The faculty of an elementary school gathered for a staff development workshop and quickly completed the first mathematics problem assigned by the facilitator: “What is \( \frac{1}{5} \times \frac{1}{4} \)?” Within a few seconds, every teacher in the room arrived at the correct answer: \( \frac{1}{20} \). The follow-up question, at first glance, seemed to be equally simple: “How did you get your answer?” “You multiply the numerators, and then you multiply the denominators,” offered a participant. The facilitator pushed harder. “That is the formula. You have described the steps you were taught to follow. I would like to challenge you to demonstrate why the formula works. In other words: Prove it!”

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The distinction between the facilitator’s two requests—“Tell me the answer” and “Demonstrate the thinking behind the answer”—is an important one. To provide a correct answer, teachers needed only use a familiar procedure. By asking teachers to “prove it,” the facilitator pushed them to dig a little deeper to the mathematical concept underlying the formula. This article describes strategies to heighten teachers’ awareness of the difference between procedural knowledge and conceptual understanding. By pressing classroom instructors to discover and then articulate the reasoning behind familiar formulas, teacher educators can demonstrate the power of teaching for understanding. Many teachers need to develop their own mathematical proficiency, as they “may know the facts and procedures they teach but often have a relatively weak understanding of the conceptual basis for that understanding” (Kilpatrick, Swafford, and Findell 2001, p. 372). At the same time, by engaging teachers as learners, staff developers can deepen their own appreciation of how much teachers actually do know about mathematics—knowledge that teachers can come to articulate when and if they are led through carefully designed activities that allow for trial and error, collaborative problem solving, and self-reflection.

Demonstrate Your Thinking

Fraction multiplication is an example of a topic frequently taught through memorization of procedural steps. The traditional approach provides learners with an efficient route to the correct answer. Even teachers who do not cover fractions as part of their curriculum can usually recall and apply the formula quickly. Knowing the formula, however, does not necessarily lead to a conceptual understanding of what it means to multiply numbers that are less than one. In the scenario above, the teachers were first given a few minutes to “prove” that $1/5 \times 1/4 = 1/20$ using pencil and paper. Without further prompting, a few teachers started sketching groups of circles in fours or fives or generated other representations of fractions. Many did not move beyond the formula—at least until the facilitator reminded participants that a multiplication sign can also be read as “of.” This hint was enough for nearly all the teachers to begin struggling individually with how to represent multiple sets of fractional numbers.

After each teacher had arrived at an answer or other “resting place,” the facilitator invited participants to share with the group. The teachers demonstrated many different diagrams and thinking processes—an observation the facilitator highlighted to underscore the diverse ways of thinking among any group of learners. The facilitator then asked one teacher to come to the easel to develop a possible diagram. First, the teacher was asked to draw a square, divide it into fourths horizontally, and represent $1/4$ by circling one row. Then, the teacher was to divide the paper into fifths by drawing vertical lines and coloring one-fifth of the already circled one-fourth. By shading one-fifth of one-fourth, the teacher colored one-twentieth of the grid (see fig. 1).

The visual representation yielded a few “aha” moments, but some participants needed another demonstration. The facilitator then physically involved the whole group in the mathematics problem, ensuring that even the teachers who had not arrived at the answer independently now became engaged in the process. Twenty teachers were chosen as “manipulatives.” The remaining four or five teachers became “mathematicians.” The task of using the manipulatives to model the explanation was given to the mathematicians, and the facilitator stepped back to allow the teachers to think, debate, and make a few false starts on their way to arriving at the correct solution. A first attempt consisted of lining up the manipulatives in a row of twenty, with
Facilitator steps in a “Prove It” activity for staff development

1. Identify a formula or mathematical procedure that teachers can apply but are unlikely to be able to derive or articulate.
2. Demonstrate teachers’ procedural fluency by providing a simple problem that requires them to apply the formula.
3. Demonstrate teachers’ need for a more conscious conceptual understanding by asking for an explanation of why the formula works. Prove it!
4. Allow time for independent exploration and intellectual struggle. As teachers work, the facilitator listens, observes, and asks questions that guide teachers from one level of understanding to the next. Do not evaluate participants’ attempts or provide answers.
5. Once teachers have reached individual solutions, allow for a brief discussion of their findings.
6. Facilitate the discovery of one or more alternative solutions using an entirely different mode of instruction. This step allows for a richer exploration of the concept as well as a greater appreciation for the importance of accessing multiple learning styles in mathematics instruction.
7. Allow time for reflection: Ask teachers to think aloud about what they learned and how they learned it; invite them to connect the learning experience to their own teaching practice; and highlight the social-emotional aspects of learning mathematics.
8. Reinforce and extend teacher learning by providing time for them to adapt the activity for their own classroom use or to begin to develop similar activities inspired by their experience.

