades they will explore the significance of the coefficient (multiplier of 2.50 in this case) and the theory of the equation of a line. At this stage, students may recognize that the number you multiply by in the equation is the unit rate. However, students should not be exposed to “slope” and “intercept” at this level.