This discussion was as important as the mathematical demonstrations that preceded it. Effective professional development enables teachers to connect their own learning experiences to the context of teaching, and it does so in a way that builds educators’ skills in analyzing instruction.

In its entirety, the “Prove It!” activity allowed the workshop participants to evaluate and enhance their own content knowledge. By asking teachers to extend themselves intellectually, emotionally, and physically, the activity pressed the teachers to reflect on their own conceptual understanding and, perhaps more important, that of their students. In the remainder of the workshop, teachers engaged in two additional activities to reinforce the theme and further illustrate the pedagogy associated with teaching for understanding. Figure 2 summarizes a facilitator process for planning and implementing professional development activities that engage teachers in discovering and explaining the logic behind familiar procedures and formulas.

The Pythagorean Theorem

A second, similar activity involved the teachers in an exploration of the Pythagorean theorem. The goal was to allow teachers to discover the powerful difference between the ability to plug numbers into the equation $a^2 + b^2 = c^2$ and the ability to articulate that the equation means that for any right triangle, the area of the square made with the hypotenuse is equal to the sum of the two squares made by the two legs of the triangle (see fig. 3).

Using an activity adapted from teacher-educator Aleta Margolis (2007), the facilitator gave teachers several labeled cardboard squares of different dimensions—three inches by three inches, four by four, five by five … ten by ten—and so on. The for-
mula, along with its textbook definition, was posted on chart paper, and the teachers were directed to use the squares to demonstrate the Pythagorean theorem. That was all the instruction given. Although unsure of what to do at first, the teachers began to manipulate the squares in different ways. Some folded the squares in half, which produced right triangles but did not lead to a model of the theorem. Others began by placing the squares one on top of another but abandoned the idea when they realized that the method did not create triangles (see fig. 4).

Eventually, many began to move toward a solution by using the squares to create triangles in the negative space between the sides of the squares (see fig. 5). By experimenting to determine which squares made right triangles and which did not, participants discovered that the triangle in the negative space made with a three-by-three-, a four-by-four-, and a five-by-five-inch square was a right triangle; but the triangle in the negative space made with a five-by-five- and two three-by-three-inch squares was not a right triangle. From a learner’s perspective, this activity helps build a visual and visceral understanding of an abstract concept.

The teachers talked as they worked, and as the activity came to a close, the facilitator noted how much mathematical vocabulary was used during the lesson: “I heard right triangle, hypotenuse, similar, different, square, equation, equals, times, multiple, and many more terms all used naturally in the course of peer-to-peer interaction.” Participants agreed that this was the sort of atmosphere they would like to create in their own classrooms—an environment where mathematical terms come up as frequently as literary terms such as book, poem, or sentence. Providing a context where students can acquire and use mathematical terminology is a crucial step toward helping students build mathematical communication skills; such skill is an important element of students’ overall mathematical proficiency (NCTM 2000).

Other reflections suggested the value of this sort of teacher support in overcoming “math phobia.” One teacher noted that the hands-on experience sparked her creativity: “I’m a follow-the-book kind of teacher when it comes to mathematics. But as we were doing this, I was thinking of all the other standards and topics I could touch on through this activity—negative and positive space, congruent shapes, all kinds of stuff. I’m always coming up with spin-offs and extensions with other subjects but not usually with mathematics, so it was nice to see I have it in me!” Just as a student’s “productive
disposition” toward mathematics is essential for mastering content and concepts, teachers also must be given the chance to develop the sense that they are capable of learning and teaching mathematics (Kilpatrick, Swafford, and Findell 2001, p. 384).

Finding \( \pi \)

The next step in the staff development workshop was to ask the teachers about \( \pi \). Volunteers quickly recalled that \( \pi \) equals approximately 3.14, and one quipped that “pie aren’t square; they’re round,” but no one could articulate that the significance of \( \pi \) is that the ratio of a circle’s circumference divided by its diameter holds true for all circles. For the next hands-on activity, adapted from Burns (2004), the group divided into pairs. Each pair of teachers was given a ball of yarn, a pair of scissors, a ruler, and a calculator. Additionally, the groups received an ordinary cylindrical object—a film canister, a box of salt, a soda can, and so on. Partners applied themselves to using the ruler to measure the diameter of their cylinder and then using the yarn to measure the circumference. (Fig. 6 shows students working through this same problem.)

Each pair of teachers recorded their measurements on a large chart that everyone could see. Finally, using a calculator, the teachers divided the circumference of each object by its diameter and recorded their results on the chart. As the final column was filled with close approximations of \( \pi \), one teacher noted, “My kids would love this. It’s like magic. You get the same answer every time.” Along with gaining a better understanding of \( \pi \)—the mathematical constant—the teachers gained a better conceptual understanding of the often-used and comfortable formula \( C = \pi d \). The follow-up discussion highlighted the teachers’ growing awareness of the importance of teaching for understanding: “These kinds of lessons would make a big difference for kids who are always mixing up which formula goes with what. I can see that doing this activity would help kids remember that \( \pi d \) is the formula for circumference, not area.”

Summary and Conclusion

The form of these teacher learning activities is as important as the substance. When staff developers allow teachers to grapple with mathematical ideas, the resulting professional learning is also personal and often powerful. Such learning cannot be achieved through any shortcut. As a high school mathematics teacher put it, “I always found that when I used discovery methods in my class, I had to sum everything up for the students at the end to make sure they really learned it. Eventually, I figured, why not skip the exploration and just go straight to the explanation? I thought of myself as saving time. But this workshop has reminded me that the discovery phase is not something you can just skip. It is not an appetizer; it’s the meal.” Asking teachers to develop physical representations of familiar formulas treated teachers as learners, engaging them intellectually. Too often, in-service teacher professional development provides practitioners with “the answers” and bypasses teachers’ thinking in the process. In contrast, the problem-based, collaborative nature of the activities described here puts teachers in action: thinking, debating, and rethinking. Learning for understanding is as critical for teachers as it is for students, for only then can the benefits of the professional development opportunity extend beyond the specific activity or technique modeled in the in-service session: “Perhaps the most important feature of learning with understanding is that such learning is generative. When students acquire knowledge with understanding, they can apply that knowledge to learn new topics and solve new and unfamiliar problems” (Carpenter and Lehrer 1999).

The temporarily destabilizing nature of the “Prove It!” process has an important emotional
impact on teachers as well. After an activity, reflecting on their experience is important for the teachers. In addition to cementing or enhancing teachers’ understanding of the mathematical concept at hand, debriefing allows the teachers to reflect on the process of being a learner. Some teachers experience confusion, frustration, and even a brief intellectual paralysis when initially tackling novel mathematical problems. For adults accustomed to the luxury of always knowing what will be expected next in the classroom, challenging learning activities can be a valuable reminder of the feelings that children experience on a daily basis as they continually face new and unfamiliar intellectual terrain. Skilled facilitation leads teachers to reflect on the social-emotional aspects of mathematical learning and connect their own learning experiences to the experiences of their students. By reentering the role of learner, teachers can reflect on past experience in order to shape their students’ future experiences.

Making such a link to classroom practice and student learning is the hallmark of effective professional development. The “Prove It” process underscores the critical difference between procedural compliance and true understanding for teachers—but the ultimate goal of the process is to lead teachers to teach for understanding in their own classrooms. Traditional, dominant mathematics instruction methods ask children to perform rote memorization or apply plug-it-in formulas. Presenting mathematics as a series of rules to be followed is like teaching history as a series of dates to be memorized—such approaches cut off learners’ thinking, sacrifice understanding for efficiency, and often stifle learners’ intrinsic interest in the subject as well. The alternative, teaching for understanding, requires more time and a bit more tolerance for the messiness of intellectual struggle but results in a fuller, more substantive appreciation of content.

References